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SYNTHETIC AND GENERALIZED SYNTHETIC ELEMENTS IN UNITAL ALGEBRAS

SAJAD KHORSHIDVANDPOUR* and PARASTOO HEIATIAN NAEINI

Department of Mathematics
Faculty of Mathematical Sciences and Computer
Shahid Chamran University of Ahvaz
Ahvaz, Iran

Department of Mathematics
Payame Noor University, Naein, Iran

Abstract

In the present paper, we introduce synthetic and generalized synthetic elements in unital algebras. These elements are two subclasses of invertible elements. Several results on synthetic and generalized synthetic elements are given. For instance, it is established that if a and b are synthetic elements, then $b^{-1} = a$ if and only if $a^2 + b^2 = -e$, where e denotes the identity. Also, we give a sufficient condition for the product of two synthetic elements to be synthetic. Finally, we prove that a matrix $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \in M_3(\mathbb{R})$ is generalized synthetic if and only if $\det B \in \mathbb{R} \setminus \{0, 1 - L + M - N\}$, where $L = \sum_{i=1}^{3} b_{ii}$, $M = \sum_{i,\, j=1}^{3} b_{ii}b_{jj}$, and $N = \sum_{j=i+1}^{3} \sum_{i=1}^{3} b_{ij}b_{ji}$.

1. Introduction and Preliminaries

The study of specific elements in various algebraic structures such as group, ring, lattice, module and algebra is an interesting topic for mathematicians. An element a in a ring R is called a regular element if there

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*Corresponding author; E-mail: skhorshidvandpour@gmail.com
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exists $b \in R$ such that $a = a^2b$. Such b is called von Neumann inverse for a. The set of all regular element of R denoted by $V_r(R)$. In [1], a characterization of such elements in $\mathbb{Z}_n$, the ring of integers modulo n, has been given. Danpattanamongkon and Kemprasitin [4], during their study on BQ-semigroups, characterized regular elements of $(\mathbb{Z}_n, .)$. Indeed, they proved that for $x \in \mathbb{Z}$, $\bar{x} \in Vr(\mathbb{Z}_n, .)$ if and only if x and $\frac{n}{(x, n)}$ are relatively prime.

Let R be a ring with a unit $1 \neq 0$ and an involution $a \to a^*$ satisfying $(a^*)^* = a$, $(a + b)^* = a^* + b^*$ and $(ab)^* = b^*c^*$. We say that $a \in R$ is Moore-Penrose invertible (or MP-invertible), if there exists $b \in R$ such that the following hold [11]

$$aba = a, \; bab = b, \; (ab)^* = ab, \; (ba)^* = ba.$$

Any b that satisfies the above conditions is called a Moore-Penrose inverse of a. It is well known that the Moore-Penrose inverse is unique when it exists. The Moore-Penrose inverse of a denoted by $a^\dagger$. Authors in [11] have studied Moore-Penrose inverses in rings with involution.

An EP element a in a ring is an element which commute with its Moore-Penrose inverse $a^\dagger$. An EP element is also called *-gMP element in [9]. In [8, Theorem 2.1], many necessary and sufficient conditions for an element of a ring with involution to be EP, have been presented. Also, a partial isometry a in a ring with involution is such that $a^* = a^\dagger$. In [7], Mosic and Djordjevic have characterized partial isometries and EP elements in rings with involution. For more information on special elements in various structures, we refer the interested reader to [2], [5], [7-14].

Throughout this paper, A denote an unital normed algebra. A^{-1} and $E(A)$ will denote the group of all invertible and idempotent elements in A, respectively. For any element $a \in A$, we define the commutant of a by $comm(a) = \{x \in A : ax = xa\}$. By $L(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = ax + b\}$ and $L(\mathbb{C}) := \{f : \mathbb{C} \to \mathbb{C} \mid f(z) = az + b\}$, we denote the algebra of all linear function with composition as product. The spectrum and spectral radius of an element $a \in A$ defined as $\sigma(a) = \{\lambda \in \mathbb{C} : \lambda e - a \notin A^{-1}\}$ and $r(a) = \inf\{\| a^n \|^{\frac{1}{n}} : n \in \mathbb{N}\}$, respectively. $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ stand for the set of all natural, real and complex numbers, respectively. Any unexplained notion can be found in [3] and [8].

