Advances and Applications in Mathematical Sciences
Volume 22, Issue 1, November 2022, Pages 169-177

EXPONENTIAL DIOPHANTINE EQUATIONS INVOLVING ISOLATED PRIMES

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Abstract

An exponential Diophantine equation is one of the special forms of Diophantine equations, in which the variables occur in exponents. So far, there exists some good exploration regarding such equations. Like those, here is our attempt, which makes use of isolated primes to resolve the equations $3^x + 67^y = z^2$, $3^x + 127^y = z^2$. This paper shows the exact solution sets for these equations along with the proof.

1. Introduction

The Pureness of Number theory has captivated mathematicians' generation after generation, each contributing some major theories in Mathematics. A basic understanding of number theory is an absolutely critical precursor to cutting edge software engineering, specifically security based software. Number Theory is the heart of cryptography, which is itself experiencing a fascinated period of rapid evolution, ranging from the famous RSA algorithm to the wildly popular block chain world. Modern Number

2020 Mathematics Subject Classification: 11A41; 11D61; 11D79; 11D45.
Keywords: Isolated Prime, Exponential Diophantine equation, integral points, Catalan's theorem, Co-Prime.
Received June 29, 2022; Accepted August 14, 2022

theory dealt with elementary, algebraic, analytical and probabilistic Number theory.

The first known study of Diophantine equations was by its namesake Diophantus of Alexandaria, a third century Mathematician who also introduced symbolisms into algebra. He was author of a series of books Arithmetica. One of the famous Diophantine problem is Hilbert's tenth problem; Given a Diophantine equation with any number of unknown quantities and with rational integral coefficients: To devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers. This problem was first solved in 1970 by Yuri Matiyasevich.

Exponential Diophantine equation is a polynomial equation with which the variables occur as exponents. In recent decades, many mathematicians dealt with the great interest of solving the Diophantine equations of type $a^x + b^y = c^z$, $x, y, z \in \mathbb{N}$. In 1933, Mahler recorded their first work. He only proved the fitness of solutions of the above equation. Some authors determined the complete solution of such equation for small values of a, b, c by using elementary congruence and catalan's conjecture.

The main theme of this paper is to analyze the solutions of two Diophantine equations $3^x + 67^y = z^2$ and $3^x + 67^y = z^2$. Notice that 67 and 127 are isolated primes; an isolated prime is a prime number p such that neither $p - 2$ nor $p + 2$ is prime.

2. Preliminaries

Proposition 2.1. $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$, *where* a, b, x *and y are integers such that* $\min\{a, b, x, y\} > 1$.

We present some Lemmas to prove the main results.

Lemma 2.2. $(1, 2)$ *is a unique solution* (x, z) *for the Diophantine equation* $3^x + 1 = z^2$, *where x and y are non-negative integers.*

Proof. Let $x, z \in \mathbb{N} \cup \{0\}$. If $x = 0$, then $z^2 = 2$ is non-viable. So take $x \geq 1$. Then $z^2 = 3^x + 1 \geq 4 \Rightarrow z \geq 2$. Consider $z^2 - 3^x = 1$. By Proposition 2.1, x must be equal to one. $z^2 = 4 \Rightarrow z = 2$.

Therefore $(1, 2)$ is the unique solution for the Diophantine equation $1 + 3^x = z^2$.

Lemma 2.3. *The Diophantine equation* $1 + 67^y = z^2$ *has no non-negative integer solution.*

Proof.

Case (i). If $y = 0$. When $y = 0$, we get an irrational solution. This is a contradiction.

Case (ii). If $y \geq 1$. Then $z^2 = 1 + 67^y \geq 68 \Rightarrow z \geq 9$. Therefore the equation $z^2 - 67^y = 1$ is solvable only if $y = 1$, by proposition 2.1. But $y = 1$, $z^2 = 68$ is not a square number.

Therefore there is no non – negative integral solution for the Exponential Diophantine equation $1 + 67^y = z^2$.

