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HERMITE-HADAMARD INEQUALITY WHOSE FIRST ORDER q-DERIVATIVES ARE m-CONVEX FUNCTIONS

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Abstract

Convexity is a fundamental and significant function in the theory of geometric functions with extensive applications in both pure and applied mathematics. In this article, an equality of quantum estimate of convex function is extended to an m-convex function. Taking so obtained result a few novel Hermite-Hadamard type integral inequalities whose first order q-derivatives are m-convex functions are established. The results are further presented in q-analogues.

1. Introduction

Quantum calculus, or q-calculus, is the study of calculus without the concept of limits, first appeared in the eighteenth century and reached its pinnacle in the twentieth due to its extensive use in physics and several mathematical disciplines. By bringing the number q into Newton's work on infinite series, Euler launched his research in the eighteenth century. The theory of q-hyper-geometric functions and Jacobi's triple product identity were discovered in the nineteenth century. Jackson (1910) established definitive q-integrals and began a systematic study of q-calculus. Multiple areas of mathematics and physics, including number theory, fundamental hyper-geometric functions, combinatorics, orthogonal polynomials,

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mathematical inequalities, quantum theory, mechanics, and the theory of relativity, all have numerous uses for the subject of quantum calculus, and, hence the q-calculus is appeared as an inter-disciplinary subject between mathematics and physics, a blend of the subjects. We refer to go through the books by Ernst [9], Kac and Cheung [10], and the paper by Gauchman [4] for some recent developments on quantum calculus and the theory of inequalities.

Convexity is a fundamental and significant function in the theory of geometric functions with extensive applications in both pure and applied mathematics. It has been noted that the theories of inequalities and convex functions are very interdependent. According to the literature, a classical or normal convex function is defined as follows:

Definition 1.1. Let $\phi : I \subset \mathbb{R} \to \mathbb{R},$ be a real valued function. Then, the function ϕ is said to be a convex function, if the inequality

$$\phi(su + (1-s)v) \leq s\phi(u) + (1-s)\phi(v) \tag{1.1}$$

holds, for all $u, v \in I,$ and, $s \in [0, 1].$

If the inequality in (1.1) is reversed, then the concavity of ϕ holds. Toader [3] introduced the concept of m-convexity of the function ϕ as follows.

Definition 1.2[3]**.** The function $\phi : [0, b] \to \mathbb{R}, b > 0,$ is said to be m-convex function if

$$\phi(su + m(1-s)v) \leq s\phi(u) + m(1-s)\phi(v)) \tag{1.2}$$

holds for all $u, v \in [0, b], s \in [0, 1], m \in [0, 1].$

The m-convexity is intermediate concept between usual convexity and star-shaped function. We recall that a function $\phi : [0, b] \to \mathbb{R}$ is called a star-shaped function, if

$$\phi(su) \leq s\phi(u)$$

holds for all $u \in [0, b], s \in [0, 1].$

Example 1.1. The function $\phi : [0, \infty) \to \mathbb{R}$ given by

$$\phi(u) = au + b$$

is an m-convex function $(m \in [0, 1])$ if $b \leq 0.$

There are numerous significant inequalities for the category of convex functions, but one of the most well-known is the so-called Hermite Hadamard integral inequality, which was independently found by Ch. Hermite and J. Hadamard in 1881 and 1893 and goes as follows.

Definition 1.3. Let $\phi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function where, $u, v \in I$ with $u < v$. Then, the following inequality holds:

$$\phi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \phi(x)dx \leq \frac{\phi(u)+\phi(v)}{2}. \tag{1.3}$$

This famous result is considered as a necessary and a sufficient condition for a function to be convex. It provides a lower and an upper estimation for the integral mean of any convex function defined on a compact interval involving the mid-point and the end points. Hermite-Hadamard's inequality has raised many scholars' attention, and thus, a variety of refinements, extensions, variants and generalizations have been found in literatures.

In this paper, we aim in developing some quantum estimates of Hermite-Hadamard type of integral inequalities for m-convex functions by considering the results of quantum estimates of convex functions.

2. Preliminary Results

In this section, we first review and state some previously understood ideas and findings on q-calculus that will be applied in the following paper.

Let $J = [u, v] \subset \mathbb{R}$, $J^0 = (u, v)$ be intervals and $0 < q < 1$ be a constant.

We define q-derivative of a function $\phi : J \to \mathbb{R}$ at a point $x \in J$ on $[u, v]$ as follows:

Definition 2.1[5]**.** Assume that $\phi : J \to \mathbb{R}$ is a continuous function and let $x \in J$. Then the expression

$${}_uD_q\phi(x) = \frac{\phi(x)-\phi(qx+(1-q)u)}{(1-q)(x-u)},\ x \neq u,\ {}_uD_q\phi(u) = \lim_{x\to u} {}_uD_q\phi(x) \tag{2.1}$$

is called the q-derivative of the function ϕ at x on J. We say that ϕ is q-differentiable on J provided ${}_uD_q\phi(x)$ exists for all $x \in J$.

