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HARDNESS RESULT OF OUTER-INDEPENDENT TOTAL ROMAN DOMINATION IN CHALLENGING FUZZY GRAPHS

S. KALAISELVI, J.GOLDEN EBENEZER JEBAMANI and P. NAMASIVAYAM

Research Scholar
Department of Mathematics
Sarah Tucker College, Tirunelveli-627007
E-mail: kalaiselvi041290@gmail.com

Department of Mathematics
Sarah Tucker College
Tirunelveli-627007, India
E-mail: goldensambeth@gamil.com

PG and Research
Department of Mathematics
The M.D.T Hindu College, Tirunelveli-627010
Affiliated to Manonmaniam Sundaranar
University, Abishekapatti, Tirunelveli-627012
Tamilnadu, India
E-mail: namasivayam@mdthinducollege

Abstract

An outer-independent Total Roman dominating function (OITRD) on a graph $\dddot{G}_{New}$ with vertex set $\ddot{V}(\dddot{G}_{New})$ is defined as a function $f : \ddot{V}(\dddot{G}_{New}) \to \{0, 1, 2\}$, such that every vertex $\ddot{v} \in \ddot{V}(\dddot{G}_{New})$ with $f(\ddot{v}) = 0$ has at least two neighbors allocate 1 under f or one neighbor $\dot{w}$ with $f(\dot{w}) = 2$ is independent. The weight of an OITRD f is the value $\dot{w}(f) = \sum_{i \in \ddot{V}(\dddot{G}_{New})} f(i)$. The

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minimum weight of an OIIDF on a graph $\dddot{G}_{New}$ is called the outer-independent Total Roman domination number $\Psi_{OITR}(\dddot{G}_{New})$ of $\dddot{G}_{New}$. In this paper, we initiate the study of the outer-independent Total Roman domination number and present the bounds on the outer-independent Total Roman domination number in terms of the proposed inverse 4−challenging fuzzy dominating set and inverse 4-challenging fuzzy domination number χ_{4CF}^{-1} of challenging fuzzy graph G_{CF}. The weight of an OITRD is the sum of its function values over all vertices, and the outer independent signed total Roman domination number (OITRD -number) $\Psi_{OITR}(\dddot{G}_{New})$ is the minimum weight of an OITRD on $\dddot{G}_{New}$. In addition, some result is obtained .

1. Introduction

Dominating set problems are among the most important class of combinatorial problems in graph optimization, from a theoretical as well as from a practical point of view [1]. For a given graph $\ddot{G} = \ddot{G}(\ddot{V}, \ddot{E})$, a subset $\ddot{D} \subset \ddot{V}$ of vertices is referred to as a dominating set if the remaining vertices, i.e., $\frac{\ddot{V}}{\ddot{D}}$ is dominated by $\ddot{D}$ according to a given topological relation (e.g., they are all adjacent to at least one vertex from $\ddot{D}$). Dominating set problems (also often called domination problems in graphs) have attracted the attention of computer scientists and plied mathematicians since the early 50s and their lose relation to covering and independent set problems has lead to the development of a whole research area. There are many applications where set domination and related concepts play a central role, including school bus routing, communication networks, radio station location, social networks analysis, biological networks analysis and also chess-problems. Variants of dominating set problems e.g., the connected dominating set problems the (weighted) independent dominating set problems, among others for further of the dominating set problems [2, 3].

2. Preliminary

In this paper, we shall only consider graphs without multiple edges or loops. For a graph $\dddot{G}_{New} = (\ddot{V}(\dddot{G}_{New}), \ddot{E}(\dddot{G}_{New})), \ddot{V}(\dddot{G}_{New})$ and $\ddot{E}(\dddot{G}_{New})$ are the sets of vertices and edges of New $\dddot{G}_{New}$, respectively. For $\Delta \subseteq \ddot{V}(\dddot{G}_{New})$

