Advances and Applications in Mathematical Sciences
Volume 20, Issue 12, 2021, Pages 3443-3514

A NEW FLEXIBLE ODD KAPPA-G FAMILY OF DISTRIBUTIONS: THEORY AND PROPERTIES

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Abstract

In this paper, we introduce a new generator based on Kappa random variable called Odd Kappa-G family whose hazard rate function (HRF) could be increasing, decreasing, *J*, reversed-*J*, bathtub, and upside-down bathtub. We present and discuss some special models. We obtain linear expansions for cumulative distribution function (CDF) and probability density function (PDF). We provide a comprehensive study of the mathematical properties of this new family. Explicit expressions for quantile function, the ordinary and incomplete moments, moment generating function, order statistics, Bonferroni and Lorenz curves, mean residual life, mean waiting time, mean deviations, entropy and other mathematical properties are derived. The maximum likelihood estimation method is given with a view to estimate unknown parameters for the members of this family. Simulation technique is established in order to validate the accuracy of estimation. Due to space constraints, applications on a member of this family will be published in a separate paper. As a result, the focus of this paper will be on theory and properties.

1. Introduction

An attractive asymmetric positively-skewed distribution called the Kappa distribution was introduced by Mielke [55] and Mielke and Johnson [56]. This distribution has recently gained more attention in applications to the hydrologic studies for use in analyzing precipitation, streamflow and wind

2010 Mathematics Subject Classification: 60Exx, 62-XX.
Keywords: Kappa distribution, Order Statistics, Bonferroni, Lorenz, MRL, MWT, Entropy.
Received July 29, 2021; Revised September 4, 2021

speed data. Traditionally, the well-known distributions gamma and log-normal have been fitted to historical rainfall data. However, these distributions are computationally inconvenient because their cumulative distribution and quantile functions cannot be obtained in closed forms. Whereas those of the Kappa distribution is closed algebraic expressions that can easily be evaluated.

The cdf of three-parameter Kappa distribution (Kappa3) (Mielke and Johnson [56]) is

$$R(t) = \frac{\left(\frac{t}{\beta}\right)^{\theta}}{\left[\alpha + \left(\frac{t}{\beta}\right)^{\alpha\theta}\right]^{\left(\frac{1}{\alpha}\right)}} \text{ for } t,\ \alpha,\ \beta \text{ and } \theta > 0, \tag{1.1}$$

where α and θ are shape parameters and β is a scale parameter. Furthermore, the pdf of three-parameter Kappa distribution (Kappa3) is

$$r(t) = \frac{\alpha\theta}{\beta}\left(\frac{t}{\beta}\right)^{\theta-1}\left[\alpha + \left(\frac{t}{\beta}\right)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)} \text{ for } t,\ \alpha,\ \beta \text{ and } \theta > 0. \tag{1.2}$$

Note that if $\theta = \beta = 1$, the three-parameter Kappa distribution (Kappa3) reduces to one-parameter Kappa distribution (Kappa1) with cdf and pdf, respectively, as follows:

$$R(t) = \frac{t}{[\alpha + (t)^{\alpha}]^{\left(\frac{1}{\alpha}\right)}} \text{ and } r(t) = \alpha[\alpha + (t)^{\alpha}]^{-\left(\frac{\alpha+1}{\alpha}\right)} \text{ for } t \text{ and } \alpha > 0. \tag{1.3}$$

Moreover, if $\theta = 1$, the three-parameter Kappa distribution (Kappa3) reduces to two-parameter Kappa distribution (Kappa2) with cdf and pdf, respectively, as the following:

$$R(t) = \frac{\left(\frac{t}{\beta}\right)}{\left[\alpha + \left(\frac{t}{\beta}\right)^{\alpha}\right]^{\left(\frac{1}{\alpha}\right)}} \text{ and } r(t) = \frac{\alpha}{\beta}\left[\alpha + \left(\frac{t}{\beta}\right)^{\alpha}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)} \text{ for } t,\ \alpha \text{ and } \beta > 0. \tag{1.4}$$