Note that if $u = 0$ in (2.1), then ${}_0D_q\phi = D_q\phi$, where D_q is the well known q-derivative of the function $\phi(x)$ defined by

$$D_q\phi(x) = \frac{\phi(x) - \phi(qx)}{(1-q)x}$$

For more details, see [5].

Definition 2.2[5]. Assume that $\phi : J \to \mathbb{R}$ is a continuous function and let $x \in J$. Then the q-integral on J is defined by

$$\int_u^x \phi(s)_u d_q s = (1-q)(x-u)\sum_{n=0}^{\infty} q^n \phi(q^n x + (1-q^n)u) \qquad (2.2)$$

for $x \in J$.

Moreover, if $c \in (u, x)$, then the definite q-integral on J is defined by

$$\int_c^x \phi(s)_c d_q s$$

$$= \int_u^x \phi(s)_u d_q s - \int_u^c \phi(s)_u d_q s = (1-q)(x-u)\sum_{n=0}^{\infty} q^n \phi(q^n x + (1-q^n)u)$$

$$- (1-q)(c-u)\sum_{n=0}^{\infty} q^n \phi(q^n c + (1-q^n)u).$$

Note that if $u = 0$, then equation (2.2) reduces to the classical q-integral of a function $\phi(x)$, defined by

$$\int_0^x \phi(s)_0 d_q s = (1-q)x\sum_{n=0}^{\infty} q^n \phi(q^n x),\ x \in [0, \infty).$$

Theorem 2.3[10]. *Assume* $\phi_1, \phi_2 : J \to \mathbb{R}$ *are continuous functions,* $\alpha \in \mathbb{R}$. *Then, for* $x \in J$

(i) $\int_u^x (\phi_1(s) + \phi_2(s))_u d_q s = \int_u^x \phi_1(s)_u d_q s + \int_u^x \phi_2(s)_u d_q s.$

(ii) $\int_u^x (\alpha\phi_1)(s)_u d_q s = \alpha \int_u^x \phi_1(s)_u d_q s.$

(iii) $\int_c^x \phi_1(s)_u D_q \phi_2(s)_u d_q s = | \phi_1 \phi_2 |_c^x$

$$- \int_c^x \phi_2(qs + (1-q)u)_u D_q \phi_1(s)_u d_q s, \; c \in (u, x).$$

Alp et al. [7] in 2018 introduced the correct form of Hermite-Hadamard type integral inequality in quantum framework as follows:

Theorem 2.4[7]**.** *Let* $\phi : J \to \mathbb{R}$ *be a continuous function on* J *and* $0 < q < 1$. *Then, we have*

$$\phi\left(\frac{qu + v}{1 + q}\right) \leq \frac{1}{v - u} \int_u^v \phi(s)_u d_q s \leq \frac{q\phi(u) + \phi(v)}{1 + q}. \tag{2.3}$$

Lemma 2.5[11]**.** *Let* $\phi : J \to \mathbb{R}$ *be a continuous function and* $0 < q < 1$. *If* ${}_u D_q \phi$ *is an integrable function on* J^0, *then the following equality holds*:

$$\frac{q\phi(u) + \phi(v)}{1 + q} - \frac{1}{v - u} \int_u^v \phi(s)_u d_q s$$

$$= \frac{q(v - u)}{1 + q} \int_0^1 (1 - (1 + q)s)_u D_q \phi(sv + (1 - s)u)_0 d_q s$$

Lemma 2.6[11]**.** *Let* $0 < q < 1$ *be a constant. Then, the following equalities hold*:

(i) $\int_0^1 1_0 d_q s = 1.$

(ii) $\int_0^1 s_0 d_q s = \frac{1}{1 + q}.$

(iii) $\int_0^1 s^2{}_0 d_q s = \frac{1}{1 + q + q^2}.$

Lemma 2.7[11]**.** *Let* $0 < q < 1$ *be a constant. Then, the following equality holds*:

$$\int_0^1 |1-(1+q)s|\,{}_0d_q s = \frac{q(2+q+q^2)}{(1+q)^3}.$$

Lemma 2.8[11]. *Let* $0 < q < 1$ *be a constant. Then, the following equality holds*:

$$\int_0^1 s|1-(1+q)s|\,{}_0d_q s = \frac{q(1+4q+q^2)}{(1+q)^3(1+q+q^2)}.$$

Lemma 2.9[11]. *Let* $0 < q < 1$ *be a constant. Then, the following equality holds*:

$$\int_0^1 (1-s)|1-(1+q)s|\,{}_0d_q s = \frac{q(1+3q^2+2q^3)}{(1+q)^3(1+q+q^2)}.$$

Theorem 2.10[11]. *Let* $\phi : J \to \mathbb{R}$ *be a continuous function. If* $|{}_uD_q\phi|$ *is convex and integrable on* J^0, *then the following inequality holds*:

$$\left|\frac{q\phi(u)+\phi(v)}{1+q} - \frac{1}{v-u}\int_u^v \phi(s)\,{}_ud_q s\right|$$

$$\leq \frac{q^2(v-u)}{(1+q)^4(1+q+q^2)}\left((1+4q+q^2)|{}_uD_q\phi(v)| + (1+3q+2q^3)|{}_uD_q\phi(u)|\right).$$

