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INTERIOR DOMINATION POLYNOMIALS OF GRAPHS

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Abstract

Let $D_{Id}(G, i)$ be the family of all interior dominating sets of a graph with cardinality i and Let $d_{Id}(G, i) = |D_{Id}(G, i)|$. Then interior domination polynomial $D_{Id}(G, x)$ of G is defined as $D_{Id}(G, x) = \sum_{i=\gamma_{Id}(G)}^{|V(G)|} d_{Id}(G, x)x^i$. Where $\gamma_{Id}(G)$ is cardinality of minimum interior domination number.

1. Introduction

Let $G = (V, E)$ be an undirected graph, without loop and multiple edges. A non empty set $D \subseteq V$ is a dominating set of G, if every vertex $V - D$ is adjacent to minimum one vertex in D. The cardinality of minimum dominating set is named as the domination number and is denoted by $\gamma(G)$ [4]. A vertex v is an interior vertex of G if for every vertex u distinct from v, there exists a vertex w such that v lies between u and w. A set $D \subseteq \gamma(G)$ is an interior dominating set if D is a dominating set of G and every vertex v

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interior vertex of G. Any pendent vertices will not be a member in interior set of a graph. $\gamma_{Id}(G)$ denoted as the cardinality of minimum interior dominating set. Let $D_{Id}(G, i)$ be the family of all interior dominating sets of a graph with cardinality i and Let $d_{Id}(G, i) = |D_{Id}(G, i)|$. Then interior domination polynomial $D_{Id}(G, x)$ of G is defined as $D_{Id}(G, x)$ $= \sum_{i=\gamma_{Id}(G)}^{|V(G)|} d_{Id}(G, x)x^i$.

2. Interior Dominating Sets and Polynomials of Graphs

Definition 2.1. Let $D_{Id}(G, i)$ be the family of all interior dominating sets of a graph with cardinality i. Then the interior domination number of G is defined as the minimum cardinality taken over all interior dominating sets of vertices in G and it is denoted by $\gamma_{Id}(G)$.

Theorem 2.2. [4] *If G is a triangle free connected graph with n vertices, then* $\left\lceil \frac{n}{\Delta + 1} \right\rceil \leq \gamma_{Id} \leq n - \Delta(G)$ *for every* $n \geq 4$.

Proof. Let G be a triangle free connected graph with n vertices.

Let $\gamma_{Id}(G)$ be a minimum interior dominating set of graph G.

If $y \in G$ is a vertex of $\Delta(G)$, then y dominates $\Delta + 1$ vertices of Graph G and $\gamma_{Id} \geq \left\lceil \frac{n}{\Delta + 1} \right\rceil$.

Next we consider upper bound.

If u is chosen so that maximum degree $\Delta(G)$, then u dominates $N[u]$. That is u dominates all but $n - \Delta(G)$ vertices of G.

Therefore $\gamma_{Id} \leq n - \Delta(G)$.

Hence $\left\lceil \frac{n}{\Delta + 1} \right\rceil \leq \gamma_{Id} \leq n - \Delta(G)$.

Corollary 2.3. *If G is a triangle free connected graph with* $n + 1$ *vertices,* *then* $\left\lceil \frac{n+1}{\Delta+1} \right\rceil \le \gamma_{Id} \le n - \Delta(G) + 1$ *for every* $n \ge 4$.

Proof. Replace n by $n+1$ of above theorem then we get $\left\lceil \frac{n+1}{\Delta+1} \right\rceil \le \gamma_{Id} \le n - \Delta(G) + 1$.

Definition 2.4. Let $D_{Id}(G, i)$ be the family of all interior dominating sets of a graph with cardinality i and Let $d_{Id}(G, i) = |D_{Id}(G, i)|$. Then interior domination polynomial $D_{Id}(G, x)$ of G is defined as $D_{Id}(G, x)$ $= \sum_{i=\gamma_{Id}(G)}^{|V(G)|} d_{Id}(G, x)x^i$. Where $\gamma_{Id}(G)$ is cardinality of minimum interior domination number.

Theorem 2.5. *Let G be a graph with* $|V(G)| = n$. *Then*

(i) $d_{Id}(G, n) = 1$ *and* $d_{Id}(G, n-1) = n$ *if* $2 \le \delta(G)$.

(ii) $d_{Id}(G, i) = 0$ *if and only if* $i < \gamma_{Id}(G)$ *or* $i > n$.

(iii) $D_{Id}(G, x)$ *has no constant term*.

(iv) *Let G be a connected graph and H be any induced subgraph of G. Then* $\deg(D_{Id}(G, x)) \ge \deg(D_{Id}(H, x))$.

(v) $D_{Id}(G, x)$ *is strictly increasing function in* $[0, \infty)$.

(vi) *Zero is a root of* $D_{Id}(G, x)$ *with multiplicity* $\gamma_{Id}(G)$.