Aucoin et al. [12] presented asymptotic properties of the two-parameter

Kappa distribution along with point and interval estimation of its parameters and quantiles using maximum likelihood. The acceptance sampling plan based on truncated life tests for the three-parameter Kappa distribution was studied by Al-Omari [3]. Furthermore, Hosking and Wallis [38] denoted that this three-parameter Kappa cumulative distribution defined in equation (1.1) is a special case of Hosking [36]'s four-parameter Kappa cumulative distribution when $\xi = \beta$, $a = \frac{\beta}{\alpha\theta}$, $k = \frac{-1}{\alpha\theta}$ and $h = -\alpha$ in its cdf that has this form:

$$R(t) = \begin{cases} \left\{1 - h\left[1 - k^{\frac{(t-\xi)}{a}}\right]^{\frac{1}{k}}\right\}^{\frac{1}{h}}, & \text{if } k \neq 0, h \neq 0 \\ \exp\left\{-\left[1 - k^{\frac{(t-\xi)}{a}}\right]^{\frac{1}{k}}\right\}, & \text{if } k \neq 0, h = 0 \\ \left\{1 - h\exp\left[-\frac{(t-\xi)}{a}\right]\right\}^{\frac{1}{h}}, & \text{if } k = 0, h \neq 0 \\ \exp\left\{-\exp\left[-\frac{(t-\xi)}{a}\right]\right\}, & \text{if } k = 0, h = 0. \end{cases} \tag{1.5}$$

In addition, there is a distribution closely related to the three-parameter Kappa distribution called Mielke's beta-kappa distribution. The cdf of this distribution is defined as $R(t) = \left(\frac{t}{\beta}\right)^{\alpha}\left[1 + \left(\frac{t}{\beta}\right)^{\theta}\right]^{\left(-\frac{\alpha}{\theta}\right)}$ for t, α, β and $\theta > 0$. This distribution is a special case of a re-parameterized generalized F distribution, see Johnston et al. [42]. Moreover, the generalized beta distributions of the second kind (Mielke and Johnson [57]) is considered as a special case the re-parameterized version of the three-parameter Kappa distribution which has been used for frequency analysis of rainfall series (Wilks [81] and Öztekin [64]) and for testing changes in extreme rainfall events (Mason et al. [54]). In addition, Tadikamalla [73] showed that the re-parameterized version of the distribution of Burr type III yields the two-parameter Kappa distribution presented in the equation (1.4).

Over the last few decades, many classical distributions have been employed to model data in different fields such as demography, economics, finance, insurance, biological studies, medical, engineering, environmental and actuarial sciences. However, the need for flexible distributions to model, describe and predict the real phenomena studied in these areas is always never ends. For every practical situation, there is no general class of distribution to model skewed data. The induction of one or more shape parameters to the base-line distribution increases the opportunities to investigate vary tail weights and skewness. Therefore, researchers have been studied many ways for creating new distribution families to be either much more flexible or even more suited to certain real-world circumstances.

Marshall and Olkin [52] introduced an interesting method for adding a new parameter to a well-established distribution to obtain more flexible new family of distribution. The extended distribution by using their method allows for more flexibility in modeling different sorts of data and includes the baseline distribution as a special case. The Marshall and Olkin family of distributions is known as a family with tilt parameter or sometimes referred to as the family of proportional odds (proportional odds model), see Marshall and Olkin [53]. Furthermore, a new method for adding two parameters to a family of distributions for control the skewness and kurtosis of the resulting family (nine types) such as stable-symmetric normal distribution due to Barakat [13]. In addition, a method for constructing full families by mixing the stable symmetric-normal family with any tractable symmetric leptokurtic cdf suggested by Barakat and Khaled [15]. Moreover, Barakat et al. [14] proposed a more tractable full family with three parameters, called MSSN (multiplicative stable-symmetric normal), based on a combination of the normal distribution and its inverse, which is better capable of fitting diverse types of data. Including one or more additional shape parameter(s) to the baseline distribution has been shown to be effective in investigating tail characteristics and improving the family's goodness of fit. For example, Mudholkar and Srivastava [59] presented the exponentiated-Weibull distribution for analyzing data with bathtub failure. However, the idea of adding a parameter by exponentiation can be dated back to Gompertz [29] who exponentiated extreme value distribution in order to graduate mortality tables see Al-Hussaini and Ahsanullah [2]. Al-Hussaini and Abdel-Hamid [1] present a survey of generation of distribution functions. Moreover, Gupta et al. [33] proposed the general class of exponentiated distributions.