Theorem 2.11[11]. *Let* $\phi : J \to \mathbb{R}$ *be a continuous function. If* $|{}_uD_q\phi|^r$ *is convex and integrable on* J^0, *and* $r \geq 1$, *then the following inequality holds*:

$$\left|\frac{q\phi(u)+\phi(v)}{1+q} - \frac{1}{v-u}\int_u^v \phi(s)\,{}_ud_q s\right| \leq \frac{q^2(2+q+q^2)(v-u)}{(1+q)^4}$$

$$\left[\frac{(1+4q+q^2)|{}_uD_q\phi(v)|^r + (1+3q^2+2q^3)|{}_uD_q\phi(u)|^r}{(1+3q^2+2q^3)(2+q+q^2)}\right]^{\frac{1}{r}}$$

Theorem 2.12[11]. *Let* ϕ_1 *and* ϕ_2 *be two real-valued, non-negative and convex functions on J. Then, the following inequality holds*:

$$\frac{1}{v-u}\int_u^v \phi_1(x)\phi_2(x)_u d_q x \leq \frac{\phi_1(u)\phi_2(u)}{1+q+q^2} + \frac{q(1+q^2)\phi_1(v)\phi_2(v) + q^2 N(u, v)}{(1+q+q^2)(1+q)}.$$

Theorem 2.13[11]**.** *Let* ϕ_1 *and* ϕ_2 *be two real-valued, non-negative and convex functions on J. Then, the following inequalities hold*:

(i) $\frac{(1+q)(1+q+q^2)}{(v-u)^2}$

$$\int_u^v\int_u^v\int_0^1 \phi_1(sy+(1-s)x)\phi_2(sy+(1-s)x)_0 d_q s_u d_q x_u d_q y \leq \frac{(1+2q+q^2)}{(v-u)}$$

$$\int_u^v \phi_1(x)\phi_2(x)_u d_q x + \frac{2q^2}{(1+q)^2}[(q^2\phi_1(u)\phi_2(u) + \phi_1(v)\phi_2(v)) + qN(u, v)]$$

(ii) $$\frac{1+q+q^2}{v-u}\int_u^v\int_0^1 \phi_1\left(sy+(1-s)\frac{u+v}{2}\right)\phi_2\left(sy+(1-s)\frac{u+v}{2}\right)_0 d_q s_u d_q y$$

$$\leq \frac{1}{v-u}\int_u^v \phi_1(x)g\phi_2(x)_u d_q x + \frac{q(1+q^2)}{4(1+q)}(M(u, v) + N(u, v))$$

$$+ \frac{q^2}{2(1+q)^2}(2(q\phi_1(u)\phi_2(u) + \phi_1(v)\phi_2(v)) + (1+q)N(u, v))$$

where $M(u, v) = \phi_1(u)\phi_2(u) + \phi_1(v)\phi_2(v)$, $N(u, v) = \phi_1(u)\phi_2(v) + \phi_1(v)\phi_2(u)$.

3. Main Results

In this section, first of all, we extend Lemma 2.5 assuming that ${}_uD_q\phi$ is q-integrable m-convex function and then present some quantum estimates of Hermite-Hadamard type integral inequalities for m-convex function on $[mu, v]$.

Lemma 3.1. *Let* $\phi : J \to \mathbb{R}$ *be a continuous function and* $0 < q < 1$. *If* ${}_uD_q\phi$ *is q-integrable m-convex function on* J^0, *where* J^0 *is interior of J, then the following equality holds*:

$$\frac{1}{v-mu}\int_{mu}^{v}\phi(s)_{mu}d_qs-\frac{q\phi(mu)+\phi(v)}{1+q}$$

$$=\frac{q(v-mu)}{1+q}\int_0^1(1-(1+q)s)_u D_q\phi(sv+m(1-s)u)_0 d_qs$$

Proof. Using q-derivative on a finite interval, we have

$$\int_0^1(1-(1+q)s)_u D_q\phi(sv+m(1-s)u)_0 d_qs=\int_0^1 1_u D_q\phi(sv+m(1-s)u)_0 d_qs$$

$$-(1+q)\int_0^1 s_u D_q\phi(sv+m(1-s)u)_0 d_qs$$

$$=\int_0^1\frac{\phi(sv+m(1-s)u)-\phi(qsv+m(1-qs)u)}{s(1-q)(v-mu)}\ {}_0d_qs$$

$$-(1+q)\int_0^1 s\cdot\frac{\phi(sv+m(1-s)u)-\phi(qsv+m(1-qs)u)}{s(1-q)(v-mu)}\ {}_0d_qs$$

$$=\frac{1}{v-mu}\left[\sum_{n=0}^{\infty}\phi(q^nv+m(1-q^n)u)-\sum_{n=0}^{\infty}\phi(q^{n+1}v-m(1-q^{n+1})u)\right]$$