Lemma 2.4. *The Diophantine equation* $1 + 127^y = z^2$ *has no non-negative integral solution.*

Proof. We cannot consider the case $y = 0$. Assume $y \geq 1$. Then $z^2 = 1 + 127^y \geq 128 \Rightarrow z \geq 12$. By proposition 2.1, the equation $1 + 127^y = z^2$ is solvable whenever $y = 1$. But if $y = 1$, then z^2 ends with the number 8 which is not a square number. Therefore there is no solution for the equation $1 + 127^y = z^2$.

3. Main Results

Theorem 3.1. $(1, 0, 2)$ *is the unique solution for the Diophantine equation* $3^x + 67^y = z^2$, $x, y, z \in \mathbb{N} \cup \{0\}$.

Proof. We split up the case y into two cases. y is odd; y is even.

Case (i). y is even. If $y = 0,$ then by Lemma 2.2, $(1, 0, 2)$ is a solution. Then $y = 2n, n \in \mathbb{N},$ our equation becomes $3^x + 67^{2n} = z^2.$ This can be written as $z^2 - 67^{2n} = 3^x.$

$$\Rightarrow (z + 67^n)(z - 67^n) = 3^{\alpha+\beta}, \text{ where } \alpha + \beta = x.$$

$$\Rightarrow z + 67^n (z - 67^n) = 3^{\alpha} - 3^{\beta}, \text{ where } \beta > \alpha.$$

$\Rightarrow 2(67^n) = 3^{\beta} - 3^{\alpha} \cdot \alpha = 0$ is the only possible value. Therefore $2(67^n) = 3^x - 1.$ Adding both sides by $-2, -2 + 2(67^n) = 3^x - 3.$ This gives that $x = 2.$ This concludes that $67^n = 4,$ this is never be true.

Case (ii). y is odd. Then $y = 2n + 1, n \in \mathbb{N} \cup \{0\}.$ Therefore $3^x + 67^y = z^2$ can be written $3^x + 67^{2n+1} = z^2 \Rightarrow 3^x + 67(67^{2n}) = z^2.$

$$\Rightarrow 3^x + 64(67^{2n}) + 3(67^{2n}) = z^2$$

$$\Rightarrow 3^x + 3(67^{2n}) = z^2 - 8^2 67^{2n}$$

$$\Rightarrow 3(3^{x-1} + 67^{2n}) = (z + 8 \times 67^{2n})(z - 8 \times 67^n).$$

We observe that z is even. $\Rightarrow 3(3^{x-1} + 67^{2n}) = (2m + 8 \times 67^{2n})$ $(2m - 8 \times 67^n)$

$$= 4(m + 4 \times 67^n) = (m - 4 \times 67^n).$$

Now we have two possibilities: (i) $n = 3 - 4 \times 67^n$ (ii) $m = 3 + 4 \times 67^n.$

Sub case (i). $m = 3 - 4 \times 67^n.$

Then

$$3(3^{x-1} + 67^{2n}) = 4(3 - 4 \times 67^n + 4 \times 67^n)(3 - 4 \times 67^n - 4 \times 67^n).$$

Therefore

$$3^{x-1} + 67^{2n} = 4(3 - 8 \times 67^n) \Rightarrow 32 \times 67^n + 67^{2n} = 3(4) - 3^{x-1}$$

$\Rightarrow 67^n(67^n + 32) = 3(4 - 3^{x-2})$.

$n = 0$ is the only viable value for the above equation. But $3^{x-1} = 21$ is not possible.

Sub case (ii). $m = 3 + 4 \times 67^n$.

Then the equation becomes $3(3^{x-1} + 67^{2n}) = 4(3)(3 + 8 \times 67^n)$. This implies that $3^{x-1} + 67^{2n} = 4(3 + 8 \times 67^n) \Rightarrow 32 \times 67^n + 67^{2n} = 3^{x-1} - 12$.

$\Rightarrow 67^n(32 - 67^n) = 3(3^{x-2} - 4) \cdot n = 0$ is the only possible value. But $3^{x-1} = 43$ is not solvable.