The rest of this article is organized as follows. In the next section, the notion of a synthetic element is introduced and many examples are given. The set of all synthetic elements in A denoted by A^s. Furthermore, some basic results on synthetic elements are established (Propositions 2.5 and 2.6). Also, we show that if $a, b \in A^s$, then $b^{-1} = a$ if and only if $a^2 + b^2 = -e$ (Theorem 2.7). Given $a, b \in A^s$, we give a sufficient condition for ab to be synthetic (Proposition 2.9). Some results about the spectrum of a synthetic element are established in Proposition 2.12. Section 3 has been devoted to introduce the notion of a generalized synthetic element in unital algebras. Some useful information about generalized synthetic elements in unital Banach algebras are presented in Proposition 3.3. Finally, generalized synthetic elements of algebras of matrices with real entries of orders 2 and 3 are characterized (Theorem 3.5).

2. Synthetic Element in Unital Algebras

We begin with the following definition.

Definition 2.1. We say that $a \in A$ is synthetic when $a \in A^{-1}$ and $a + a^{-1} = e$.

By A^s, we denote the set of all synthetic elements in A.

Obviously, if $a \in A^s$, then $a^{-1} \in A^s$. Furthermore, if A is a $*$-algebra and $a \in A^s$, then $a^* \in A^s$. Also it is easy to see that $A^s \subset Vr(A)$. Further, if $a \in Vr(A)$ and b is a von Neumann inverse for a, then $a \in A^s$ if and only if $b \in A^s$. If $a \in A^s$, then a is Moore-Penrose invertible and $a^{\dagger} = a^{-1}$. It is routine to check that every synthetic element is EP. Also $a \in A^s$ is partial isometry if and only if $a^* = a^{-1}$.

Example 2.2. (i) Consider $A = \mathbb{C}$, the complex numbers algebra. Then $\mathbb{C}^s = \left\{\frac{1 + i\sqrt{3}}{2}, \frac{1 - i\sqrt{3}}{2}\right\}$.

(ii) $A = \mathbb{R}$, the real numbers algebra, has no synthetic element.

(iii) $L(\mathbb{R})^s = \phi$.

(iv) $L(\mathbb{C})^s = \{iz + \frac{1-i}{2}, -iz + \frac{1+i}{2}\}$.

Proposition 2.3. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$ *such that* $b, c \neq 0$, $ad \neq 1$ *and* $a \neq d$. *Then* $A \in M_2(\mathbb{R})^s$ *if and only if* $\det A = 1$ *and* $d + a = 1$.

Proof. Assume that $A \in M_2(\mathbb{R})^s$. Then we have

$$A + A^{-1} = I_2 \tag{2.1}$$

where I_2 denotes the identity matrix of order 2. Therefore we have $b(ad - bc) = b$ and $c(ad - bc) = c$. Since $b, c \neq 0$, so $\det A = 1$. Also, it follows from (2.1) and $\det A = 1$ that $\frac{d}{1-a} = \frac{a}{1-d}$. The rest of what we need follows from $ad \neq 1$ and $a \neq d$. The converse is straightforward. □

Example 2.4. Consider $A = \begin{bmatrix} \frac{1}{4} & -1 \\ \frac{13}{16} & \frac{3}{4} \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} \frac{3}{4} & 1 \\ \frac{-13}{16} & \frac{1}{4} \end{bmatrix}$.

By Proposition 2.3, $A \in M_2(\mathbb{R})^s$.

In the following, some basic results on a synthetic element are given.

Proposition 2.5. (i) *Let A be an unital normed algebra. Then* A^s *is a norm-closed subset of* A^{-1}.

(ii) *Let* A *and* B *be unital algebras and* $\theta : A \to B$ *homomorphism. If* $a \in A^s$, *then* $\theta(a) \in B^s$.

(iii) $E(A) \cap A^s = \phi$.

(iv) *Let* $a \in A^s$. *Then* $a^{n+2} = a^{n+1} - a^n$ *for any integer number* n.