$$-\frac{1+q}{v-mu}\left[\sum_{n=0}^{\infty}q^n\phi(q^nv+m(1-q^n)u)-\sum_{n=0}^{\infty}q^n\phi(q^{n+1}v-m(1-q^{n+1})u)\right]$$

$$=\frac{\phi(v)-\phi(mu)}{v-mu}-\frac{1+q}{v-mu}\sum_{n=0}^{\infty}q^n\phi(q^nv+m(1-q^n)u)$$

$$+\frac{1+q}{v-mu}\sum_{n=0}^{\infty}q^n\phi(q^{n+1}v+m(1-q^{n+1})u)$$

$$=\frac{\phi(v)-\phi(mu)}{v-mu}-\frac{1+q}{v-mu}\sum_{n=0}^{\infty}q^n(q^nv+m(1-q^n)u)$$

$$+\frac{1+q}{q(v-mu)}\left[\phi(v)-\phi(u)\sum_{n=0}^{\infty}q^n\phi(q^{n+1}v+m(1-q^{n+1})u)\right]$$

$$= \frac{\varphi(v)-\varphi(mu)}{v-mu} - \frac{(1+q)\varphi(v)}{q(v-mu)} + \frac{1+q}{q(v-mu)} \sum_{n=0}^{\infty} q^n \varphi(q^n v + m(1-q^n)u)$$

$$- \frac{1+q}{v-mu} \sum_{n=0}^{\infty} q^n \phi(q^n v + m(1-q^n)u)$$

$$= -\frac{q\phi(mu)+\phi(v)}{q(v-mu)} + \frac{(1+q)(1-q)(v-mu)}{q(v-mu)(v-mu)} \sum_{n=0}^{\infty} q^n \phi(q^n v + m(1-q^n)u)$$

$$= -\frac{q\phi(mu)+\phi(v)}{q(v-mu)} + \frac{(1+q)}{q(v-mu)^2} \int_{mu}^{v} \phi(s)_{mu} d_q s$$

$$= \frac{1+q}{q(v-mu)} \left[\frac{1}{v-mu} \int_{mu}^{v} \phi(s)_{mu} d_q s - \frac{q\phi(mu)+\phi(v)}{1+q} \right].$$

Thus, we have

$$\frac{1}{v-mu} \int_{mu}^{v} \phi(s)_{mu} d_q s - \frac{q\phi(mu)+\phi(v)}{1+q}$$

$$= \frac{q(v-mu)}{1+q} \int_0^1 (1-(1+q)s)_u D_q \phi(sv + m(1-s)u)_0 d_q s.$$

The proof is now complete. □

Remark 3.2. If $m = 1$, and then it reduces to the result as given in Lemma 2.5, and if $q \to 1$, $m = 1$, then it reduces to the following result:

$$\frac{\phi(u)+\phi(v)}{2} - \frac{1}{v-u} \int_u^v \phi(x)dx = \frac{v-u}{2} \int_0^1 (1-2s)\phi'(sv + (1-s)u)ds.$$

Theorem 3.3. *Let* $\phi : J \to \mathbb{R}$ *be a continuous function. If* $|\, _{mu}D_q \phi \,|$ *is* m*-convex and integrable on* J^0*, then the following inequality holds*:

$$\left| \frac{q\phi(mu)+\phi(v)}{1+q} - \frac{1}{v-mu} \int_{mu}^{v} \phi(s)_u d_q s \right|$$

$$\leq \frac{q^2(v-mu)}{(1+q)^4(1+q+q^2)} \left((1+4q+q^2) |\, _{mu}D_q \phi(v) \,| + (1+3q+2q^3) |\, _{mu}D_q \phi(u) \,| \right).$$

Proof. Using lemma 3.1 and m-convexity of ${}_{mu}D_q\phi$ on J^0 and lemmas 2.8 and 2.9, we have

$$\left|\frac{1}{v-mu}\int_{mu}^{v}\phi(s)_{mu}d_qs - \frac{q\phi(mu)+\phi(v)}{1+q}\right|$$

$$= \left|\frac{q(v-mu)}{1+q}\int_0^1 (1-(1+q)s)_u D_q\phi(sv+m(1-s)u)_0 d_qs\right|$$

$$\leq \frac{q(v-mu)}{1+q}$$

$$\int_0^1 |1-(1-q)s|[s|_{mu}D_q\phi(v)+m(1-s)|\,{}_{mu}D_q\phi(u)|]_0 d_qs$$

$$\leq \frac{q(v-mu)}{1+q}$$

$$[|\,{}_{mu}D_q\varphi(v)|\int_0^1 s|1-(1+q)s|_0\, d_qs + m|\,{}_{mu}D_q\varphi(u)|\int_0^1 (1-s)|1-(1+q)s|_0\, d_qs]$$

$$= \frac{q(v-mu)}{1+q}\left[|\,{}_{mu}D_q\phi(v)|\frac{q(1+4q+q^2)}{(1+q)^3(1+q+q^2)} + \frac{mq(1+3q^2+2q^3)}{(1+q)^3(1+q+q^2)}|\,{}_{mu}D_q\phi(u)|\right]$$

$$= \frac{q^2(v-mu)}{(1+q)^4(1+q+q^2)}[|\,{}_{mu}D_q\phi(v)|(1+4q+q^2)+m(1+3q^2+2q^3)|\,{}_{mu}D_q\phi(u)|].$$

The proof is complete. □

Remark 3.4. If $m = 1,$ then it reduces to the previous result as in Theorem 2.10.