Eugene et al. [27] proposed a new family of distributions generated from the beta distribution. They introduced the beta-normal distribution by taking $F(x)$ to be the CDF of a normal distribution. A number of novel distributions have been defined and investigated by utilizing this technique. Jones [45] introduced a broad family of univariate distributions using a random variable beta. Jones [43] and Cordeiro and de Castro [22] introduced an extension of the beta-generated method by using the Kumaraswamy distribution (Kumaraswamy [47]) as a generator instead of beta distribution. Moreover, Al-Shomrani et al. [4] introduced Topp-Leone family of distributions by using the Topp-Leone distribution see Topp and Leone [78] and Nadarajah and Kotz [60]. Lee et al. [49] gave a review of beta-generated distributions and other generalizations.

Other well-known generators, for example but not limited, are gamma-G type 1 by Zografos and Balakrishnan [82], gamma-G type 2 by Ristic and Balakrishnan [67], gamma-G type 3 by Torabi and Hedesh [79], McDonald-G (Mc-G) by Alexander et al. [5], Transformed-Transformer (T-X) by Alzaatreh et al. [7], exponentiated (T-X) by Alzaghal et al. [9], Weibull-G by Bourguignon et al. [16] and log-gamma-G by Amini et al. [10], logistic-G by Torabi and Montazeri [80], exponentiated half logistic generated family by Cordeiro et al. [21], T-X{Y}-quantile based approach by Aljarrah et al. [6], T-R{Y} by Alzaatreh et al. [8], alpha power transformation introduced by Mahdavi and Kundu [51]. For more review of other generators see Lee et al. [49], Famoye et al. [28], Jones [44], Tahir and Nadarajah [76] and Tahir and Cordeiro [75].

Some extensions of Kappa distribution were introduced. Hussain [39] proposed three extended forms of Kappa distribution namely Kumaraswamy generalized Kappa (KGK), Exponentiated generalized Kappa (EGK) and McDonald generalized Kappa (McGK) distributions. For these new extended forms, the author explored their various statistical properties, used different methods to estimate their unknown parameters and applied these models to real-life data sets. Nawaz et al. [63] (deduced from Hussain [39]) introduced Kumaraswamy generalized Kappa distribution (KGK) and Javed et al. [40]

proposed Marshall-Olkin Kappa distribution (MOK). They presented various properties, estimated the unknown parameters and provided an illustration of applications of these proposed distributions to different real-life data sets.

In general, most of the above classes or generators can be expressed within one formulation as follows (see Tahir et al. [74]): Alzaatreh et al. [7] proposed a new class of distributions called "Transformed-Transformer" or "T-X" distributions as an extension of the beta-generated family of distributions by using any distribution of non-negative continuous random variable, say T, as the generator in place of the beta random variable. As a result, different upper limit can be defined for generating different types of T-X distributions.

Let $r(t)$ and $R(t)$ be the pdf and cdf of the random variable $T \in [a, b]$ where $-\infty \leq a < b \leq \infty$. Suppose $W(G(x))$ be a function of the cdf $G(x)$ of any random variable X so that $W(G(x))$ is the upper limit that satisfies the following conditions:

$$\left.\begin{array}{l} \text{[i]}\ W(G(x)) \in [a, b] \\ \text{[ii]}\ W(G(x)) \text{ is differentiable and monotically non-decreasing} \\ \text{[iii]}\ W(G(x)) \to a \text{ as } x \to -\infty \text{ and } W(G(x)) \to b \text{ as } x \to \infty. \end{array}\right\} \tag{1.6}$$

The cdf of a new family of distributions is defined by Alzaatreh et al. [7] as

$$F(x) = \int_a^{W(G(x))} r(t)\, dt = R(W(G(x))), \tag{1.7}$$

where $W(G(x))$ satisfies the conditions [i]-[iii].

