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MIXED BOUNDARY VALUE PROBLEMS ON THE QUARTER PLANE

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Abstract

Explicit representation for the solution of mixed boundary value problem arising from the combination of Schwarz- $(n-1)$ Dirichlet is written on the quarter plane.

1. Introduction

In real world situations, such problems arise very frequently in which mixed boundary conditions need to be applied to get mathematical formulation of physical problems. Here, generalized method of solving such problems is discussed in the form an example. In particular, explicit solution of mixed boundary value problem arising from the combination of "Schwarz- $(n-1)$ Dirichlet boundary conditions for the inhomogeneous polyanalytic equation" is given on the quarter plane. In case of bounded domains, iteration-substitution method works, but in unbounded domains it fails due to arising divergent integrals. Therefore, on such domains a new method is developed which uses "complex form of Gauss theorem and Cauchy Pompeiu representation formula for higher orders" to solve boundary value problems.

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2. Mixed Boundary Value Problem

For the polyanalytic equation $\partial_{\bar{z}}^{n} w = f$ different combinations off Dirichlet, Schwarz boundary conditions are investigated in [4, 9, 14, 15, 16, 17, 18] on the disc $\mathbb{D}$, upper half plane $\mathbb{H}$ and quarter plane $\mathbb{Q}$. In this section we treat the case Schwarz and $(n-1)$ Dirichlet problem for the quarter plane $\mathbb{Q}$. We begin with some representation formulae.

Lemma 2.1. *For a function* $w \in C^{1}(\mathbb{Q}_1;\, \mathbb{C}) \cap C(\overline{\mathbb{Q}_1};\, \mathbb{C})$, *satisfying* $\partial_{\bar{z}}^{n} w(z) \in L_1(\mathbb{Q}_1;\, \mathbb{C})$ *and for which* $z^{\delta} w(z)$ *for some* $0 < \delta$ *is bounded in* $\mathbb{Q}_1$, *according to Gauss theorem and because* $-\bar{z},\, \bar{z},\, -z \notin \mathbb{Q}_1$ *if* $z \in \mathbb{Q}_1$ *then we have*

(i) $$w(z) = \sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{\lambda!} \int_{0}^{+\infty} \partial_{\varsigma}^{\lambda} w(s)\,(s-\bar{z})^{\lambda} \frac{ds}{(s-z)}$$

$$- \sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{\lambda!} \int_{0}^{+\infty} \partial_{\varsigma}^{\lambda} w(is)\,(is+\bar{z})^{\lambda} \frac{ds}{(s+iz)}$$

$$+ \frac{1}{\pi} \frac{(-1)^{n}}{(n-1)!} \int_{\mathbb{Q}_1} f(\varsigma)\,(\overline{\varsigma - z})^{n-1} \frac{d\xi}{d\eta}\,(\varsigma - z) \tag{2.1}$$

(ii) $$\frac{(-1)^{n}}{(n-1)!} \frac{1}{\pi} \int_{\mathbb{Q}_1} \partial_{\bar{\varsigma}}^{n} w(\varsigma)\,(\varsigma + \bar{z})^{n-1} \frac{d\xi d\eta}{(\varsigma + z)}$$

$$= -\sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{(\lambda)!} \int_{0}^{+\infty} \partial_{\bar{\varsigma}}^{\lambda} w(s) \frac{(s+\bar{z})^{\lambda}}{(s+z)}\, ds$$

$$+ \sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{(\lambda)!} \int_{0}^{+\infty} \partial_{\bar{\varsigma}}^{\lambda} w(is)\,(-is+z)^{\lambda} \frac{ds}{(s-iz)} \tag{2.2}$$