Remark 3.5. If $m = 1, q \to 1$ then, the above result reduces to the following already established result

$$\left|\frac{\phi(u)+\phi(v)}{2} - \frac{1}{v-u}\int_u^v \phi(s)ds\right| \leq \frac{(v-u)}{8}[|\phi'(v)|+|\phi'(u)|].$$

We, now present the second result of q-integral inequality for m-convex function on $[mu, v]$.

Theorem 3.6. *Let* $\phi : J \to \mathbb{R}$ *be a continuous function. If* $|\, {}_{mu}D_q\phi \,|^r$ *is* m*-convex and integrable on* J^0 *and* $r \geq 1$*, then the following inequality holds*:

$$\left| \frac{q\phi(mu) + \phi(v)}{1+q} - \frac{1}{v - mu}\int_{mu}^{v} \phi(s)_u d_q s \right| \leq \frac{q^2(2 + q + q^2)(v - mu)}{(1+q)^4}$$

$$\left(\frac{(1 + 4q + q^2)|\, {}_{mu}D_q\phi(v)\,|^r + (1 + 3q^2 + 2q^3)|\, {}_{mu}D_q\phi(u)\,|^r}{(1 + q + q^2)(2 + q + q^3)} \right)^{\frac{1}{r}}$$

Proof. From lemma 3.1 and using power-mean inequality and m-convexity of $|\, {}_{mu}D_q\phi \,|^r$, and lemmas 2.7, 2.8 and 2.9, we have

$$\left| \frac{1}{v - mu}\int_{mu}^{v} \phi(s)_{mu} d_q s - \frac{q\phi(mu) + \phi(v)}{1+q} \right|$$

$$= \left| \frac{q(v - mu)}{1+q}\int_0^1 (1 - (1+q)s)_u D_q\phi(sv + m(1-s)u)_0 d_q s \right|$$

$$\leq \frac{q(v - mu)}{(1+q)}\int_0^1 (1 - (1+q)s)_u D_q\phi(sv + m(1-s)u)_0 d_q s$$

$$\leq \frac{q(v - mu)}{1+q}\left(\int_0^1 |\, 1 - (1+q)s \,|_0 d_q s \right)^{1-\frac{1}{r}}$$

$$\cdot \left(\int_0^1 |\, 1 - (1+q)s \,|_{ma} D_q\phi(sb + m(1-s)a)\,|_0^r \; d_q s \right)^{\frac{1}{r}}$$

$$\leq \frac{q(v - mu)}{1+q}\left(\int_0^1 |\, 1 - (1+q)s \,|_0 d_q s \right)^{1-\frac{1}{r}}$$

$$\left(|\, {}_{mu}D_q\phi(v)\,|^r \int_0^1 s|\, 1 - (1+q)s_0 d_q s \,| + m|\, {}_{mu}D_q\phi(u)\,|^r \int_0^1 (1-s)|\, 1 - (1+q)s \,|_0 d_q s \right)^{\frac{1}{r}}$$

$$\leq \frac{q(v - mu)}{1+q}\left(\frac{q(2 + q + q^2)}{(1+q)^3} \right)^{1-\frac{1}{r}}.$$

$$\left(\frac{q}{(1+q+q^2)(1+q)^3}((1+4q+q^2)|\,_{mu}D_q\varphi(v)\,|^r + m(1+3q^2+2q^3)|\,_{mu}D_q\varphi(u)\,|^r)\right)^{\frac{1}{r}}$$

$$= \left| \frac{q\phi(mu)+\phi(v)}{1+q} - \frac{1}{v-mu}\int_{mu}^{v}\phi(s)_u d_q s \right| \le \frac{q^2(2+q+q^2)(v-mu)}{(1+q)^4}$$

$$\left(\frac{(1+4q+q^2)|\,_{mu}D_q\phi(v)\,|^r + (1+3q^2+2q^3)|\,_{mu}D_q\phi(u)\,|^r}{(1+q+q^2)(2+q+q^3)}\right)^{\frac{1}{r}}$$

This completes the proof. □

Remark 3.7. If $m = 1,$ then it reduces to Theorem 2.11.