and $\ddot{v} \in \ddot{V}(\ddot{G}_{New})$, the open neighborhood of $\ddot{v}$ in Δ is denoted by $N_{\Delta}[\ddot{v}]$. That is to say $N_{\Delta}[\ddot{v}] = \{\dot{u} | \dot{u}\dot{v} \in \ddot{E}(\ddot{G}_{New}), \dot{u} \in \Delta\}$. The closed neighborhood $N_{\Delta}[\ddot{v}]$ of $\ddot{v}$ in Δ is defined as $N_{\Delta}[\ddot{v}] = \{\ddot{u}\} \cup N_{\Delta}(\dot{v})$. If $\Delta = \ddot{V}(\ddot{G}_{New})$, then $N_{\Delta}[\ddot{v}]$ and $N_{\Delta}[\ddot{v}]$ are denoted by $N_{\Delta}[\ddot{v}]$ and $N_{\Delta}[\ddot{v}]$, respectively. Let $\Delta = \ddot{V}(\ddot{G}_{New})$, we write $N_{\ddot{G}_{New}}[\Delta] = \cup N_{\ddot{G}_{New}}(X)$. The degree of $\ddot{v}$ is $D(\dot{v}) = | N(\dot{v}) |$. A set $\Delta \subseteq \ddot{V}(\ddot{G}_{New})$ of $\ddot{G}_{New}$ is independent if any two vertices in Δ are not adjacent in $\ddot{G}_{New}$. A leaf of $\ddot{G}_{New}$ is a vertex of degree one and a support vertex of $\ddot{G}_{New}$ is a vertex adjacent to a leaf. The set of leaves of $\ddot{G}_{New}$ is denoted by $\gamma(\ddot{G}_{New})$ and the set of support vertices by $\Delta(\ddot{G}_{New})$. Since outer-independent total domination and outer independent total Roman domination is not defined for graphs having isolated vertices, so all the graphs considered herein have no isolated vertices [14]. Given an OITDS $\ddot{D}$ of a graph $\ddot{G}_{New}$ a vertex $\dot{v} \in D$ is said to have a private neighbor if there exists a vertex $\dot{w} \in N(\dot{v}) \cap \left(\frac{\ddot{V}(\ddot{G}_{New})}{D}\right)$ for which $N(\dot{w}) \cap D = \{\dot{v}\}$.

2.1. Proposal OITR Graphs

If G is a disconnected graph and $g_1, g_2, \ldots, g_R$ are the connected components of g, where $R \geq 2$, then by Theorem 1, $\beta_{OITR}(g_k)$ for each g_k Hence, g is an OITR graph if and only if each g_k is an OITR graph. So, in what follows, we only consider connected OITR graphs.

Lemma 1. *Let g be a connected OITR graph and d'' be β_{OIT}-set of g. Then every vertex in d'' has a private neighbor.*

Proof. Let ver be any vertex in d''. By the definition of an OITDS, $N(ver) \cap d'' \neq \varphi$. Suppose that ver has no private neighbour. Then every vertex in $\frac{ver(g)}{d''}$ is adjacent to a vertex in $\frac{d''}{\{ver\}}$.

Consider a function $f = (\ddot{V}(g)/d'', \{ver\}, d''/\{ver\})$. Then f is an OITRD function on g and $\alpha(f) = 1 + 2(|d''| - 1) = 2|d''| - 1 = 2\beta_{OIT}(g) - 1$, contradicting that g is an OIT-Roman graph. Thus, every vertex in d'' has a private neighbor.

Lemma 2. *Let g be a connected OIT-Roman graph and d'' be a β_{OIT}-set of $ver \in d'''$ Then any vertex $ver \in d''$ has a leaf neighbor.*

Proof. By Lemma 1, verhas a private neighbour, say $\vec{u}$. Hence, $N(\vec{u}) \cap d'' = \{ver\}$. Note that $\ddot{V}(g)/d''$ is independent we obtain that $\vec{u}$ has only one neighbor ver in g. Thus, $\vec{u}$ is a leaf neighbour of ver.

2.2. Converse 4- challenging fuzzy domination in anti fuzzy graphs G_{CF}.

Definition 2.2.1. A subset vertex ρ of $\ddot{V}(G_{CF})$ is said to be 4-challenging fuzzy dominating (4_{CFd}) set of G_{CF} if for every vertex $A \in \ddot{V} - \rho$ there is at least two vertices $B_1, B_2 \in \rho$ such that $\alpha(A, B_1) = \phi(A)\ddot{V}\phi(B_1)$ and $\alpha(A, B_1) = \phi(A)\ddot{V}\phi(B_2)$.

