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SIMPLICIAL VERTICES IN THE DIRECT SUM, STRONG PRODUCT, LEXICOGRAPHIC PRODUCT AND MAXIMAL PRODUCT OF TWO FUZZY GRAPHS

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Abstract

In this paper, simplicial vertices in some products of two fuzzy graphs are studied. Conditions for a vertex to be simplicial in the direct sum, strong product, Lexicographic product and Maximal product of two crisp graphs are derived. Using them and the effective edge properties, conditions for a vertex to be simplicial in the above operations of two fuzzy graphs are obtained.

1. Introduction

Azriel Rosenfeld [10] introduced Fuzzy graph theory in 1975. He has developed fuzzy analogue of many graphs' theoretic concepts. Since then, it has been growing fast and has numerous applications in various fields. Terry A. Mckee [11] introduced the concept of simplicial and non-simplicial complete subgraphs. K. Radha and S. Arumugam introduced and discussed the effective properties of direct sum [3], strong product [6], lexicographic

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products [4] and maximal product [5] of two fuzzy graphs. K. Radha and N. Kumaravel [8, 9] discussed the degree of an edge in union, join, cartesian product and composition of two fuzzy graphs. K. Radha and P. Indumathi [7] discussed about properties of simplicial vertices in some graphs. In this paper, simplicial vertices in some products of two fuzzy graphs are studied.

Conditions for a vertex to be simplicial in the direct sum, strong product, Lexicographic product and Maximal product of two crisp graphs are derived. Using them and the effective edge properties, conditions for a vertex to be simplicial in the above operations of two fuzzy graphs are obtained.

2. Basic Concepts

Definition 2.1 [8]. Let V be a non-empty finite set and $E \subseteq V \times V$. A fuzzy graph $G : (\sigma, \mu)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ such that $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$. Underlying crisp graph of $G : (\sigma, \mu)$ is denoted by $G^* : (V, E)$.

Definition 2.2 [8]**.** A fuzzy graph $H : (P, \sigma', \mu')$ is called fuzzy subgraph of $G : (V, \sigma, \mu)$ if $\sigma'(u) \leq \sigma(u), \forall u \in P$ and $\mu'(uv) = \mu(uv), \forall u, v \in P \cdot (\sigma', \mu')$ is a spanning fuzzy subgraph of (σ, μ) if $\sigma = \sigma'$ and $\mu' \subseteq \mu$, that is, if $\sigma(u) = \sigma'(u)$ for every $u \in V$ and $\mu'(e) \leq \mu(e)$ for every $e \in E \cdot (\sigma', \mu')$ is an induced fuzzy subgraph of (σ, μ) if $\sigma(u) = \sigma'(u)$ for every $u \in P$ and $\mu'(uv) = \mu(uv)$ for every $u, v \in P$.

Definition 2.3 [3]**.** An edge uv is effective if $\mu(xy) = \sigma(x) \wedge \sigma(y)$. G is an effective fuzzy graph if $\mu(xy) = \sigma(x) \wedge \sigma(y)$ for all $xy \in E$. G is a complete fuzzy graph if $\mu(xy) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, that is, if all its edges are effective edges.

Definition 2.4 [3]**.** Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying crisp graphs $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. Let $V = V_1 \cup V_2$ and Let $E = \{uv/u, v \in V; uv \in E_1$ or $uv \in E_2$ but not both$\}$. The direct sum [3] of G_1 and G_2 is a fuzzy graph $G_1 \oplus G_2 = G : (\sigma, \mu)$ given by

$$\sigma(u) = \begin{cases} \sigma_1(u), & \text{if } u \in V_1 \\ \sigma_2(u), & \text{if } u \in V_2 \\ \sigma_1(u) V \sigma_2(u), & \text{if } u \in V_1 \cup V_2 \end{cases} \quad \text{and} \quad \mu(uv) = \begin{cases} \mu_1(uv), & \text{if } uv \in E_1 \\ \mu_2(uv), & \text{if } uv \in E_2 \end{cases}$$