The corresponding pdf associated with the cdf in equation (1.7) is given by

$$f(x) = r(W(G(x)))\left\{\frac{d}{dx} W(G(x))\right\}. \tag{1.8}$$

This family of distributions named as "transformed-transformer" family which was used as the pdf $r(t)$ in equation (1.7) is "transformed" into a new cdf $F(x)$ through the function $W(G(x))$ that acts as a "transformer". The new pdf $f(x)$ is the pdf transformed from the random variable T through the "transformer" random variable X.

The rest of the paper is organized as follows. In section 2, we introduce the new Odd Kappa-G family. In section 3, we present ten special models and plots of their pdfs and hrfs. We present asymptotic and shapes of pdf and hrf of this family in section 4. We give a very useful linear representation for the pdf and cdf of the family in section 5. In section 6, we obtain the quantile function of the family. In section 7, moments including ordinary and incomplete moments, skewness and kurtosis, mean deviations, residual life and generating function are introduced. The order statistics of the family are presented in section 8. In section 9, we obtain some of its general mathematical properties including extreme values, probability-weighted moments (PWMs), stress-strength, stochastic ordering, and bivariate extension. Entropy is introduced in section 10. In section 11, the estimate of model parameters using maximum likelihood is discussed. In section 12, the results of a simulation study to evaluate the performance of the maximum likelihood estimation approach are given.

2. Odd Kappa-G Family

If the function $W(G(x))$ in equations (1.7) and (1.8) is defined as the odds ratio $G(x)/(1-G(x))$ and the pdf $r(t)$ is considered to be the pdf of Kappa distribution defined in equation (1.2), then the Odd Kappa-G family of distributions' CDF is as follows:

$$F(x; \alpha, \beta, \theta, \vartheta) = \int_0^{\frac{G(x;\, \vartheta)}{1-G(x;\, \vartheta)}} \frac{\alpha\theta}{\beta}\left(\frac{t}{\beta}\right)^{\theta-1}\left[\alpha+\left(\frac{t}{\beta}\right)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)} dt,$$

$$= \left[\frac{\left(\frac{G(x;\, \vartheta)}{\beta[1-G(x;\, \vartheta)]}\right)^{\alpha\theta}}{\alpha+\left(\frac{G(x;\, \vartheta)}{\beta[1-G(x;\, \vartheta)]}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)}$$

$$= \frac{\left(\frac{G(x; \vartheta)}{\beta[1-G(x; \vartheta)]}\right)^{\theta}}{\left[\alpha + \left(\frac{G(x; \vartheta)}{\beta[1-G(x; \vartheta)]}\right)^{\alpha\theta}\right]^{\left(\frac{1}{\alpha}\right)}}, x \in R \tag{2.1}$$

where $T \sim$ Kappa (α, β, θ).

Moreover, the Odd Kappa-G family of distributions' pdf as follows:

$$f(x; \alpha, \beta, \theta, \vartheta) = \frac{\alpha\theta}{\beta}\left(\frac{g(x; \vartheta)}{[1-G(x; \vartheta)]^2}\right)\left(\frac{G(x; \vartheta)}{\beta[1-G(x, \vartheta)]}\right)^{\theta-1}$$

$$\times\left[\alpha + \left(\frac{G(x; \vartheta)}{\beta[1-G(x; \vartheta)]}\right)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)}, \tag{2.2}$$