Definition 2.2.2. A 4_{CFd} set of G_{CF} with minimum number of vertices is called minimum 4-challenging fuzzy dominating (4_{CFd}) set of G_{CF}. A challenging fuzzy (4_{CFd}) domination number is define is take maximum cardinality for all $(Max4_{CFd})$ set and denoted by χ_{4CF}.

Definition 2.2.3. Let $G_{CF} = (\varphi, \alpha)$ be any challenging fuzzy graph without isolated vertex and ρ be a $(Max4_{CFd})$ set of G_{CF}, if $\ddot{V} - \rho$ contains a 4_{CFd} set ρ' then D' is called inverse 4_{CFd} set of $G_{CF}\, w.r.t\, \rho$. The maximum cardinality taken over all minimum inverse 4_{CFd} sets of is called converse 4-domination number of G_{CF} and denoted by $\chi^{-1}_{4CF}(G_{CF})$.

Example. Consider a challenging fuzzy graph G_{CF}, which is given in Figure 1. The $Max4_{CFd}$ sets of G_{CF} are $\rho_1 = \{U_1, U_2, V_1, V_2\}$ $\rho_2 = \{A, B, U, V\}$.

$\chi_{4CF} = \max[| \rho_1 |, | \rho_2 |] = \max[2.3, 4.5] = 4.5 = | \rho_2|$, then ρ_1 is inverse 4_{CFd} set, thus $\chi_{4CF}^{-1} = | \rho_1 | = 2.3$.

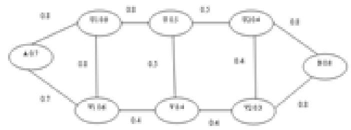

Figure1. converse 4-challenging fuzzy dominating set.

Preposition 2.2.4 (i). $\chi_{4CF}(Y_\varphi) = \max[\varphi(A) + \varphi(B)]\ \forall A, B \in \ddot{V}(Y_\varphi)$, $z \geq 5$.

(ii) *Let* $G_{CF} \equiv I_z$ *vertices and* $z \geq 3$. *and every edge is effective then*

$$\chi_{4CF}(Y_z) = \max\left\{\sum_{a=0}^{\frac{z}{2}-1} \varphi(U_{b+2a}); b = 1, 2\right\}.$$

(iii) If $G_{CF} \equiv I_{\varphi_1, \varphi_2}$ *is a complete bipartite challenging fuzzy graph with* i, j *vertices then*

$$G_{CF}(I_{\varphi_1,\varphi_2}) = \begin{cases} \max\begin{cases} | A | & if\ 4 \leq i \leq 8 \\ | B | & 4 \leq j \leq 8 \end{cases} \\ \max[\varphi(A_a) + \varphi(A_b) + \varphi(Y_b)] i, j \geq 4, a_a, a_b \in A\, and\, b_a, b_b \in B \end{cases}$$

Theorem 2.2.5. *For any challenging fuzzy graph* G_{CF} *has an inverse* 4_{CFd} *set, then a vertex* $A \in \dddot{V} - \rho$ *belongs to every inverse* 2_{CFd} *set of* G_{CF} *if a has either two or three neighbors*.

Suggestion 1. Let $G_{CF} = (\varphi, \alpha)$ be any anti graph has no isolated vertex, if inverse 4_{CFd} exist then G_{CF} contains at least four vertices.

Proof. Let $\ddot{D}$ be a $Maxx4_{CFd}$ set of G_{CF}, since G_{CF} has no isolated vertex, so $\ddot{D}$ contains at least two vertices. If inverse 4_{CFd} set exists then $\dddot{V} - \ddot{D}$ contains 4_{CFd} set with respect to $\ddot{D}$. Thus $V - \ddot{D}$. has at least two vertices.

Observation 2.2.6. For any anti fuzzy graph has no isolated vertex, every inverse 4_{CFd} set is inverse fuzzy dominating set.

Suggestion 2. if there is inverse 4_{CFd} set of G_{CF} then $\chi_{4CF} + \chi_{4CF}^{-1} \leq P$.