Proof. (i) Since G has n vertices, there is only one way to choose all the vertices.

Therefore, $d_{Id}(G, n) = 1$.

Since G has n-1 vertices, there is we choose n ways of all the vertices.

Therefore, $d_{Id}(G, n-1) = n$.

(ii) Since $D_{Id}(G, i) = \varphi$ if $i < \gamma_{Id}(G)$ and $D_{Id}(G, n+k) = \varphi$, $k = 1, 2, 3, \ldots$, we have $d_{Id}(G, i) = 0$ if and only if $i < \gamma_{Id}(G)$ or $i < n$.

(iii) Single and double vertex cannot interior dominate itself. So, the set of all vertices of G. is interior dominated by atleast every vertices of degree two of G. Hence the interior domination polynomial has no constant term.

(iv) Since the number of vertices in $H \leq$ number of vertices in G.

$\deg(D_{Id}(G, x)) \geq \deg(D_{Id}(H, x))$.

The proof of (v) and (vi) follows from the definition of interior domination polynomial.

Theorem 2.6. *For any star with n vertices, for every* $\geq 3 D_{Id}(S_n, x) = x$.

Proof. Let S_n be a star with n vertices.

Let $D = \{D\}$ where u is the interior vertex of S_n. And also u is the central vertex of S_n.

Then D is a minimum interior dominating set of S_n.

Only one interior dominating set is $\{u\}$.

Therefore $\gamma_{Id}(S_n) = 1$.

Therefore $d_{Id}(S_n, 1) = 1$ for every $n \geq 3$.

Hence $D_{Id}(S_n, x) = x$.

Theorem 2.7. *Let* $K_{m, n}$ *be a complete bipartite graph with* $m + n$ *vertices. Then the interior domination polynomial of* $K_{m, n}$ *is* $D_{Id}(K_{m, n}, x) = [(1 + x)^m][(1 + x)^n] + x^m + x^n$.

Proof. Let $G = K_{m, n}$ be a complete bipartite graph with partite set V_1 and V_2.

$| V_1 | = m$ and $| V_2 | = n$.

Every vertex of V_1 is adjacent to every vertex of other set V_2.

Let $D = \{u, v\}$, $u \in V_1$ and $v \in V_2$.

The vertex u dominates all the vertices of V_2 and it is the interior dominating vertex. Similarly vertex v dominates all the vertices of V_1 and it is the interior dominating vertex.

Therefore D is a minimum interior dominating set and hence $\gamma_{Id}(K_{m,n}) = 2$

We have domination polynomial of $K_{m,n}$ is $D_{Id}(K_{m,n}, x) = [(1+x)^m]$ $[(1+x)^n] + x^m + x^n$ also $D(K_{m,n}, x) = D_{Id}(K_{m,n}, x)$.

Therefore interior domination polynomial of $K_{m,n}$ is $D_{Id}(K_{m,n}, x)$ $= [(1+x)^m][(1+x)^n] + x^m + x^n$.

Theorem 2.8. *Let $B_{m,n}$ be a bi-star with $m+n+2$ vertices. Then the interior domination polynomial of $B_{m,n}$ is $D_{Id}(B_{m,n}, x) = x^2$.*

Proof. Let $B_{m,n}$ be a bi-star with $m+n+2$ vertices.

Label the vertices of $B_{m,n}$ as $v_1, v_2, v_3, \ldots, v_m, v_{m+1}, v_{m+2}, \ldots, v_{m+n+2}$ as given in

Let $D = \{v_{m+1}, v_{m+2}\}$.

The vertex v_{m+1} and v_{m+2} dominate all the vertices and also v_{m+1} and v_{m+2} are interior vertex of G.

Then D is a minimum interior dominating set of $B_{m,n}$.

Therefore $\gamma_{Id}(B_{m,n}) = 2$ and $d_{Id}(B_{m,n}, 2) = 1$.

Hence $D_{Id}(B_{m,n}, x) = x^2$.

Corollary 2.9. *Let $B_{n,n}$ be a bi-star with $2n+2$ vertices. Then the interior domination polynomial of $B_{n,n}$ is $D_{Id}(B_{n,n}, x) = x^n$.*

Theorem 2.10. *Let G be any connected triangle free graph of order n.*

Then $D_{Id}(G \circ \overline{K}_m, x) = x^n$.

Proof. Since G has n vertices, $G \circ \overline{K}_m$ has $n(m+1)$ vertices.

Clearly $\{v_1, v_2, v_3, \ldots, v_n\}$ is the minimum interior dominating set of $G \circ \overline{K}_m$.

Therefore $\gamma_{Id}(G \circ \overline{K}_m) = n$ and $D_{Id}(G \circ \overline{K}_m, n) = 1$.

Hence $D_{Id}(G \circ \overline{K}_m, x) = x^n$.

3. Conclusion

In this paper we have described the interior dominating sets and some interior domination polynomials of graphs.

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