The strong product [6] $G_1 \circ G_2(\sigma, \mu)$ of G_1 and G_2 on (V, E), where $V = V_1 \times V_2$ and $E = \{((u, v)(x, y)) \mid u = x, vy \in E_2$ or $v = y, ux \in E_1$ or $ux \in E_1, vy \in E_2\}$, is given by

$\sigma((u, v)) = \sigma_1(u) \wedge \sigma_2(u)$ and

$$\mu((u, v)(x, y)) = \begin{cases} \sigma_1(u) \wedge \mu_2(vy), & \text{if } u = x, vy \in E_2 \\ \sigma_2(v) \wedge \mu_1(ux), & \text{if } v = y, ux \in E_1 \\ \mu_1(ux) \wedge \mu_2(uy), & \text{if } ux \in E_1, vy \in E_2 \end{cases}.$$

The lexicographic min-product [4] $G_1[G_2]\min : (\sigma, \mu)$ of G_1 with G_2 on (V, E), where $V = V_1 \times V_2$ and $E = \{((u, v)(x, y)) \mid ux \in E_1$ or $v = x$, $uy \in E_2\}$, is given by $\sigma((u, v)) = \sigma_1(u) \wedge \sigma_2(v)$ and

$$\mu((u, v)(x, y)) = \begin{cases} \mu_1(ux), & \text{if } ux \in E_1 \\ \sigma_1(u) \wedge \mu_2(vy), & \text{if } u = x, uy \in E_2 \end{cases}$$

The lexicographic max-product[4] $G_1[G_2]\min : (\sigma, \mu)$ of G_1 with G_2 on (V, E), where $V = V_1 \times V_2$ and $E = \{((u, v)(x, y)) \mid ux \in E_1$ or $v = x$, $uy \in E_2\}$, is given by $\sigma((u, v)) = \sigma_1(u) \wedge \sigma_2(u)$ and

$$\mu((u, v)(x, y)) = \begin{cases} \mu_1(ux), & \text{if } ux \in E_1 \\ \sigma_1(u) \wedge \mu_2(vy), & \text{if } u = x, uy \in E_2 \end{cases}.$$

The maximal product [5] $G_1 \circ G_2(\sigma, \mu)$ of G_1 and G_2 on (V, E), where $V = V_1 \times V_2$ and $E = \{((u, v)(x, y)) \mid u = x, vy \in E_2$ or $v = y, ux \in E_1\}$, is given by $\sigma((u, v)) = \sigma_1(u) \wedge \sigma_2(u)$ and

$$\mu((u, v)(x, y)) = \begin{cases} \sigma_1(u) \vee \mu_2(vy), & \text{if } u = x, vy \in E_2 \\ \sigma_2(v) \vee \mu_1(ux), & \text{if } v = y, ux \in E_1 \end{cases}.$$

3. Simplicial Vertices in Direct Sum

If a vertex is simplicial in either of the two fuzzy graphs, then it need not be simplicial in their direct sum. The vertex u in the following figure 3.1 is simplicial in both G_1 and G_2. But it is not simplicial in their direct sum.

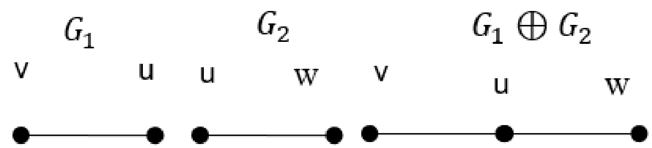

Figure 3.1.

Theorem 3.1. *Let $G_1^* : (V, E)$ and $G_1^* : (V, E)$ be two graphs. Let u be a vertex in $V_1 - V_2$ such that all edges of $N_{G_1}^*[u]$ are in E_1 only. Then u is simplicial in G_1^* if and only if u is simplicial in $G_1^* \oplus G_2^*$.*

Proof of Theorem 3.1. Since all the edges of $N_{G_1}^*[u]$ are in E_1 only, they all appear in $G_1 \oplus G_2$. Since $u \in V_1 - V_2$, no other edge is incident at u in the direct sum. Therefore $N_{G_1^*}[u] = N_{G_1^* \oplus G_2^*}[u]$. Hence the result follows.