(iii) $$\frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1}\partial_{\bar{\varsigma}}^n\overline{w(\varsigma)}(\varsigma+z)^{n-1}\frac{d\xi d\eta}{\bar{\varsigma}+z}$$

$$=\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{(-1)^\lambda}{(\lambda)!}\int_0^{+\infty}\frac{\partial_{\bar{\varsigma}}^\lambda\overline{w(s)}(s+z)^\lambda}{(s+z)}ds$$

$$+\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{(-1)^\lambda}{(\lambda)!}\int_0^{+\infty}\frac{\partial_{\bar{\varsigma}}^\lambda\overline{w(is)}(is+z)^\lambda}{(s+iz)}ds \tag{2.3}$$

(iv) $$\frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1}\partial_{\bar{\varsigma}}^n\overline{w(\varsigma)}(\varsigma+z)^{n-1}\frac{d\xi d\eta}{\bar{\varsigma}-z}$$

$$=\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{(-1)^\lambda}{(\lambda)!}\int_0^{+\infty}\frac{\partial_{\bar{\varsigma}}^\lambda\overline{w(s)}(s-z)^\lambda}{(s-z)}ds$$

$$+\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{(-1)^\lambda}{(\lambda)!}\int_0^{+\infty}\frac{\partial_{\bar{\varsigma}}^\lambda\overline{w(is)}(is-z)^\lambda}{(s-iz)}ds \tag{2.4}$$

Proof. (i) Equation (2.1) is Cauchy-Pompeui representation formula on quarter plane [14]. For (ii), we first obtain for $z\in\mathbb{Q}_{1,R}=\mathbb{Q}_1\cap\{|z|<R\}$,

$$\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{(-1)^\lambda}{\lambda!}\int_{\partial\mathbb{Q}_{1,R}}\partial_{\bar{\varsigma}}^\lambda w(\varsigma)\frac{(\bar{\varsigma}+z)^\lambda}{(\varsigma+z)}d\varsigma$$

$$+\frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_{1,R}}\partial_{\bar{\varsigma}}^n w(\varsigma)\frac{(\bar{\varsigma}+z)^{n-1}}{(\varsigma+z)}d\xi d\eta=0 \tag{2.5}$$

The above formula (2.5) can be proved using induction and applying Gauss theorem for regular domains [15]. Considering limit as $R\to\infty$, we get the formula (ii). The proof of (iii) and (iv) can be formulated as in (ii). □

Observing the above lemma, we have following set of mixed boundary value problems.

Theorem 2.2. *For a function* $w\in C^1(\mathbb{Q}_1;\mathbb{C})\cap C(\overline{\mathbb{Q}_1};\mathbb{C})$ *satisfying* $\partial_{\bar{z}}^n w(z)\in L_1(\mathbb{Q}_1;\mathbb{C})$ *and for which* $z^\delta w(z)$ *for some* $0<\delta$ *is bounded in* $\mathbb{Q}_1$, *according to Gauss theorem and because* $-\bar{z}, \bar{z}, -z \notin \mathbb{Q}_1$ *if* $z \in \mathbb{Q}_1$ *then we have Schwarz-*$(n-1)$ *Dirichlet problem for the inhomogeneous polyanalytic equation in the quarter plane* $\mathbb{Q}_1$

$$\partial_{\bar{z}}^{n} w = f \text{ in } Q, \; Rew = \beta_0 \text{ on } 0 < y < \infty, \; x = 0,$$

$$Imw = \lambda_0 0 < x < \infty, \; y = 0, \; \partial_{\bar{z}}^{\lambda} w = \beta_\lambda \text{ on } 0 < y < \infty, \; x = 0,$$

$$\partial_{\bar{z}}^{n} w = \gamma_\lambda \text{ on } 0 < x < \infty, \; y = 0, \; 1 \le \lambda \le n-1 \tag{2.6}$$

is solvable if and only if, for $1 \le v \le n-1$ *we have*

$$\sum_{\lambda=v}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda-v}}{(\lambda-v)!} \int_0^{+\infty} \gamma_\lambda(s)(s-z)^{\lambda-v} \frac{ds}{(s-\bar{z})}$$