(v) *Let* $a \in A^s$ *and* $2 \leq n \in \mathbb{N}$. *Then* $\| a^n + (a^n)^{-1} \| = \begin{cases} 1 & n \text{ is even} \\ 2 & n \text{ is odd.} \end{cases}$

Proof. (i) Let $(x_n) \subset A^s$ be an arbitrary sequence such that $x_n \to x \in A^{-1}$. as $n \to \infty$. We have $x_n + x_n^{-1} = e$ for any $n \in \mathbb{N}$. Since $x_n \to x$, as $n \to \infty$, so $x_n^{-1} \to x^{-1}$. Therefore $e = x_n + x_n^{-1} \to x + x^{-1}$. Hence, $x + x^{-1} = e$, as desired.

(ii) Since $a \in A^s$, so $a + a^{-1} = e_A$. Notice that $\theta(a) \in B^{-1}$. Also, it is easy to show that $(\theta(a))^{-1} = \theta(a^{-1})$. On the other hand, $\theta(a) + \theta(a^{-1})$ $= \theta(a + a^{-1}) = \theta(e_A) = e_B$, as desired.

(iii) Suppose that $a \in E(A) \cap A^s$. It follows that $a = a^{-1}$. Also, since $a \in A^s$, we conclude that $a = \frac{1}{2}e$ which is neither synthetic nor idempotent.

(iv) It follows from $a \in A^s$ that $a - a^2 = e$. Therefore, $a^2 - a^3 = a^4$. Continuing the process, implies that $a^{n+2} = a^{n+1} - a^n$ for any integer number n.

(v) Let $2 \leq n \in \mathbb{N}$. Since $a \in A^s$, it can be seen easily that $a^n + (a^n)^{-1} = \begin{cases} -e & n \text{ is even} \\ -2e & n \text{ is odd.} \end{cases}$ This proves the assertion. □

In the following proposition, we consider A as a noncommutative ring with identity.

Proposition 2.6. *Let* a, b *and* $ab \in A^s$. *Suppose further*, $b^{-1} \in Comm(a)$. *Then*

(i) $b^{-1} = a^2$.

(ii) $a^n + a^{n+3} = 0$, *for each positive integer n*.

(iii) $b^n + b^{n+3} = 0$, *for each positive integer n*.

Proof. (i) Since $ab \in A^s$, we have $ab + (ab)^{-1} = e$. Then $ab + ab^{-1} - ab^{-1} + b^{-1}a^{-1} = e$ which implies $a(b + b^{-1}) = ab^{-1} - b^{-1}a^{-1} + e$. Since $b \in A^s$ and $b^{-1} \in comm(a)$, we have $a - e = b^{-1}a - b^{-1}a^{-1}$. Since $a \in A^s$, we have

$$(b^{-1} - e)a^{-1} = b^{-1}a \tag{2.2}$$

which yields that $e - a^2 = b$. Now it follows from the assumption $b \in A^s$ that $b^{-1} = a^2$. Hence the result.

(ii) First we conclude from part (i) that $a^{-1}b^{-1}a^{-1} = e$. Therefore, $a + a^{-1} - a^{-1}b^{-1}a^{-1} = 0$. This infers that $a^2 + ab = 0$. Equivalently, $a^2b^{-1} + a = 0$. By part (i), $b^{-1} = a^2$. Hence $a^4 + a = 0$ which gives the desired conclusion.

(iii) It is easy to show from (2.2) that $b^{-1} - e = b^{-1}a^2$. Since $b \in A^s$, so $b^{-1}a^2 + b = 0$. Thus $a^2 + b^2 = 0$. Now applying (i) yields that $b^3 = -e$. This gives the assertion. □

In the following, we give a necessary and sufficient condition for a synthetic element to be the inverse of another synthetic element.

Theorem 2.7. *Let* $a, b \in A^s$. *Then* $b^{-1} = a$ *if and only if* $a^2 + b^2 = -e$.

Proof. Assume that $b^{-1} = a$. By the proof of Theorem 2.7, $a^2 + (a^{-1})^2 = -e$. This completes the proof of this direction. Conversely, let $a^2 + b^2 = -e$. Since $a, b \in A^s$, it immediately follows that $a^2 + a^{-1} = 0$ and $b^2 + b^{-1} = 0$. Therefore, $a^2 + b^2 + a^{-1} + b^{-1} = 0$ and so $a^{-1} + b^{-1} = e$. Now, since $a, \in A^s$, we conclude that $b^{-1} = a$. □

Corollary 2.8. *Let* $a, b \in A^s$. *If* $a^2 + b^2 = -e$, *then* $a, b \notin A^s$.