Theorem 3.2. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two effective fuzzy graphs on $G_1^* : (V, E)$ and $G_2^* : (V, E)$ respectively. Let $u \in V_1 - V_2$ be such that all the edges of $N_{G_1^*}[u]$ are in E_1 only and each edge vw of $N_{G_1^*}[u]$ with one end $v \in V_1 \cap V_2$ satisfies $\sigma_1(v) \geq \sigma_1(w)$. Then u is simplicial in the fuzzy graph G_1 if and only if u is simplicial in $G_1 \oplus G_2$.*

Proof of Theorem 3.2. By theorem 3.1, u is simplicial in G_1^* if and only if u is simplicial in $G_1 \oplus G_2$. Also $N_{G_1^*}[u] = N_{G_1^* \oplus G_2^*}[u]$. Let u be simplicial in G_1. Let $e = uw$ be any edge of $N_{G_1^* \oplus G_2^*}[u]$. Then $\mu_1(vw) = \sigma_1(v) \wedge \sigma_1(w)$ and $(\mu_1 \oplus \mu_2)(vw) = \mu_1(vw)$ By hypothesis, both v and w cannot be in $V_1 \cap V_2$.

Therefore, either both $v, w \in V_1$ or one of the vertices, say $v \in V_1 \cap V_2$ and $w \in V_1$.

Case 1. $v, w \in V_1$

Using the definition of direct sum,

$(\mu_1 \oplus \mu_2)(vw) = \mu_1(vw) = \sigma_1(v) \wedge \sigma_1(w) = (\sigma_1 \oplus \sigma_2)(v) \wedge (\sigma_1 \oplus \sigma_2)(w)$

Case 2. $v \in V_1 \cap V_2$ and $w \in V_1$

By hypothesis $\sigma_1(v) \geq \sigma_1(w)$.

Therefore, $(\sigma_1 \oplus \sigma_2)(v) = \sigma_1(v) \vee \sigma_2(w) \geq \sigma_1(v) \geq \sigma_1(w) = (\sigma_1 \oplus \sigma_2)(w)$

Hence $(\sigma_1 \oplus \sigma_2)(v) \wedge (\sigma_1 \oplus \sigma_2)(w) = (\sigma_1 \oplus \sigma_2)(w)$

Now $(\mu_1 \oplus \mu_2)(vw) = \mu_1(vw) = \sigma_1(v) \wedge \sigma_2(w) = \sigma_1(w)$

$= (\sigma_1 \oplus \sigma_2)(w) = (\sigma_1 \oplus \sigma_2)(v) \wedge (\sigma_1 \oplus \sigma_2)(w)$

Hence in both cases, vw is an effective edge.

Therefore $N_{G_1} \oplus G_2[u]$ is complete. Hence u is simplicial in $G_1 \oplus G_2$.

Conversely let u be simplicial in $G_1 \oplus G_2$. Let vw be any edge of $N_{G_1}[u]$.

Then $(\mu_1 \oplus \mu_2)(vw) = (\sigma_1 \oplus \sigma_2)(v) \wedge (\sigma_1 \oplus \sigma_2)(w)$ (3.1)

If $v, w \in V_1$, then the above relation (3.1) becomes $(\mu_1)(vw) = \sigma_1(v) \wedge \sigma_1(w)$

If $v \in V_1 \cap V_2$ and $w \in V_1$, then $(\sigma_1 \oplus \sigma_2)(v) \geq (\sigma_1 \oplus \sigma_2)(w)$

Therefore $(\sigma_1 \oplus \sigma_2)(v) \wedge (\sigma_1 \oplus \sigma_2)(w) = (\sigma_1 \oplus \sigma_2)(w) = \sigma_1(w)$

$= \sigma_1(v) \wedge \sigma_1(w) (\because \sigma_1(v) \geq \sigma_1(w))$

Therefore (3.1) becomes $(\mu_1)(vw) = \sigma_1(v) \wedge \sigma_1(w)$.

Thus in both cases, vw is an effective edge.