$$- \sum_{\lambda=v}^{n-1} \frac{1}{2\pi i} \frac{1}{(\lambda-v)!} \int_0^{+\infty} \beta_\lambda(s)(is+z)^{\lambda-v} \frac{ds}{(s+i\bar{z})}$$

$$+ \frac{(-1)^{n-v}}{(n-v-1)!} \frac{1}{\pi} \int_{\mathbb{Q}_1} f(\varsigma)(\bar{\varsigma}-z)^{n-v-1} \frac{d\xi d\eta}{\varsigma - \bar{z}} = 0, \tag{2.7}$$

$$\sum_{\lambda=v}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{\lambda!} \int_0^{+\infty} \gamma_\lambda(s)(s+z)^{\lambda} \frac{ds}{(s+\bar{z})}$$

$$- \sum_{\lambda=v}^{n-1} \frac{1}{2\pi i} \frac{(-1)}{\lambda!} \int_0^{+\infty} \beta_\lambda(s)(z-is)^{\lambda} \frac{ds}{(s-i\bar{z})}$$

$$+ \frac{(-1)^{n-v}}{(n-v-1)!} \frac{1}{\pi} \int_{\mathbb{Q}_1} f(\varsigma)(\bar{\varsigma}+z)^{n-v-1} \frac{d\xi d\eta}{\varsigma + \bar{z}} = 0, \tag{2.8}$$

$$\sum_{\lambda=v}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{\lambda!} \int_0^{+\infty} \gamma_\lambda(s)(s+\bar{z})^{\lambda} \frac{ds}{(s+z)}$$

$$- \sum_{\lambda=v}^{n-1} \frac{1}{2\pi i} \frac{1}{\lambda!} \int_0^{+\infty} \beta_\lambda(s)(-is+\bar{z})^{\lambda} \frac{ds}{(s-iz)}$$

$$+\frac{(-1)^{n-v}}{(n-v-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1} f(\varsigma)(\overline{\varsigma+z})^{n-v-1}\frac{d\xi d\eta}{\varsigma+\bar{z}}=0 \tag{2.9}$$

The solution if it exists is given by

$$w(s)=\frac{1}{\pi}\int_0^{+\infty}\gamma_0(s)\left[\frac{1}{(s-z)}+\frac{1}{(s+z)}\right]ds$$

$$-\frac{1}{\pi i}\int_0^{+\infty}\beta_0(s)\left[\frac{1}{(s+iz)}+\frac{1}{(s-iz)}\right]ds$$

$$+\sum_{\lambda=1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{(\lambda)!}\int_0^{+\infty}\gamma_\lambda(s)\left[\frac{(s-\bar{z})^{\lambda}}{(s-z)}+\frac{(s+z)^{\lambda}}{(s+z)}\right]ds$$

$$-\sum_{\lambda=1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{(\lambda)!}\int_0^{+\infty}\overline{\gamma_\lambda(s)}\left[\frac{(s-z)^{\lambda}}{(s-z)}+\frac{(s+z)^{\lambda}}{(s+z)}\right]ds$$

$$-\sum_{\lambda=1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{(\lambda)!}\int_0^{+\infty}\beta_\lambda(s)\left[\frac{(is+\bar{z})^{\lambda}}{(s+iz)}+\frac{(-is+z)^{\lambda}}{(s-iz)}\right]ds$$

$$-\sum_{\lambda=1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{(\lambda)!}\int_0^{+\infty}\overline{\beta_\lambda(s)}\left[\frac{(is+z)^{\lambda}}{(s+iz)}+\frac{(is-z)^{\lambda}}{(s-iz)}\right]ds$$

$$+\frac{(-1)^{n-v}}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1}\left[f(\varsigma)\left(\frac{(\overline{\varsigma-z})^{n-1}}{(\varsigma-z)}+\frac{(\bar{\varsigma}+z)^{n-1}}{(\varsigma+z)}\right)\right.$$

$$\left.+\overline{f(\varsigma)}\left(\frac{(\varsigma-z)^{n-1}}{(\bar{\varsigma}-z)}-\frac{(\varsigma+z)^{n-1}}{(\bar{\varsigma}+z)}\right)\right]d\xi d\eta \tag{2.10}$$