where $f(x; \alpha, \beta, \theta, \vartheta) = \frac{d}{dx}F(x; \alpha, \beta, \theta, \vartheta)$ and $g(x; \vartheta) = \frac{d}{dx}G(x; \vartheta) \cdot G(x; \vartheta)$ and $g(x; \vartheta)$ are the cdf and pdf of the baseline distribution with parameter(s) ϑ. Let $F(x) = F(x; \alpha, \beta, \theta, \vartheta), f(x) = f(x; \alpha, \beta, \theta, \vartheta), G(x) = G(x; \vartheta)$ and $g(x) = g(x; \vartheta)$. These shortcuts will be used throughout the paper when there is no confusion involved. Equation (2.2) is most tractable when the cdf $G(x)$ and the pdf $g(x)$ have simple analytic expressions.

Remark 2.1. Clearly, if $\alpha = \beta = \theta = 1$ then $F(x; 1, 1, 1, \vartheta) = G(x; \vartheta)$ and $f(x; 1, 1, 1, \vartheta) = g(x; \vartheta)$, i.e., the Odd Kappa-G family of distributions reduce to the baseline G distributions.

The survival reliability function (sf or srf) or reliability function (rf):

$$S(x) = 1 - \left[\frac{\left(\frac{G(x; \vartheta)}{\beta[1-G(x; \vartheta)]}\right)^{\alpha\theta}}{\alpha + \left(\frac{G(x; \vartheta)}{\beta[1-G(x; \vartheta)]}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)}. \tag{2.3}$$

The hrf which is an essential quantity characterizing life phenomena of a system is given by:

$$h(x)=\frac{\frac{\alpha\theta}{\beta}\left(\frac{g(x;\vartheta)}{[1-G(x;\vartheta)]^2}\right)\left(\frac{G(x;\vartheta)}{\beta[1-G(x;\vartheta)]}\right)^{\theta-1}\left[\alpha+\left(\frac{G(x;\vartheta)}{\beta[1-G(x;\vartheta)]}\right)^{\alpha\theta}\right]^{-1}}{\left[\alpha+\left(\frac{G(x;\vartheta)}{\beta[1-G(x;\vartheta)]}\right)^{\alpha\theta}\right]^{\left(\frac{1}{\alpha}\right)}-\left(\frac{G(x;\vartheta)}{\beta[1-G(x;\vartheta)]}\right)^{\theta}}. \quad (2.4)$$

By adopting the interpretation of the CDF of the odd Weibull family as given in Cooray [20], the following is the physical interpretation of the Odd Kappa-G family: Let Y be a continuous lifespan random variable with cdf, $G(x;\vartheta)\cdot G(x;\vartheta)/[1-G(x;\vartheta)]$ represents the odds that a person (a component) following the lifetime Y will die (fail) at time x. Assume that the variability (randomness) of the odds of death (failure) is expressed by the random variable X, which follows the Kappa model with scale β and shapes α and θ, as seen in equation (1.1). We can have

$$P(Y\leq x)=P\left(X\leq\frac{G(x;\vartheta)}{[1-G(x;\vartheta)]}\right)=F(x;\alpha,\beta,\theta,\vartheta)$$

which is exactly the cdf of the Odd Kappa-G family given by equation (2.1).

Proposition 2.1. *Let* $X\sim$ *Odd Kappa-G* $(\alpha,\beta,\theta,\vartheta)$.

1. *If* $T=G(x;\vartheta)$ *then* $F_T(t)=\left[\frac{\left(\frac{t}{\beta[1-t]}\right)^{\alpha\theta}}{\alpha+\left(\frac{t}{\beta[1-t]}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)}$ *where* $0<t<1$.

2. *If* $T=\frac{G(X;\vartheta)}{[1-G(X;\vartheta)]}$ *then* $F_T(t)=\left[\frac{\left(\frac{t}{\beta}\right)^{\alpha\theta}}{\alpha+\left(\frac{t}{\beta}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)}$ *or*

   $T\sim$ *Kappa* (α,β,θ).