Therefore $N_{G_1}[u]$ is complete. Hence u is simplicial in $G_1 \oplus G_2$.

4. Simplicial Vertices in Strong Product

Theorem 4.1. *Let* $G_1^* : (V, E)$ *and* $G_2^* : (V, E)$ *be two graphs. A vertex* (u, v) *is simplicial in the strong product* $G_1^* \circ G_2^*$ *if and only if is simplicial in* G_1^* *and* v *is simplicial in* G_2^*.

Proof of Theorem 4.1. Let $G^* = G_1^* \circ G_2^*$. Let (u, v) be simplicial in G^*. Then $G^*[N_{G^*}[(u, v)]]$ is complete. Suppose that u is not simplicial in G_1^*. Then there exists $x, y \in N_{G_1^*}(u)$ such that $xy \notin E_1$. Since $ux, uy \in E_1$, the vertices (x, v) and (y, v) are adjacent to (u, v) in G^*. But (x, v) and (y, v) are not adjacent in G^* since $xy \notin E_1$. Therefore $G^*[N_{G^*}[(u, v)]]$ is not complete which is a contradiction. Hence u is simplicial in G_1^*. Similarly, v is simplicial in G_2^*.

Conversely suppose that u is simplicial in G_1^* and v is simplicial in G_2^*. Consider any two vertices (x, y) and (s, t) in $N_{G^*}[(u, v)]$. If one of them is (u, v), then they are adjacent. So let $(x, y) \neq (u, v)$ and $(s, t) \neq (u, v)$. Since (u, v) is adjacent to both (x, y) and (s, t), $x = u, yv \in E_2$ or $y = v, xu \in E_1$ or $xu \in E_1, yv \in E_2$; $s = u, tv \in E_2$ or $t = v, us \in E_1$ or $su \in E_1, tv \in E_2$. Therefore, there are nine cases to consider for (x, y) and (s, t).

Case 1. $x = u, yv \in E_2, s = u, tv \in E_2$

Then $x = s$. Since $yv \in E_1, tv \in E_2$ and v is simplicial in G_2^*, $yt \in E_2$.

Hence (x, y) and (s, t) are adjacent in G^*.

Case 2. $x = u, yv \in E_2, t = v, us \in E_1$

Then $xs \in E_1$ and $yt \in E_2$. Hence (x, y) and (s, t) are adjacent in G^*.

Case 3. $x = u, yv \in E_2, su \in E_1, tv \in E_2$

Then $sx \in E_1$. Also since $yv \in E_2, tv \in E_2$ and v is simplicial in G_2^*, $yt \in E_2$.

Hence (x, y) and (s, t) are adjacent in G^*.

Case 4. $y = v, xu \in E_2, s = u, tv \in E_2$

Then $xs \in E_1$, $yt \in E_2$. Hence (x, y) and (s, t) are adjacent in G^*.

Case 5. $y = v, xu \in E_1, t = v, us \in E_1$

Then $y = t$. Also since $xu \in E_1, us \in E_1$ and u is simplicial in G_1^*, $xs \in E_1$.

Hence (x, y) and (s, t) are adjacent in G^*.

Case 6. $y = v, xu \in E_1, su \in E_1, tv \in E_2$.

Then $tv \in E_2$. Also since $xu \in E_1, su \in E_1$ and u is simplicial in G_1^*, $xs \in E_1$.

Hence (x, y) and (s, t) are adjacent in G^*.

Case 7. $xu \in E_1, yv \in E_2, s = u, tv \in E_2$.

Then $xs \in E_1$. Since $yv \in E_2, tv \in E_2$ and v is simplicial in G_2^*, $yt \in E_2$.

Hence (x, y) and (s, t) are adjacent in G^*.

Case 8. $xu \in E_1, yv \in E_2, t = v, us \in E_1$.

Then $yt \in E_2$. Also since $xu \in E_1, us \in E_1$ and u is simplicial in G_1^*, $xs \in E_1$.

Hence (x, y) and (s, t) are adjacent in G^*.

Case 9. $xu \in E_1, yv \in E_2, su \in E_1, tv \in E_2$.