Proof. Suppose by contradiction that $a, b \in A^s$. According to Proposition 2.6 (i), $b^{-1} = a^2$. Now, Theorem 2.7 implies that $a \in E(A)$ which contradicts to Proposition 2.5 (iii). □

The product of two synthetic elements need not be synthetic. For example, $A = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ -\frac{7}{5} & \frac{4}{5} \end{bmatrix}$ and $B = \begin{bmatrix} \frac{1}{3} & -\frac{8}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$.

According to Proposition 2.3, $A, B \in M_2(\mathbb{R})^s$ but $AB \notin M_2(\mathbb{R})^s$. Now, we give a sufficient condition for the product of two synthetic elements to be synthetic.

Proposition 2.9. *Let* $a, b \in A^s$. *Suppose further*, *a and b commute and* $a^2 + b^2 = 0$. *Then* $ab \in A^s$.

Proof. It follows from the assumption $a^2 + b^2 = 0$ that $ab^{-1} + a^{-1}b = 0$. Since, a and b are synthetic, we have $ab + ab^{-1} + a^{-1}b + a^{-1}b^{-1} = e$. Finally, using $a^{-1} \in comm(b^{-1})$ completes the proof. □

In the following, we study the commutant of a synthetic element.

Proposition 2.10. *Let* $a, b \in A^s$. *Suppose that* $c = aba$ *and* $d = bab$. *Then*

(i) $c \in Comm(a)$ *and* $d \in Comm(b)$.

(ii) $a^2 \in Comm(b^n)$ *and* $b^2 \in Comm(a^n)$, *for any* $n \in \mathbb{N}$.

Proof. (i) It follows from $c = aba$ that $a^{-1}ca^{-1} = b$. Since $b \in A^s$, one can obtain $ba^{-1}ca^{-1} = b - e$. Therefore $b^2a = b(a - e)$. By using the assumption $a \in A^s$ we have $b^2a = -ba^{-1}$. Now it can be seen easily that $a^2 = -b^{-1}$. Therefore, $ca = aba^2 = -a$ and $ac = a^2ba = -a$. Hence $c \in comm(a)$. In exactly the same way, $d \in comm(b)$.

(ii) First, we use from $c \in comm(a)$ in part (i) to obtain $aba = ba^2$ and $a^2b = aba$ and so $a^2 \in comm(b)$. Now we have

$$a^2b^2 = ba^2b \overset{a^2 \in Comm(b)}{\Rightarrow} a^2b^2 = b^2a^2.$$

Continuing this process gives us $a^2 \in comm(b^n)$, for any $n \in \mathbb{N}$. In a similar manner, one can prove that $b^2 \in Comm(a^n)$, for any $n \in \mathbb{N}$. □

Theorem 2.11. *Let A be an unital Banach algebra,* $a \in A^s$ *and* $r(a) < 1$. *Then we have*

$$a^{-1} + \sum_{n=1}^{\infty} a^n = 0$$

Proof. By [3, Theorem 2.9], $e - a$ is invertible and $(e - a)^{-1} = e + \sum_{n=1}^{\infty} a^n$. On the other hand, $e - a = a^{-1}$, since $a \in A^s$. Therefore $a^{-1} + \sum_{n=1}^{\infty} a^n = 0$. □

Finally, we have the following result about the spectrum of a synthetic element and its powers.

Proposition 2.12. *Let* $a \in A^s$. *Then*

(i) $\sigma(a) \subseteq \mathbb{C} \setminus \{1\}$.

(ii) $\sigma(a^n) \subseteq \mathbb{C} \setminus \{-1\}$, *for each even* $n \in \mathbb{N}$.

(iii) $\sigma(a^n) \subseteq \mathbb{C} \setminus \{-2\}$, *for each odd* $n \in \mathbb{N}$.

Proof. (i) Since $a \in A^s$, $e - a \in A^{-1}$ and so $1 \notin \sigma(a)$.

(ii) If $n \in \mathbb{N}$ is even, then by the proof of Proposition 2.5 (v), $-e - a^n \in A^{-1}$ which means that $-1 \notin \sigma(a^n)$.