Thus in any cases, we cannot find the integral solution. Therefore $(1, 0, 2)$ is the unique solution for the Diophantine equation $3^x + 67^y = z^2$.

Theorem 3.2. $(1, 0, 2)$ *is the unique non-negative integer solution for the Diophantine equation* $3^x + 127^y = z^2$.

Proof.

Case (i). y is even. If $y = 0$, then by Lemma 2.4, $(1, 0, 2)$ is a solution for the given equation. Assume $y = 2n, n \in \mathbb{N}$. Then the equation will become $z^2 - 127^{2n} = 3^x \Rightarrow (z + 127^n)(z - 127^n) = 3^x \Rightarrow z + 127^n - z + 127^n = 3^\beta - 3^\alpha$, where $\alpha + \beta = x$ and $\beta > \alpha$. This gives that $2(127^n) = 3^\beta - 3^\alpha \cdot \alpha = 0$ is the only possible value. Therefore $2(127^n) = 3^x - 1$. Adding -2 on both sides, we get, $2(127^n - 1) = 3(3^{x-1} - 1)$. This gives that $x = 2$. But $127^n = 4$, which is absurd.

Case (ii). y is odd. That is $y = 2n + 1$, then our equation becomes $3^x + 127^{2n+1} = z^2$.

$3^x + 127^{2n+1} = z^2 \Rightarrow 3^x + 127(127^{2n}) = z^2$

$\Rightarrow 3^x + 10^2 \times 127^{2n} = z^2 - 27 \times 127^{2n}$

$$\Rightarrow 3^x + 27 \times 127^{2n} = z^2 - 10^2 \times 127^{2n}$$

$$\Rightarrow 3^x + 3^3 \times 127^{2n} = (z + 10 \times 127^n)(z - 10 \times 127^n).$$

Since z is even, $3^x + 3^3 \times 127^{2n} = 4(m + 5 \times 127^n)(m - 5 \times 127^n)$

$$\Rightarrow 3^u(3^{x-u} + 3^{3-u} \times 127^{2n}) = 4(m + 5 \times 127^n)(m - 5 \times 127^n) \, (1)$$

There are two possible cases. (i) $m = 3^u + 5 \times 127^n$ (ii) $m = 3^u + 5 \times 127^n$.

Sub case (i). $m = 3^u + 5 \times 127^n$. Then (1) becomes.

$4(3^u)(3^u + 10 \times 127^n) = 3^u(3^{x-u} + 3^{3-u} \times 127^{2n}) \Rightarrow 4(3^u + 10 \times 127^n) = 3^{x-u} + 3^{3-u} \times 127^{2n} \Rightarrow 3^{3-u} \times 127^{2n} - 40 \times 127^n = 4(3^u) - 3^u \Rightarrow 127^n(3^{3-u} \times 127^{2n} - 40) = 4(3^u) - 3^{x-u} \cdot n = 0$ is the only possible value. Therefore $3^{3-u} - 40 = 4(3^u) - 3^{x-u} \Rightarrow 3^3 - 40 \times 3^u = 4 \times 3^{2u} - 3^x \Rightarrow 4(3^{2u} + 10 \times 3^u - 7) = 3^x - 1 = (3^k + 1)(3^k - 1)$, since $x = 2k$. Then either $k = 0$ or $k = 1$ is possible for finding the solutions. When $k = 0, 3^{2u} + 10 \times 3^u - 7 = 0$, a contradiction. When $k = 1, x = 2$. $4 \times 3^{2u} + 40 \times 3^u - 28 = 8 \Rightarrow 3^{2u} + 10 \times 3^u = 9 \Rightarrow 3^u(3^u + 10) = 3^2$, not possible to find integer solution for such equations. Hus we conclude when $k = 1, u = 0$ is the only possible value. But $u = 0, 3^{x-1} = 16$ which is non-viable.