Since u and v are simplicial in G_1^* and G_2^* respectively,

$xu \in E_1$, $su \in E_1 \Rightarrow xs \in E_1$ and $yv \in E_2$, $tv \in E_2 \Rightarrow yt \in E_2$.

Hence (x, y) and (s, t) are adjacent in G^*.

Therefore, any two vertices in $N_{G^*}[(u, v)]$ are adjacent. Hence $G^*[N_{G^*}[(u, v)]]$ is complete. This implies that (u, v) is simplicial in G^*.

Theorem 4.2. *Consider two fuzzy graphs* $G_1 : (\sigma_1, \mu_1)$ *and* $G_{22} : (\sigma_2, \mu_2)$ *on crisp graphs* $G_1^* : (V, E)$ *and* $G_2^* : (V, E)$ *respectively. A vertex* (u, v) *is simplicial in the strong product* $G_1 \circ G_2$ *if and only if u is simplicial in the fuzzy graph* G_1 *and v is simplicial in the fuzzy graph* G_2.

Proof of Theorem 4.2. Let $G = G_1 \circ G_2$ be the strong product of the fuzzy graphs G_1 and G_2.

Let (u, v) be simplicial in G. Then (u, v) is simplicial in G^* and all the edges of $G^*[N_{G^*}[(u, v)]]$ are effective. By theorem 4.1, u is simplicial in G_1^* and v is simplicial in G_2^*. Let xy be any edges in $G_1^*[N_{G_1^*}[u]]$. Suppose xy is not effective in G_1. Then $\mu_1(xy) < \sigma_1(x) \wedge \sigma_1(y)$.

For the edge $(x, v)(y, v)$ in $G^*[N_{G^*}[(u, v)]]$,

$$(\mu_1 \circ \mu_2)((x, v)(y, v)) = \mu_1(xy) \wedge \sigma_2(v) < \sigma_1(x) \wedge \sigma_1(y) \wedge \sigma_2(v)$$

$$= (\sigma_1 \circ \sigma_2)(x, v) \wedge (\sigma_1 \circ \sigma_2)(y, v)$$

Thus the edge $(x, v)(y, v)$ is not effective in G. This is a contradiction.

Hence xy is effective in G_1.

Thus all the edges in $G_1^*[N[u]]$ are effective. Similarly, all the edges in $G_2^*[N[u]]$ are also effective. Hence u is simplicial in G_1 and v is simplicial in G_2.

Conversely assume that u is simplicial in G_1 and v is simplicial in G_2.

Then u is simplicial in G_1^* and all the edges of $G_1^*[N[u]]$ are effective; v is simplicial in G_2^* and all the edges of $G_2^*[N[u]]$ are effective.

By theorem 4.1, (u, v) is simplicial in G^*. Let $(x, y)(s, t)$ be any edge in $G^*[N[(u, v)]]$. Then $x, s \in G_1^*[N[(u, v)]]$ and $y, t \in G_2^*[N[(u, v)]]$.

Case 1. $x = s,\ yt \in E_2$

Since v is simplicial in G_2, yt is effective in G_2. So

$(\mu_1 \circ \mu_2)((x,\ y)(x,\ t)) = \sigma_1(x) \wedge \mu_2(yt) = \sigma_1(x) \wedge \sigma_2(y) \wedge \sigma_2(t)$

$= (\sigma_1(x) \wedge \sigma_2(y)) \wedge (\sigma_1(x) \wedge \sigma_1(t)) = (\sigma_1 \circ \sigma_2)(x,\ y) \wedge (\sigma_1 \circ \sigma_2)(x,\ t)$

Case 2. $y = t,\ xs \in E_1$

Since u is simplicial in G_1, xs is effective in G_1. Now

$(\mu_1 \circ \mu_2)((x,\ y)(s,\ y)) = \mu_1(xs) \wedge \sigma_2(y) = \sigma_1(x) \wedge \sigma_2(s) \wedge \sigma_2(y)$

$= (\sigma_1 \circ \sigma_2)(x,\ y) \wedge (\sigma_1 \circ \sigma_2)(s,\ y)$

Case 3. $xs \in E_1, yt \in E_2$

Here xs is effective in G_1 and yt is effective in G_2.