(iii) Suppose that $n \in \mathbb{N}$ is odd ≥ 3. Again, the proof of Proposition 2.5 (v) infers that $-2 - a^n \in A^{-1}$ which means that $-2 \notin \sigma(a^n)$. □

3. Generalized Synthetic Element in Unital Algebras

In the present section, we introduce the notion of a generalized synthetic element in unital algebras.

Definition 3.1. Let A be an unital algebra. We say that $a \in A$ is generalized synthetic when $a \in A^{-1}$ and there exists an invertible element b in A such that $a + b = e$.

There is at most one b such that above condition holds and such b is denoted by A^{gs}. The set of all generalized synthetic elements of A is denoted by A^{gs}. Obviously, $A^{s} \subset A^{gs}$. See the figure below.

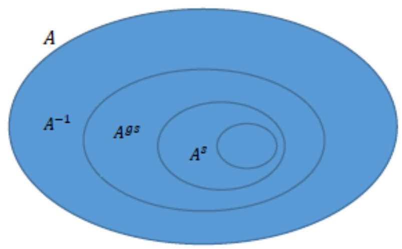

Figure 1.

Example 3.2 (i) A quaternion is an expression of the form $a + bi + cj + dk$ are consisting of a scalar part and a vector part a is called the scalar part and $bi + cj + dk$ is the vector part. The algebra of quaternions is often denoted by H. Indeed, $H = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ and has $\{1, i, j, k\}$ as a basis. For details on this algebra see [6]. It can be seen easily that $H^{gs} = H - \{0, 1\}$ where $0 = 0 + 0i + 0j + 0k$ and $1 = 1 + 0i + 0j + 0k$.

(ii) $L(R)^{gs} = \{f(x) = ax + b \mid a \in \mathbb{R} = \{0, 1\}$ and $b \in \mathbb{R}\}$.

The following proposition presents some useful information on generalized synthetic elements.

Proposition 3.3. (i) *For every* $a \in A\{0, e\}$, *we have* $(a^{gs})^{gs} = a$.

(ii) *If A is a unital normed algebra, then* A^{gs} *is a closed subset of* A^{-1}.

(iii) *If* $a, b, a+b \in A^{gs}$, *then* $(a+b)^{gs} = a^{gs} + b^{gs} - e$.

(iv) *If* $a, b \in A^{gs}$, *then* $ab \in A^{gs}$.

(v) *Let A be an unital normed algebra. Then* $\| ab \| \leq (1 + \| a \|)(1 + \| b \|)$, *for every* $a, b \in A^{gs}$.

(vi) *Let* $a \in A^{gs}$. *Then* $a \in E(A)$ *if and only if* $a^{gs} \in E(A)$.

(vii) *Let A be a division algebra. Then* $A^{gs} = A - \{0, e\}$.

Proof. (i) is straightforward.

(ii) Suppose that $(x_n) \subset A^{gs}$ be an arbitrary sequence such that $x_n \to x \in A^{-1}$, as $n \to \infty$. Then, there exists a sequence (y_n) of elements in A^{-1} such that $x_n + y_n = e$ for each $n \in \mathbb{N}$. On the other hand, since A^{-1} is closed, so $y_n \to y$, as $n \to \infty$ for some $y \in A^{-1}$. Therefore, $\lim_{n\to\infty}(x_n + y_n) = e$ which implies that $x + y = e$. This means that $x \in A^{gs}$, as required.

(iii) is straightforward.

(iv) Firstly, $ab \in A^{-1}$. Secondly, we have $a + a^{gs} = e$ and $b + b^{gs} = e$. Thirdly, $(a + a^{gs})(b + b^{gs}) = e$. It follows that $ab + ab^{gs} + a^{gs}b + a^{gs}b^{gs} = e$. Now, since $ab^{gs} + a^{gs}b + a^{gs}b^{gs} \in A^{-1}$, thus $ab \in A^{gs}$. Furthermore, we obtain $(ab)^{gs} = ab^{gs} + a^{gs}b + a^{gs}b^{gs}$.

(v) We have $\| ab \| = \| (e - a^{gs})(e - b^{gs}) \| = \| e - (a^{gs} + b^{gs}) + a^{gs}b^{gs} \|$

$\leq 1 + \| a^{gs} \| + \| b^{gs} \| + \| a^{gs} \| \| b^{gs} \| = (1 + \| a \|)(1 + \| b \|)$.