3. *If* $T=\frac{[1-G(X,\vartheta)]}{G(X;\vartheta)}$ *then* $F_T(t)=1-[1+\alpha(\beta t)^{\alpha\theta}]^{\left(-\frac{1}{\alpha}\right)}$ *where* $t>0$.

Proof. It is straightforward by using the cdf technique. Also, see appendix A.

Some basic motivations for using the Odd Kappa-G family in practice are the following:

1. Providing consistently better fits than other generated models under the same baseline distribution.

2. Generating distributions with symmetric, right-skewed, left skewed and reversed-J shaped.

3. Constructing heavy-tailed distributions that are not longer-tailed for modeling real data.

4. Producing a skewness based on symmetrical baseline distributions.

5. Making the kurtosis more flexible relative to the baseline model.

6. Defining special models with all types of the hrf.

3. Special Models

We provide in this section ten special models of the Odd Kappa-G (OK-G) distributions selected from Table 1.

Table 1. Distributions and corresponding $G(x; \vartheta)/\bar{G}(x; \vartheta)$ functions.

Distribution	$G(x; \vartheta)$	$G(x; \vartheta)/\bar{G}(x; \vartheta)$	ϑ
Uniform $(0 < x < \lambda)$	$x/\lambda,\ \lambda > 0$	$x/(\lambda - x)$	λ
Exponential $(x > 0)$	$1 - e^{-\lambda x},\ \lambda > 0$	$e^{\lambda x} - 1$	λ
Lindley $(x > 0)$	$1 - \frac{(1+\lambda+\lambda x)}{[(1+\lambda)e^{\lambda x}]}, \lambda > 0$	$[(1+\lambda)e^{\lambda x}]/(1+\lambda+\lambda x) - 1$	λ
Pareto $(x > \lambda)$	$1 - \left(\frac{\lambda}{x}\right)^{\delta},\ \delta > 0$ and $\lambda > 0$	$(x/\lambda)^{\delta} - 1$	(δ, λ)
Weibull $(x > 0)$	$1 - e^{\lambda x^{\alpha}},\ \alpha > 0$ and	$e^{\lambda x^{\alpha}} - 1$	(λ, α)

	$\lambda > 0$		
Frechet $(x > 0)$	$e^{-\left(\frac{x}{\lambda}\right)^{-\delta}}$, $\delta > 0$ and $\lambda > 0$	$(e^{(x/\lambda)^{\delta}} - 1)^{-1}$	(δ, λ)
Gompertz $(x > 0)$	$1-e^{-\delta(e^{\lambda x}-1)}$, $\delta>0$ and $\lambda > 0$	$e^{\delta(e^{\lambda x}-1)} - 1$	(δ, λ)
Lomax $(x > 0)$	$1-\left(1+\frac{x}{\lambda}\right)^{-\delta}$, $\delta > 0$ and $\lambda > 0$	$(1 + x/\lambda)^{\delta} - 1$	(δ, λ)
Log-Logistic $(x > 0)$	$\frac{1}{\left[1+\left(\frac{x}{\lambda}\right)^{-\eta}\right]}$, $\eta > 0$ and $\lambda > 0$	$(x/\lambda)^{\eta}$	(η, λ)
Logistic $(-\infty < x < \infty)$	$1/\left[1+\exp\left(-\frac{(x-\mu)}{\sigma}\right)\right]$ $-\infty < \beta < \infty$ and $\sigma > 0$	$\exp((x-\mu)/\sigma)$	(μ, σ)
Gumbel $(-\infty < x < \infty)$	$\exp\left[-\exp\left(-\frac{(x-\mu)}{\sigma}\right)\right]$, $-\infty < \mu < \infty$ and $\sigma > 0$	$1/\{\exp[\exp(-(x-\mu)/\sigma)]-1\}$	(μ, σ)
Normal $(-\infty < x < \infty)$	$\Phi\left(\frac{x-\mu}{\sigma}\right)$, $-\infty < \mu < \infty$ and $\sigma > 0$	$\Phi((x-\mu)/\delta)/[1-\Phi((x-\mu)/\sigma)]$	(μ, σ)
Burr XII $(x > 0)$	$1-\left(1+\left(\frac{x}{\lambda}\right)^{\eta}\right)^{-\delta}$, $\eta > 0$, $\delta > 0$ and $\lambda > 0$	$(1 + (x/\lambda)^{\eta})^{\delta} - 1$	(η, δ, λ)