$(\mu_1 \circ \mu_2)((x,\ y)(x,\ t)) = \mu_1(x,\ s) \wedge \mu_2(y,\ t) = \sigma_1(x) \wedge \sigma_1(s) \wedge \sigma_2(y) \wedge \sigma_2(t)$

$= (\sigma_1(x) \wedge \sigma_2(y)) \wedge (\sigma_1(s) \wedge \sigma_2(t)) = (\sigma_1 \circ \sigma_2)((x,\ y)) \wedge (\sigma_1 \circ \sigma_2)((s,\ t))$

Hence $(x,\ y)(x,\ t)$ is effective in $G_1 \circ G_2$ all the three cases.

Hence $G[N[(u, v)]]$ is complete. Thus (u, v) is simplicial in $G_1 \circ G_2$.

Theorem 4.3. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs on $G_1^* : (V, E)$ and $G_2^* : (V, E)$ respectively. Then*

(i) *The edge $(u,\ y)(u,\ t)$ is effective in $G_1 \circ G_2$ if and only if the edge yt is effective in G_2.*

(ii) *The edge $(x,\ v)(s,\ v)$ is effective in $G_1 \circ G_2$ if and only if the edge xs is* *effective in* G_1.

(iii) *The edge* $(x, y)(s, t)$ *is effective in* $G_1 \circ G_2$ *if and only if the edge* xs *is effective in* G_1 *and* yt *is effective in* G_2.

5. Simplicial Vertices in Lexcicographic Products

Theorem 5.1. *Let* $G_1^* : (V, E)$ *and* $G_2^* : (V, E)$ *be two fuzzy graphs. A vertex* (u, v) *is simplicial in* $G_1^*[G_2^*]$ *if and only if* u *is simplicial in* G_1^* *and* v *is simplicial in* G_2^*.

Proof of Theorem 5.1. Let (u, v) be simplicial in $G_1^*[G_2^*]$. Suppose u is not simplicial in G_1^*. Then there exists $w \in N_{G_1^*}[u]$ such that $uw \notin E_1$. Then the edge $(u, v)(w, v)$ is not in $G_1^*[G_2^*]$. So (u, v) is not simplicial in $G_1^*[G_2^*]$ which is a contradiction. Hence u is simplicial in G_1^*. Similarly, v is simplicial in G_2^*.

Conversely assume that u is simplicial in G_1^* and v is simplicial in G_2^*. Suppose (u, v) is not simplicial in $G_1^*[G_2^*]$. Then there exist non-adjacent vertices $(x, y)(s, t) \in N_{G_1^*[G_2^*]}((u, v))$. Therefore either $xs \notin E_1$ or $x = s$ and $yt \notin E_2$. This implies that either u is not simplicial in G_1^* or v is not simplicial in G_2^*. This contradiction shows that (u, v) is simplicial in $G_1^*[G_2^*]$.

Remark 5.2. If u and v are simplicial in fuzzy graphs G_1 and G_2 respectively, then the vertex (u, v) need not be simplicial in the lexicographic min-product $G_1[G_2]$. For example, in the following figure 5.1, u_1 is simplicial in G_1 and v_1 is simplicialin G_2, but (u_1, v_1) is not simplicial in $G_1[G_2]_{\min}$.

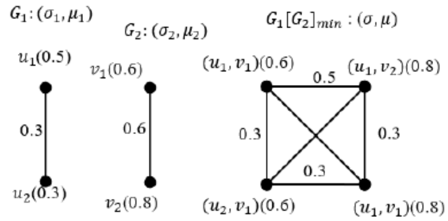

Figure 5.1.