Sub case (ii). $m = 3^u + 5 \times 127^n$. Then equation (1) becomes $4 \times 3^u(3^u + 10 \times 127^n) = 3^u(3^{x-u} + 3^{3-u} \times 127^{2n}) \Rightarrow 4(3^u + 10 \times 127^n) = 3^{x-u} + 3^{3-u} \times 127^{2n} \Rightarrow 3^{3-u} \times 127^{2n} - 40 \times 127^n 4 \times 3^u - 3^{x-u} \Rightarrow 127^n (3^{3-u} \times 127^{2n} - 40) = 4(3^u) - 3^{x-u} \cdot n = 0$ is the one and only possible value. $3^{3-u} - 40 = 4 \times 3^u - 3^{x-u} \Rightarrow 4 \times 3^u(3^u - 10) = 3^3 + 3^x$. But $x = 2k$. Then $4(3^{2u} - 10 \times 3^u - 7) = (3^k + 1)(3^k - 1)$. Then either $k = 0$ or $k = 1$ is possible. When $k = 0, 3^{2u} + 10 \times 3^u - 7 = 0$, which is not possible. When $k = 1$, then $x = 2$. That is $4 \times 3^u(3^u - 10) = 36 \Rightarrow 3^{2u} = 76$, which is not viable. Thus we conclude our discussion with $(1, 0, 2)$ is the unique solution for the Diophantine equation $3^x + 127^y = z^2$.

Corollary 3.3. *The Diophantine equation* $3^x + 67^y = w^4$ *has no non-negative integer solution.*

Proof. Let x, y and w be the non-negative integers such that $3^x + 67^y = w^4$. Let $z = w^2$. Then by Theorem 3.1, $3^x + 67^y = z^2$ has a unique solution $(1, 0, 2)$.

That is $w^2 = 2 \Rightarrow w = \sqrt{2}$, but $w \in \mathbb{N} \cup \{0\} \cdot 3^x + 67^y = w^4$ has no non – negative integer solutions.

Corollary 3.4. *The Diophantine equation* $9^w + 67^y = z^2$ *has no non-negative integer solution.*

Proof. Let w, y and z be the non-negative integers such that $9^w + 67^y = z^2$. Let $x = 2w$. Then by Theorem 3.1, $3^x + 67^y = z^2$ has a unique non-negative integer solution $(1, 0, 2)$. That is $x = 2 \Rightarrow w = \frac{1}{2} \in \mathbb{Q}$. Therefore, there is no non-negative integer solution for the Diophantine equation $9^w + 67^y = z^2$.

Corollary 3.5. *There is no non-negative integral solution for the Diophantine equation* $3^x + 127^y = w^4$.

Proof. Let x, y and w be the non-negative integers such that $3^x + 127^y = z^2$. Let $z = w^2$. Then by Theorem 3.2, $3^x + 127^y = z^2$ has a unique solution $(1, 0, 2)$. That is $w^2 = 2 \Rightarrow w = \sqrt{2}$, but $w \in \mathbb{N} \cup \{0\} \cdot 3^x + 127^y = w^4$ has no non-negative integer solutions.

Corollary 3.6. *There is no non-negative integral solution for the Diophantine equation* $9^w + 127^y = z^2$.

Proof. Let w, y and z be the non – negative integers such that $9^w + 127^y = z^2$. Let $x = 2w$. Then by Theorem 3.2, $3^x + 127^y = z^2$ has a unique non-negative integer solution $(1, 0, 2)$. That is $x = 2 \Rightarrow w = \frac{1}{2} \in \mathbb{Q}$. Therefore, there is no non-negative integer solution for the Diophantine equation $9^{w} + 127^{y} = z^{2}$.

Conclusion

In this paper, we have shown that $(1, 0, 2)$ is the unique non-negative solution for the Diophantine equations $3^{x} + 67^{y} = z^{2}$ and $3^{x} + 127^{y} = z^{2}$. Here we solve for isolated primes 67 and 127. We may ask, is the solution for the Diophantine equation $3^{x} + p^{y} = z^{2}$ is unique, where p is an isolated prime number?

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