(vi) We have

$$a \in E(A) \Leftrightarrow (e - a^{gs})^2 = e - a^{gs} \Leftrightarrow e - a^{gs} - a^{gs} + (a^{gs})^2 = e - a^{gs}$$
$$\Leftrightarrow (a^{gs})^2 = a^{gs} \Leftrightarrow a^{gs} \in E(A).$$

(vii) It is well-known that A is isomorphic to a copy of $\mathbb{R}^n$, the n-dimensional real algebra, up to an isomorphism τ ([3]). It is routine to show that $(\mathbb{R}^n)^{gs} = \mathbb{R}^n - \{0, e\}$. Now, it follows from the fact that τ preserves generalized synthetic elements. Hence the result. □

Remark 3.4. A glimpse at the structure of A^s shows that it is only a set which is sometimes empty; while A^{gs} is a semi group endowed with the product of A (it follows from Proposition 3.3 (iv)).

Finally, generalized synthetic elements of the algebras $M_2(\mathbb{R})$ and $M_3(\mathbb{R})$ are characterized.

Theorem 3.5. (i) $A = \begin{bmatrix} q & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})^{gs}$ *if and only if* $detA \in \mathbb{R} \setminus \{0, 1 + d - 1\}$.

(ii) $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \in M_3(\mathbb{R})^{gs}$ *if and only if* $detB \in \mathbb{R} \setminus \{0, 1 - L + M - N\}$, *where*

$$L = \sum_{i=1}^{3} b_{ii},$$

$$M = \sum_{\substack{i,\, i=1 \\ i \neq j}}^{3} b_{ii} b_{jj},$$

and

$$N = \sum_{i=1}^{3} \sum_{i=1}^{3} b_{ij} b_{ji}.$$

Proof. (i) Suppose that $A = \begin{bmatrix} q & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})^{gs}$. Hence there exists $B \in M_2(\mathbb{R})^{-1}$ such that $A + B = I_2$.

Indeed, $B = \begin{bmatrix} 1-a & -b \\ -c & 1-d \end{bmatrix}$. Since B is invertible, so $\det B \neq 0$ which implies that $ad - bc \neq a + d - 1$. This means that $\det A \neq a + d - 1$. Furthermore, $\det A \neq 0$ follows from the invertibility of A. The converse is obvious.

(ii) First assume that $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \in M_3(\mathbb{R})^{gs}$. Then there exists $C \in M_3(\mathbb{R})^{-1}$ such that $B + C = I_3$. Then $C = \begin{bmatrix} 1-b_{11} & -b_{12} & -b_{13} \\ -b_{21} & 1-b_{22} & -b_{23} \\ -b_{31} & -b_{32} & 1-b_{33} \end{bmatrix}$. By simple calculations we have

$$\det B = (b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32}) - (b_{12}b_{21}b_{33} + b_{11}b_{23}b_{32} + b_{13}b_{22}b_{31})$$

and

$$\det C = ((1-b_{11})(1-b_{22})(1-b_{33}) - (b_{12}b_{23}b_{31}) - (b_{13}b_{21}b_{32})) - (b_{12}b_{21}(1-b_{33}) + b_{23}b_{32}(1-b_{11}) + b_{13}b_{31}(1-b_{22}).$$

It can be seen easily that

$$\begin{aligned}\det C = (1 - (b_{11} + b_{22} + b_{33}) + (b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33}) \\ -(b_{12}b_{21} + b_{23}b_{32} + b_{13}b_{31})) \\ -((b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32}) \\ -(b_{12}b_{21}b_{33} + b_{11}b_{23}b_{32} + b_{13}b_{22}b_{31})).\end{aligned}$$

Now, by taking $L = \sum_{i=1}^{3} b_{ii}$, $M = \sum_{i,\, j=1}^{3} b_{ii}b_{jj}$, and $N = \sum_{j=i+1}^{3} \sum_{i=1}^{3} b_{ij}b_{ji}$, we have $\det C = (1 - L + M - N) - \det B$.

On the other hand, it follows from invertibility of B and C that $\det B$, $\det C \neq 0$. Therefore, $\det B \neq 1 - L + M - N$. The converse is similar.□

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