Proof. Consider $\ddot{D}$ and $\ddot{D}'$ are $Max4_{CFd}$ set and inverse 4_{CFd} set of G_{CF} respectively. Then $\ddot{D}' \subseteq \ddot{V} - \ddot{D}$. So $\ddot{D}' \leq |\ddot{V} - \ddot{D}|$. Thus $\chi_{4CF}^{-1} \leq \rho \leq \chi_{4CF}$. Hence, $\chi_{4CF} + \chi_{4CF}^{-1} \leq P$.

Corollary 3.2. *For any challenging fuzzy graph if there is inverse* 4_{CFd} *set, then* $[\varphi(a) + \varphi(b)] \leq \chi_{4CF}^{-1} \leq \sum U_a$ *where* $a, b \in \rho'$ *and* $U_a \in \ddot{V} - \ddot{D}$.

Preposition 2.2.7. *If* $I_{\phi 1}, \phi_2$ *is a complete bipartite anti fuzzy graph with* i, j *vertices then*

$$\chi_{4CF}^{-1}(I_{\varphi 1, \phi_2}) = \begin{cases} \max \begin{cases} |A| & \text{if } 4 \leq i \leq 8 \text{ and } i < j \\ |B| & \text{if } 4 \leq j \leq 8 \text{ and } j < i \\ \min[|A|, |B|] & i = j < 4 \end{cases} \\ \{[|A|, |B|], \max[\varphi(A_a) + \varphi(A_b) + \varphi(Y_b)] i, j \geq 4, a_a, a_b \\ \in A \text{ and } b_a, b_b \in B - \ddot{D} \end{cases}$$

Proof. Let $\ddot{V}(I_{\phi_1, \phi_2}) = A \cup B$, where $A = \{a_1, a_2, \ldots, a_i\}$, $B = \{b_1, b_2, \ldots, b_j\}$. And let $\ddot{D}$ is $Max4_{CFd}$ set of $I_{\phi 1, \phi_2}$. There are three cases:

Case1. If *i* or *j* less than four, then there are two subcases as follows.

Subcase 1. If $4 = i < j$ then it is clear that $\ddot{D} = \{a_1, a_2\}$ where $a_1, a_2 \in A$ is $Max4_{CFd}$ set. Such that $|\ddot{D}| = \chi_{4CF}(I_{\varphi_1, \varphi_2}) = |A|$. Thus the set B is inverse 4_{CFd} set of I_{φ_1, φ_2} such that $\chi_{4CF}^{-1}(I_{\varphi 1, \varphi_2}) = |B|$. Similarly if $4 = i < j$, $\chi_{4CF}^{-1}(I_{\varphi 1, \varphi_2}) = |A|$.

Subcase 2. If $i < j < 8$ it is clear $\chi_{4CF}^{-1}(I_{\varphi 1, \varphi_2}) = \min[|A|, |B|]$.

Case 2. If $i = 8$, then

$\chi^{-1}_{4CF}(I_{\varphi_1, \varphi_2}) = \{| B |, \max a[\varphi(a_x) + \varphi(a_y) + \varphi(b_x) + \varphi(b_y)]\}, a_x, a_y \in A - \ddot{D}$ and $b_x, b_y \in B - \ddot{D}$. Likewise if $j = 8$ then

$$\chi^{-1}_{4CF}(I_{\varphi_1, \varphi_2}) = \{| A |, \max a[\varphi(a_x) + \varphi(a_y) + \varphi(b_x) + \varphi(b_y)]\}, a_x, a_y \in A - \ddot{D}$$

Case 3. If $j = 9$. It is clear by preposition a $Max4_{CFd}$ set of $I_{\phi 1, \phi_2}$ contains four vertices take two vertices of each sets A and B with maximum cardinality value. Thus the inverse 4_{CFd} set of $I_{\phi 1, \phi_2}$ is $Max4_{CFd}$ set of $I_{\phi 1-2, \phi_2-2}$. Thus,

$\chi^{-1}_{4CF}(I_{\phi 1, \phi_2}) = \chi_{4CF}(I_{\phi 1-2, \phi_2-2}) = \max[\varphi(a_x) + \varphi(a_y) + \varphi(b_x) + \varphi(b_y), i, j > 8$ and $a_x, a_y \in A - \ddot{D}, b_x, b_y \in B - \ddot{D}]$. Hence, the results is obtains.