Proof. Using Lemma (2.1), i.e., adding (2.1), (2.2), (2.3) and (2.4), we have

$$\frac{(-1)^{n-v}}{(n-v)!}\frac{1}{\pi}\int_{\mathbb{Q}_1}[f(\varsigma)\left(\frac{(\overline{\varsigma-z})^{n-1}}{(\varsigma-z)}+\frac{(\bar{\varsigma}+z)^{n-1}}{(\varsigma+z)}\right)$$

$$+\overline{f(\varsigma)}\left(\frac{(\varsigma-z)^{n-1}}{(\bar{\varsigma}-z)}-\frac{(\varsigma+z)^{n-1}}{(\bar{\varsigma}+z)}\right)]d\xi d\eta$$

$$= w(z) - \sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{(\lambda)!} \int_0^{+\infty} \partial_{\varsigma}^{\lambda} w(s) \frac{(s-\bar{z})^{\lambda}}{(s-z)} ds$$

$$+ \sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{1}{(\lambda)!} \int_0^{+\infty} \partial_{\varsigma}^{\lambda} w(is) \frac{(is+\bar{z})^{\lambda}}{(s+iz)} ds - \sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{(\lambda)!} \int_0^{+\infty} \partial_{\varsigma}^{\lambda} w(s) \frac{(s+z)^{\lambda}}{(s+z)} ds$$

$$+ \sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{1}{(\lambda)!} \int_0^{+\infty} \partial_{\varsigma}^{\lambda} w(is) \frac{(-is+\bar{z})^{\lambda}}{(s-iz)} ds + \sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{(\lambda)!} \int_0^{+\infty} \partial_{\varsigma}^{\lambda} \overline{w(s)} \frac{(s-z)^{\lambda}}{(s-z)} ds$$

$$+ \sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{1}{(\lambda)!} \int_0^{+\infty} \partial_{\varsigma}^{\lambda} \overline{w(is)} \frac{(is-\bar{z})^{\lambda}}{(s-iz)} ds + \sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{(\lambda)!} \int_0^{+\infty} \frac{\partial_{\varsigma}^{\lambda} \overline{w(s)} (s+z)^{\lambda}}{(s+z)} ds$$

$$+ \sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{(\lambda)!} \int_0^{+\infty} \frac{\partial_{\varsigma}^{\lambda} \overline{w(s)} (is+z)^{\lambda}}{(s+iz)} ds$$

$$= w(z) - \sum_{\lambda=1}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{(\lambda)!} \int_0^{+\infty} \partial_{\varsigma}^{\lambda} w(s) \frac{(t-\bar{z})^{\lambda}}{(s-z)} ds$$

$$+ \sum_{\lambda=1}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{(\lambda)!} \int_0^{+\infty} \partial_{\varsigma}^{\lambda} w(is) \frac{(is+\bar{z})^{\lambda}}{(s+iz)} ds$$

$$- \sum_{\lambda=1}^{n-1} \frac{1}{2\pi i} \frac{1}{(\lambda)!} \int_0^{+\infty} \partial_{\varsigma}^{\lambda} w(s) \frac{(s+\bar{z})^{\lambda}}{(s+z)} ds + \sum_{\lambda=1}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{(\lambda)!} \int_0^{+\infty} \partial_{\varsigma}^{\lambda} w(is) \frac{(-is+z)^{\lambda}}{(s-iz)} ds$$

$$+ \sum_{\lambda=1}^{n-1} \frac{1}{2\pi i} \frac{1}{(\lambda)!} \int_0^{+\infty} \partial_{\varsigma}^{\lambda} \overline{w(s)} \frac{(s-z)^{\lambda}}{(s-z)} ds + \sum_{\lambda=1}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{(\lambda)!} \int_0^{+\infty} \partial_{\varsigma}^{\lambda} \overline{w(is)} \frac{(is-z)^{\lambda}}{(s-iz)} ds$$