Dagum $(x > 0)$	$\left(1+\left(\frac{x}{\lambda}\right)^{-\eta}\right)^{-\delta}$ $\delta > 0,\ \eta > 0$ and $\lambda > 0$	$[(1+(x/\lambda)^{-\eta})^{\delta}-1]^{-1}$	$(\delta,\ \eta,\ \lambda)$
Kappa $(x > 0)$	$\left\{\frac{\left(\frac{x}{\lambda}\right)^{ab}}{\left[a+\left(\frac{x}{\lambda}\right)^{ab}\right]}\right\}^{\frac{1}{a}}$, $a > 0,\ b > 0$ and $\lambda > 0$	$\{[1+a(\lambda/x)^{ab})^{1/a}-1\}^{-1}$	$(a,\ \lambda,\ b)$

3.1. Odd Kappa-Uniform (OK-U)

Suppose that the baseline distribution is uniform on the interval $(0, \lambda)$ with the corresponding cdf and pdf, respectively, as follows: $G(t; \lambda) = \frac{t}{\lambda}$ and $g(t; \lambda) = \frac{1}{\lambda}$ for $0 < t < \lambda$ and $\lambda > 0$, then the cdf, pdf and hrf of the Odd Kappa-Uniform (OK-U) distribution, respectively, are given by

$$F(x;\ \alpha,\ \beta,\ \theta,\ \lambda) = \left[\frac{\left(\frac{x}{\beta(\lambda-x)}\right)^{\alpha\theta}}{\alpha+\left(\frac{x}{\beta(\lambda-x)}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)} \tag{3.1}$$

$$f(x;\ \alpha,\ \beta,\ \theta,\ \lambda) = \left(\frac{\alpha\theta\lambda}{\beta(\lambda-x)^2}\right)\left(\frac{x}{\beta(\lambda-x)}\right)^{\theta-1}\left[\alpha+\left(\frac{x}{\beta(\lambda-x)}\right)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)} \tag{3.2}$$

$$h(x;\ \alpha,\ \beta,\ \theta,\ \lambda) = \frac{\left(\frac{\alpha\theta\lambda}{\beta(\lambda-x)^2}\right)\left(\frac{x}{\beta(\lambda-x)}\right)^{\theta-1}\left[\alpha+\left(\frac{x}{\beta(\lambda-x)}\right)^{\alpha\theta}\right]^{-1}}{\left[\alpha+\left(\frac{x}{\beta(\lambda-x)}\right)^{\alpha\theta}\right]^{\left(\frac{1}{\alpha}\right)}-\left(\frac{x}{\beta(\lambda-x)}\right)^{\theta}}, \tag{3.3}$$

where $0 < x < \lambda$ and $\alpha,\ \beta,\ \theta,\ \lambda > 0$.

In Figure 1-(a), we display plots of the pdf and hrf of the Odd Kappa-Uniform (OK-U) distribution.

3.2. Odd Kappa-Exponential (OK-E)

Let the baseline distribution be exponential with the corresponding cdf and pdf, respectively, as following: $G(t;\lambda) = 1 - e^{-\lambda t}$ and $g(t;\lambda) = \lambda e^{-\lambda t}$ for $t > 0$ and $\lambda > 0$, then the cdf, pdf and hrf of the OK-E distribution, respectively, are obtained as

$$F(x;\alpha,\beta,\theta,\lambda) = \left[\frac{\left(\frac{e^{\lambda x}-1}{\beta}\right)^{\alpha\theta}}{\alpha + \left(\frac{e^{\lambda x}-1}{\beta}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)} \tag{3.4}$$