Remark 3.8. If $m = 1$ and $q \to 1,$ then we have the following previously known result

$$\left| \frac{\phi(u)+\phi(v)}{2} - \frac{1}{v-u}\int_{u}^{v}\phi(s)ds \right| \le \frac{v-u}{4}\left[\frac{|\,\phi'(u)\,|^r + |\,\phi'(v)\,|^r}{2}\right]^{\frac{1}{r}}.$$

Theorem 3.9. *Let* ϕ_1 *and* ϕ_2 *be two real-valued, non-negative m-convex functions on J. Then, the following inequality holds*:

$$\frac{1}{v-mu}\int_{mu}^{v}\phi_1(x)\phi_2(x)_{mu}d_q x \le \frac{\phi_1(u)\phi_2(v)}{1+q+q^2}$$

$$+ \frac{m^2q(1+q^2)\phi_1(v)\phi_2(v) + mq^2N(u, v)}{(1+q+q^2)(1+q)}.$$

where, $N(u, v) = \phi_1(u)\phi_2(v) + \phi_1(v)\phi_2(u).$

Proof. Using m-convexity of ϕ_1 and ϕ_2 and for all $s \in [0, 1],$ we have

$$\phi_1(sv + m(1-s)u) \le s(\phi_1(v) + m(1-s)\phi_1(u)).$$

$$\phi_2(sv + m(1-s)u) \le s(\phi_2(v) + m(1-s)\phi_2(u)).$$

Multiplying above inequalities, we have

$$\phi_1(sv + m(1-s)u)\phi_2(sv + m(1-s)u) \le s^2\phi_1(v)\phi_2(v) + m^2(1-s)^2\phi_1(u)\phi_2(u)$$

$$+ ms(1-s)(\phi_1(u)\phi_2(v) + \phi_1(v)\phi_2(u)).$$

Taking q-integral with respect to s over $[0, 1]$ and using lemma 2.6, we have

$$\int_0^1 \phi_1(sv + m(1-s)u)\phi_2(sv + m(1-s)u)_0 d_q s$$

$$\leq \int_0^1 (s^2\phi_1(v)\phi_2(v) + m^2(1-s)^2\phi_1(u)\phi_2(u)$$

$$+ ms(1-s)(\phi_1(u)\phi_2(v) + \phi_1(v)\phi_2(u)))_0 d_q s.$$

$$\int_0^1 \phi_1(sv + m(1-s)u)\phi_2(sv + m(1-s)u)_0 d_q s$$

$$\leq \frac{\phi_1(v)\phi_2(v)}{1+q+q^2} + \frac{m^2q(1+q^2)\phi_1(u)\phi_2(u)}{(1+q+q^2)(1+q)} + \frac{mq^2N(u, v)}{(1+q+q)(1+q)}.$$

Substituting $x = sv + m(1-s)u$, on left-side of above inequality, we have

$$\frac{1}{v-mu}\int_{mu}^{v} \phi_1(x)\phi_2(x)_{mu} d_q x \leq \frac{\phi_1(u)\phi_2(u)}{1+q+q^2}$$

$$+ \frac{m^2q(1+q^2)\phi_1(v)\phi_2(v) + mq^2N(u, v)}{(1+q+q^2)(1+q)}$$

where, $N(u, v) = \phi_1(u)\phi_2(v) + \phi_1(v)\phi_2(u)$.

This completes the proof. □

Remark 3.10. If $q \to 1$, then the above inequality reduces to

$$\frac{1}{v-mu}\int_{mu}^{v} \phi_1(x)\phi_2(x)dx \leq \frac{1}{3}[\phi_1(v)\phi_2(v) + m^2\phi_1(u)\phi_2(u)] + \frac{1}{6}N(u, v).$$

Remark 3.11. If $q \to 1$ and $m = 1$, then the inequality reduces to

$$\frac{1}{v-u}\int_{u}^{v} \phi_1(x)\phi_2(x)dx \leq \frac{1}{3}M(u, v) + \frac{1}{6}N(u, v).$$

Theorem 3.12. *Let ϕ_1 and ϕ_2 be two real-valued, non-negative m-convex functions on J. Then, the following inequalities hold*:

(i) $\frac{(1+q)(1+q+q^2)}{(v-u)^2}$

$$\int_u^v\int_u^v\int_0^1 \phi_1(sy+m(1-s)x)\phi_2(sy+(1-s)x)_0 d_qs_ud_qx_ud_qy \leq \frac{1+q+m^2q(1+q^2)}{v-u}$$

$$\int_u^v \phi_1(x)\phi_2(x)_u d_qx + \frac{2q^2m}{(1+q)^2}(q^2\phi_1(u)\phi_2(u) + qN(u, v) + \phi_1(v)\phi_2(v)).$$

(ii)

$$\frac{1+q+q^2}{v-u}\int_u^v\int_0^1 \phi_1\left(sy+m(1-s)\frac{u+v}{2}\right)\phi_2\left(sy+m(1-s)\frac{u+v}{2}\right)_0 d_qs_ud_qy$$

$$\leq \frac{1}{v-u}\int_u^v \phi_1(y)\phi_2(y)_u d_qy + \frac{m^2q(1+q^2)}{4(1+q)}(M(u, v) + N(u, v))$$

$$+\frac{mq^2}{2(1+q)^2}(2(q\phi_1(u)\phi_2(u) + \phi_1(v)\phi_2(v)) + (1+q)N(u, v))$$

where $M(u, v) = \phi_1(u)\phi_2(u) + \phi_1(v)\phi_2(v)$, *and* $N(u, v) = \phi_1(u)\phi_2(v) + \phi_1(v)\phi_2(u)$.