Preposition 2.2.8. *Every inverse* 4_{CFd} *set of* $G_{CF} = (\varphi, \alpha)$ *is inverse* 4-*dominating set of challenging crisp graph* $G^*_C = (\varphi^*, \alpha^*)$.

Proof. Let $\ddot{D}''$ be an inverse 4_{CFd} set of G_{CF}, then for every vertex $b \in \dddot{V} - \ddot{D}''$ has at least two effective neighbors in $\ddot{D}''$, i.e. There are $a_1, b_2 \in \ddot{D}''$ such that $\alpha(b, a_1) \vee \varphi(b) > 0$ thus $(b, a_1) \in \alpha^*$ and $\alpha(b, a_2) = \varphi(a_2) \vee \varphi(b) > 0, (b, a_2) \in \alpha^*$. Therefore $\ddot{D}''$, contains two neighbours of b. Hence, the theorem is prove.

Example. Consider $G_{CF} = (\varphi, \alpha)$ and $G^*_C = (\varphi^*, \alpha^*)$ in figure $2(a, b)$ respectively Clearly Hardness Result of Outer-Independent Total Roman Domination in Challenging Fuzzy Graphs $\{(b, e), (c, f)\}$ is inverse 4-dominating set of $G^*_C = (\varphi^*, \alpha^*)$ but not inverse in $G_{CF} = (\varphi, \alpha)$.

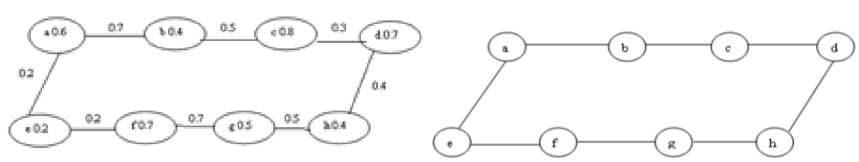

Figure 2. (a) Challenging fuzzy graph and (b) crisp graph.

Theorem 2.2.9. *Let* $G_{CF} = (\varphi, \alpha)$ *be a challenging fuzzy graph. Then* $\ddot{D}''$ *be inverse* 4_{CFd} *set* G_{CF} *such that* $|\ddot{D}''| = \chi^{-1}_{4CF}$ *is minimal if for each* vertex $a \in \ddot{D}''$, *either*, 1, $N(a) \cap D'' < 4$ *or* 2. *There exists a vertex* $b \in \ddot{V} - \ddot{D}''$ *such that* $N(b) \cap D'' < 4\, a\, b \in N(b)$.

Proof. Let $\ddot{D}''$ be an inverse 4_{CFd} set of G_{CF} such that $|\ddot{D}''| = \chi^{-1}_{4CF}$. Assume that the above conditions are not holds, i.e. there exist $a \in \ddot{D}''$ such that $N(a) \cap D'' \geq 4$ and for each vertex $b \in \ddot{V} - \ddot{D}''$ either $N(b) \cap D'' > 4$ or $b \notin N(a)$. Consider $a = \ddot{D}'' - \{a\}$, since a has at least two neigphbors in $\ddot{D}''$ Thus X is inverse 4_{CFd} set of G_{CF}, which contradiction with minimality $\ddot{D}''$.

Conversely. Let $\ddot{D}''$ be an inverse 4_{CFd} set of G_{CF} satisfying the conditions (1) and (2).Consider $X = \ddot{D}'' - \{a\}$ for any vertex $a \in \ddot{D}''$ If condition (1) holds then X is not inverse 4_{CFd} set, and if (2) holds then X has one neigphbor of b .then b is not inverse 4_{CFd} set. Hence, $\ddot{D}''$ is minimal inverse 4_{CFd} set of G_{CF}.

3. Conclusion

This paper considers the properties of the outer-independent domination. More recently, known as Roman-f2g domination, Total Roman domination was proposed in this paper. We show bounds relating the outer-independent Total Roman domination number to the proposed challenging fuzzy number, order and diameter. The results partially answer theorem 1 and 2 proposed by this work respectively. Moreover, we have characterized all OIT-Roman domination graphs and given a proposed challenging Fuzzy graph for recognizing an OIT-Roman Domination graph, which answers theorem 3.

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