Theorem 5.3. *Let* $G_1:(\sigma_1, \mu_1)$ *and* $G_2:(\sigma_2, \mu_2)$ *be two fuzzy graphs on* $G_1^*:(V, E)$ *and* $G_2^*:(V, E)$ *respectively such that* $\sigma_1, \mu_1, \sigma_2, \mu_2$ *are all constant functions of same constant value. Then a vertex* (u, v) *is simplicial in the lexicographic min-product (or max-product) of* G_1 *with* G_2 *if and only if* u *is simplicial in* G_1 *and* v *is simplicial in* G_2.

Proof of Theorem 5.3. Since $\sigma_1, \mu_1, \sigma_2, \mu_2$ are all constant functions of same constant value, $G_1, G_2, G_1[G_2]_{\min}$ and $G_1[G_2]_{\max}$ are all effective fuzzy graphs. Hence the result follows from theorem 5.1.

6. Simplicial Vertices in Maximal Product

If u is simplicial in a graph G_1^* and v is simplicial in a graph G_2^*, then the vertex (u, v) need not be simplicial in maximal product $G_1^* \bullet G_2^*$. Here in figure 6.1, u_1 and u_2 are simplicial in G_1^* and v_1 and v_2 are simplicial in G_2^*. But no vertex is simplicial in the maximal product $G_1^* \bullet G_2^*$.

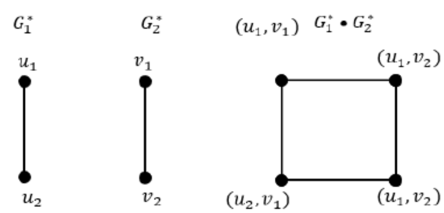

Figure 6.1.

Theorem 6.2. *Let* $G_1^*:(V_1, E_1)$ *and* $G_2^*:(V_2, E_2)$ *be two graphs with* $p_1 \geq 2$ *and* $p_2 \geq 2$ *vertices respectively. Then their maximal product* $G_1^* \bullet G_2^*$ *has no simplicial vertex.*

Proof of Theorem 6.2. Let (u, v) be any vertex in $G_1^* \bullet G_2^*$. Let u be adjacent to a vertex x in G_1 and v be adjacent to a vertex y in G_2. Then (u, v) is adjacent to (u, y) and (x, v) in $G_1^* \bullet G_2^*$. But by the definition of maximal product, (u, y) and (x, v) cannot be adjacent. Therefore (u, v) is not simplicial in $G_1^* \bullet G_2^*$.

Theorem 6.3. *Let G_1^* and G_2^* be two graphs such that G_1^* has exactly one vertex u. Then a vertex (u, v) in the maximal product $G_1^* \bullet G_2^*$ is simplicial if and only if v is simplicial in G_2^*.*

Proof of Theorem 6.3. If G_1^* has exactly one vertex u, then $G_1^* \bullet G_2^*$ is isomorphic to G_2^*. Hence the theorem follows.

Theorem 6.4. *If G_2^* has exactly one vertex v, then a vertex (u, v) in the maximal product $G_1^* \bullet G_2^*$ is simplicial if and only if u is simplicial in G_1^*.*

Theorem 6.5. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs on $G_1^* : (V, E)$ and $G_2^* : (V, E)$ respectively such that $V_1 = \{u\}$ and σ_1, σ_2 and μ_2 are all constant functions of same constant value. Then (u, v) is simplicial in $G_1 \bullet G_2$ if and only if v is simplicial in G_2.*

Proof of Theorem 6.5. Since σ_1, σ_2 and μ_2 are all constant functions of same constant value, G_2 and $G_1 \bullet G_2$ are effective fuzzy graphs. Hence the result follows from theorem 6.3.

Theorem 6.6. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs on $G_1^* : (V, E)$ and $G_2^* : (V, E)$ respectively such that $V_2 = \{u\}$ and σ_1, σ_2 and μ_1 are all constant functions of same constant value. Then (u, v) is simplicial in $G_1 \bullet G_2$ if and only if u is simplicial in G_1.*

7. Conclusion

In this paper, the conditions for a vertex to be simplicial in the direct sum, strong product, Lexicographic product and Maximal product of two crisp graphs are derived. Using them and the effective edge properties, the conditions for a vertex to be simplicial in the above operations of two fuzzy graphs are discussed.

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