$$+ \sum_{\lambda=1}^{n-1} \frac{1}{2\pi i} \frac{1}{(\lambda)!} \int_0^{+\infty} \frac{\partial_{\varsigma}^{\lambda} \overline{w(s)} (s+z)^{\lambda}}{(s+z)} ds + \sum_{\lambda=1}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda}}{(\lambda)!} \int_0^{+\infty} \frac{\partial_{\varsigma}^{\lambda} \overline{w(is)} (is+z)^{\lambda}}{(s+iz)} ds$$

$$- \frac{1}{\pi i} \int_0^{+\infty} \left(\frac{w(s) - \overline{w(s)}}{2i} \right) \left[\frac{1}{(s-z)} + \frac{1}{(s+z)} \right] ds$$

$$+\frac{1}{\pi i}\int_0^{+\infty}\left(\frac{\overline{w(is)}-w(is)}{2}\right)\left[\frac{1}{(s-z)}+\frac{1}{(s+z)}\right]ds$$

$$= w(z)-\sum_{\lambda=1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{(\lambda)!}\int_0^{+\infty}\gamma_\lambda(s)\left[\frac{(s-\bar{z})^{\lambda}}{(s-z)}+\frac{(s+z)^{\lambda}}{(s+z)}\right]ds$$

$$+\sum_{\lambda=1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{(\lambda)!}\int_0^{+\infty}\overline{\gamma_\lambda(s)}\left[\frac{(s-\bar{z})^{\lambda}}{(s-z)}+\frac{(s+z)^{\lambda}}{(s+z)}\right]ds$$

$$+\sum_{\lambda=1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{(\lambda)!}\int_0^{+\infty}\beta_\lambda(s)\left[\frac{(is+\bar{z})^{\lambda}}{(s+iz)}+\frac{(-is+z)^{\lambda}}{(s-iz)}\right]ds$$

$$+\sum_{\lambda=1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{(\lambda)!}\int_0^{+\infty}\overline{\beta_\lambda(s)}\left[\frac{(is+z)^{\lambda}}{(s+iz)}+\frac{(is-z)^{\lambda}}{(s-iz)}\right]ds$$

$$-\frac{1}{\pi}\int_0^{+\infty}\gamma_0(s)\left[\frac{1}{(s-z)}+\frac{1}{(s+z)}\right]ds+\frac{1}{\pi i}\int_0^{+\infty}\beta_0(s)\left[\frac{1}{(s-z)}+\frac{1}{(s+z)}\right]ds$$

Thus, we obtain (2.10). The solvability conditions (2.7)-(2.9) can be obtained as in [11]. □

Using Lemma 2.1 and adding (2.1), (2.2) and subtracting (2.3), (2.4) we obtain, the solution of the following boundary value problem.

Theorem 2.3. *Let w be as in theorem* 2.1. *Then Schwarz-*$(n-1)$ *Dirichlet problem for the inhomogeneous polyanalytic equation in the quarter plane* $\mathbb{Q}_1$

$\partial_{\bar{z}}^{n}w = f$ *in* Q_1, $\operatorname{Im} w = \beta_0$ *on* $0 < y < \infty$, $x = 0$, *Rew* $= \gamma_0$ *on* $0 < x < \infty$, $y = 0$

$\partial_{\bar{z}}^{n}w = \beta_\lambda$ *on* $0 < y < \infty$, $x = 0$, $\partial_{\bar{z}}^{\lambda}w = \gamma_\lambda$ *on* $0 < x < \infty$, $y = 1, \leq \lambda \leq n-1$

is solvable if and only if

$$\sum_{\lambda=v}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda-v}}{(\lambda-v)!}\int_0^{+\infty}\gamma_\lambda(s)(s-z)^{\lambda-v}\frac{ds}{(s-\bar{z})}$$

$$-\sum_{\lambda=v}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda-v}}{(\lambda-v)!}\int_0^{+\infty}\beta_\lambda(s)(is+z)^{\lambda-v}\frac{ds}{(s-i\bar{z})}$$