Proof. From the definition of m-convexity of ϕ_1 and ϕ_2, for all $s \in [0, 1]$, $x, y \in J$, we have

$$\phi_1(sy+m(1-s)x) \leq s\phi_1(y) + m(1-s)\phi_1(x).$$

$$\phi_2(sy+m(1-s)x) \leq s\phi_2(y) + m(1-s)\phi_2(x).$$

Multiplying above inequalities, we have

$$\phi_1(sy+m(1-s)x)\phi_2(sy+m(1-s)x) \leq s^2\phi_1(y)\phi_2(y) + m^2(1-s)^2\phi_1(x)\phi_2(x)$$

$$+ms(1-s)(\phi_1(x)\phi_2(y) + \phi_1(y)\phi_2(x)).$$

Taking q-integral with respect to s over $[0, 1]$ and using lemma 2.6

$$\int_0^1 \phi_1(sy+m(1-s)x)\phi_2(sy+m(1-s)x)_0 d_qs$$

$$\leq \int_0^1 (s^2\phi_1(y)\phi_2(x) + m^2(1-s)^2\phi_1(x)\phi_2(x)$$

$$+ms(1-s)(\varphi_1(x)\varphi_2(y)+\varphi_1(y)\varphi_2(x)))_0 d_q s.$$

$$\int_0^1 \phi_1(ty+m(1-t)x)\phi_2(ty+m(1-t)x)_0 d_q t$$

$$\leq \frac{\varphi_1(y)\varphi_2(x)}{1+q+q^2}+\frac{m^2q(1+q^2)\varphi_1(x)\varphi_2(x)}{(1+q+q^2)(1+q)}+\frac{mq^2N(u,v)}{(1+q+q^2)(1+q)}.$$

Next, taking double q-integral to both sides of the above inequality with respect to x, y on $[u, v]$, we have

$$\int_u^v\int_u^v\int_0^1 \phi_1(sy+m(1-s)x)\phi_2(sy+(1-s)x)_0 d_q s_u d_q x_u d_q y$$

$$\leq \int_0^1 s^2\varphi_1(y)\varphi_2(y)+m^2(1-s)^2\varphi_1(x)\varphi_2(x)+ms(1-s)(\varphi_1(x)\varphi_2(y)+\varphi_1(y)\varphi_2(x))_0 d_q s.$$

$$\int_0^1 \phi_1(sy+m(1-s)x)\phi_2(sy+m(1-s)x)_0 d_q s$$

$$\leq \frac{v-u}{1+q+q^2}\int_u^v \phi_1(y)\phi_2(y)_u d_q y+\frac{m^2q(1+q^2)(v-u)}{(1+q+q^2)(1+q)}\int_u^v \phi_1(x)\phi_2(x)_u d_q x$$

$$+\frac{mq^2}{(1+q+q^2)(1+q)}\left(\int_u^v \phi_2(y)_u d_q y\int_u^v \phi_1(x)_u d_q x+\int_u^v f(y)_u d_q y\int_u^v g(x)_u d_q x\right)$$

$$\leq (v-u)\left(\frac{1}{1+q+q^2}+\frac{m^2q(1+q^2)}{(1+q)(1+q+q^2)}\right)$$

$$\int_u^v f(x)g(x)_u d_q x+\frac{mq^2}{(1+q+q^2)(1+q)}$$

$$\left((v-u)\cdot\frac{qg(u)+g(v)}{1+q}\cdot(v-u)\frac{qf(u)+f(v)}{1+q}\right.$$

$$\left.+(v-u)\cdot\frac{qf(u)+f(v)}{1+q}\cdot(v-u)\frac{qg(u)+g(v)}{1+q}\right).$$

On simplifying, we have

$$= \frac{(v-u)(1+q+m^2q(1+q^2))}{(1+q)(1+q+q^2)} \int_u^v \phi_1(x)\phi_2(x)_u d_q x$$

$$+ \frac{2mq^2(v-u)^2}{(1+q)^3(1+q+q^2)} (q^2\phi_1(u)\phi_2(u) + qN(u, v) + \phi_1(v)\phi_2(v)).$$

Multiplying both sides by $\frac{(1+q)(1+q+q^2)}{(v-u)^2}$, we have

$$\frac{(1+q)(1+q+q^2)}{(v-u)^2} \int_u^v \int_u^v \int_0^1 \phi_1(sy + m(1-s)x)\phi_2(sy + (1-s)x)_0 d_q s_u d_q x_u d_q y$$

$$\leq \frac{1+q+m^2q(1+q^2)}{v-u}$$

$$\int_u^v \varphi_1(x)\varphi_2(x)_u d_q x + \frac{2q^2m}{(1+q)^2} (q^2\varphi_1(u)\varphi_2(u) + qN(u, v) + \varphi_1(v)\varphi_2(v)))$$

Now, we begin the proof of (ii) part:

From the definition of m-convexity of ϕ_1 and ϕ_2, for all $s \in [0, 1]$, $x, y \in J$, we have

$$\phi_1\left(sy + m(1-s)\frac{u+v}{2}\right) \leq s\phi_1(y) + m(1-s)\phi_1\left(\frac{u+v}{2}\right)$$

$$\phi_2\left(sy + m(1-s)\frac{u+v}{2}\right) \leq s\phi_2(y) + m(1-s)\phi_2\left(\frac{u+v}{2}\right).$$

Multiplying the above inequalities, we obtain

$$\phi_1\left(sy + m(1-s)\frac{u+v}{2}\right)\phi_2\left(sy + m(1-s)\frac{u+v}{2}\right)$$

$$\leq s^2\phi_1(y)\phi_2(y) + m^2(1-s)^2\phi_1\left(\frac{u+v}{2}\right)\phi_2\left(\frac{u+v}{2}\right)$$

$$+ ms(1-s)\left(\phi_1(y)\phi_2\left(\frac{u+v}{2}\right) + \phi_1\left(\frac{u+v}{2}\right)\right).$$

Taking q-integral with respect to s over $[0, 1]$, we have

$$\int_0^1 \phi_1\left(sy + m(1-s)\frac{u+v}{2}\right)\phi_2\left(sy + m(1-s)\frac{u+v}{2}\right)\, {}_0d_q s$$

$$\leq \frac{\phi_1(y)\phi_2(y)}{1+q+q^2} + \frac{m^2 q(1+q^2)}{(1+q+q^2)(1+q)}\phi_1\left(\frac{u+v}{2}\right)\phi_2\left(\frac{u+v}{2}\right)$$

$$+ \frac{mq^2}{(1+q+q^2)(1+q)}\left(\phi_1(y)\phi_2\left(\frac{u+v}{2}\right) + \phi_1\left(\frac{u+v}{2}\right)\phi_2(y)\right).$$

Applying q-integral with respect to y over $[u, v]$ and using m-convexity of functions of ϕ_1 and ϕ_2, we observe

$$\int_u^v \int_0^1 \phi_1\left(sy + m(1-s)\frac{u+v}{2}\right)\phi_2\left(sy + m(1-s)\frac{u+v}{2}\right)\, {}_0d_q s\, {}_ud_q y$$

$$\leq \frac{1}{1+q+q^2}\int_u^v \phi_1(y)\phi_2(y)\, {}_ud_q y + \frac{m^2 q(1+q^2)}{(1+q+q^2)(1+q)}\int_u^v \phi_1\left(\frac{u+v}{2}\right)\phi_2\left(\frac{u+v}{2}\right) {}_ud_q y$$

$$+ \frac{mq^2}{(1+q+q^2)(1+q)}\left(\int_u^v \phi_1(y)\, {}_ud_q y\, \phi_2\left(\frac{u+v}{2}\right) + \phi_1\left(\frac{u+v}{2}\right)\int_u^v \phi_2(y)\, {}_ud_q y\right)$$

$$\leq \frac{1}{1+q+q^2}\int_u^v \phi_1(y)\phi_2(y)\, {}_ud_q y + \frac{m^2 q(1+q^2)(u-v)}{(1+q+q^2)(1+q)}$$

$$\left(\frac{\phi_2(u)+\phi_2(v)}{2}\cdot\frac{\phi_1(u)+\phi_2(v)}{2}\right) + \frac{mq^2}{(1+q+q^2)(1+q)}$$

$$\left(\frac{\phi_2(u)+\phi_2(v)}{2}\cdot\frac{q\phi_1(u)+\phi_1(v)}{1+q} + \frac{\phi_1(u)+\phi_1(v)}{2}\cdot\frac{q\phi_2(u)+\phi_2(v)}{1+q}\right)$$

$$= \frac{1}{1+q+q^2}\int_u^v \phi_1(y)\phi_2(y)\, {}_ud_q y + \frac{m^2 q(1+q^2)}{4(1+q+q^2)(1+q)}(M(u, v) + N(u, v))$$

$$+ \frac{mq^2(v-u)}{2(1+q+q^2)(1+q)^2}(2(q\phi_1(u)\phi_2(u) + \phi_1(v)\phi_2(v)) + (1+q)N(u, v)).$$

Multiplying both sides by $\frac{(1+q)(1+q+q^2)}{v-u}$, we get

$$\frac{(1+q)(1+q+q^2)}{(v-u)^2}\int_u^v\int_u^v\int_0^1 \phi_1(sy+m(1-s)x)\phi_2(sy+(1-s)x)_0 d_q s_u d_q x_u d_q y$$

$$\leq \frac{1+q+m^2q(1+q^2)}{v-u}$$

$$\int_u^v \phi_1(x)\phi_2(x)_u d_q x + \frac{2q^2m}{(1+q)^2}(q^2\phi_1(u)\phi_2(u) + qN(u, v) + f(v)g(v)).$$

This completes the proof. □

Remark 3.13. If $m = 1$, then it reduces to the result as given in Theorem 2.6.

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