$$+\frac{(-1)^{n-v}}{(n-v-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1}f(\varsigma)(\bar{\varsigma}-z)^{n-v-1}\frac{d\xi d\eta}{\varsigma-\bar{z}}=0, \qquad (2.11)$$

$$\sum_{\lambda=v}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{\lambda!}\int_0^{+\infty}\gamma_\lambda(s)(s+z)^{\lambda}\frac{ds}{(s+\bar{z})}$$

$$-\sum_{\lambda=v}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{\lambda!}\int_0^{+\infty}\beta_\lambda(s)(z-is)^{\lambda}\frac{ds}{(s-i\bar{z})}$$

$$+\frac{(-1)^{n-v}}{(n-v-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1}f(\varsigma)(\bar{\varsigma}+z)^{n-v-1}\frac{d\xi d\eta}{\varsigma+\bar{z}}=0, \qquad (2.12)$$

$$\sum_{\lambda=v}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{\lambda!}\int_0^{+\infty}\gamma_\lambda(s)(s+\bar{z})^{\lambda}\frac{ds}{(s+z)}$$

$$-\sum_{\lambda=v}^{n-1}\frac{1}{2\pi i}\frac{1}{\lambda!}\int_0^{+\infty}\beta_\lambda(s)(-is+\bar{z})^{\lambda}\frac{ds}{(s-iz)}$$

$$+\frac{(-1)^{n-v}}{(n-v-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1}f(\varsigma)(\overline{\varsigma+z})^{n-v-1}\frac{d\xi d\eta}{\varsigma+\bar{z}}=0, \qquad (2.13)$$

The solution if it exists is given by

$$w(z)=\frac{1}{\pi}\int_0^{+\infty}\gamma_0(s)\left[\frac{1}{(s-z)}+\frac{1}{(s+z)}\right]ds-\frac{1}{\pi i}\int_0^{+\infty}\beta_0(s)\left[\frac{1}{(s+iz)}+\frac{1}{(s-iz)}\right]ds$$

$$+\sum_{\lambda=1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{(\lambda)!}\int_0^{+\infty}\gamma_\lambda(s)\left[\frac{(s-\bar{z})^{\lambda}}{(s-z)}+\frac{(s+z)^{\lambda}}{(s+z)}\right]ds$$

$$+\sum_{\lambda=1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{(\lambda)!}\int_0^{+\infty}\overline{\gamma_\lambda(s)}\left[\frac{(s-z)^{\lambda}}{(s-z)}+\frac{(s+z)^{\lambda}}{(s+z)}\right]ds$$

$$+\sum_{\lambda=1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{(\lambda)!}\int_0^{+\infty}\beta_\lambda(s)\left[\frac{(is+\bar{z})^{\lambda}}{(s+iz)}+\frac{(-is+z)^{\lambda}}{(s-iz)}\right]ds$$

$$+\sum_{\lambda=1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{(\lambda)!}\int_0^{+\infty}\overline{\beta_\lambda(s)}\left[\frac{(is+z)^{\lambda}}{(s+iz)}+\frac{(is-z)^{\lambda}}{(s-iz)}\right]ds$$

$$+\frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1}\left[f(\varsigma)\left(\frac{(\overline{\varsigma-z})^{n-1}}{(\varsigma-z)}+\frac{(\bar{\varsigma}+z)^{n-1}}{(\varsigma+z)}\right)\right.$$

$$\left.+\overline{f(\varsigma)}\left(\frac{(\varsigma-z)^{n-1}}{(\bar{\varsigma}-z)}+\frac{(\varsigma+z)^{n-1}}{(\bar{\varsigma}+z)}\right)\right]d\xi d\eta \tag{2.14}$$

3. Conclusion

Other mixed boundary value problems can also be solved in the similar way arising from the combination of Dirichlet, Schwarz and Neumann boundary conditions for polyanalytic functions.

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