$$f(x;\alpha,\beta,\theta,\lambda) = \left(\frac{\alpha\theta\lambda e^{\lambda x}}{\beta}\right)\left(\frac{e^{\lambda x}-1}{\beta}\right)^{\theta-1}\left[\alpha + \left(\frac{e^{\lambda x}-1}{\beta}\right)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)} \tag{3.5}$$

$$h(x;\alpha,\beta,\theta,\lambda) = \frac{\left(\frac{\alpha\theta\lambda e^{\lambda x}}{\beta}\right)\left(\frac{e^{\lambda x}-1}{\beta}\right)^{\theta-1}\left[\alpha + \left(\frac{e^{\lambda x}-1}{\beta}\right)^{\alpha\theta}\right]^{-1}}{\left[\alpha + \left(\frac{e^{\lambda x}-1}{\beta}\right)^{\alpha\theta}\right]^{\left(\frac{1}{\alpha}\right)} - \left(\frac{e^{\lambda x}-1}{\beta}\right)^{\theta}}, \tag{3.6}$$

where $x > 0$ and $\alpha, \beta, \theta, \lambda > 0$.

In Figure 1-(b), we present plots of the pdf and hrf of the Odd Kappa-Exponential (OK-E) distribution or more clearly the Odd three-parameter Kappa-Exponential (OK3-E) distribution. Special cases, if $\theta = \beta = 1$, the Odd three-parameter Kappa-Exponential (OK3-E) distribution reduces to the Odd one-parameter Kappa-Exponential (OK1-E) distribution. In addition, if $\theta = 1$ the Odd three-parameter Kappa-Exponential (OK3-E) distribution reduces to the Odd two-parameter Kappa-Exponential (OK2-E) distribution. Moreover, this is also true for any baseline distribution from this family like exponential distribution. However, for short and simplicity from now on, the Odd three-parameter Kappa-Exponential (OK3-E) distribution will be called the Odd Kappa-Exponential (OK-E) distribution.

3.3. Odd Kappa-Weibull (OK-W)

Considering the baseline distribution as Weibull with the corresponding cdf and pdf, respectively, as follows: $G(t;\lambda, a) = 1 - e^{-\lambda t^a}$ and $g(t;\lambda, a) = \lambda a t^{a-1} e^{-\lambda t^a}$ for $t > 0$ and $\lambda, a > 0$, then the cdf, pdf and hrf of the OK-W distribution, respectively, are expressed as

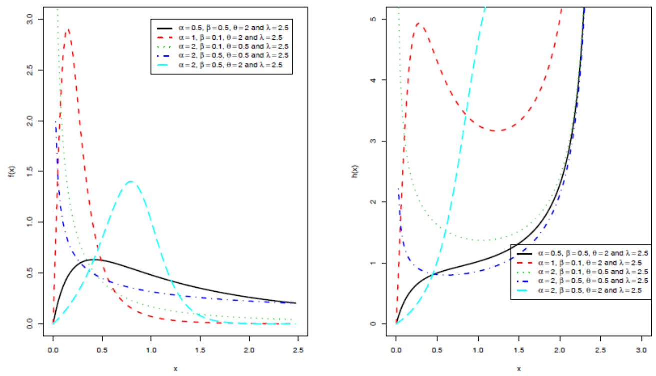
$$F(x;\alpha,\beta,\theta,\lambda,a) = \left[\frac{\left(\frac{e^{\lambda x^a}-1}{\beta}\right)^{\alpha\theta}}{\alpha + \left(\frac{e^{\lambda x^a}-1}{\beta}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)} \tag{3.7}$$

$$f(x;\alpha,\beta,\theta,\lambda,a) = \left(\frac{\alpha\theta\lambda a x^{a-1} e^{\lambda x^a}}{\beta}\right)\left(\frac{e^{\lambda x^a}-1}{\beta}\right)^{\theta-1}\left[\alpha + \left(\frac{e^{\lambda x^a}-1}{\beta}\right)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)} \tag{3.8}$$

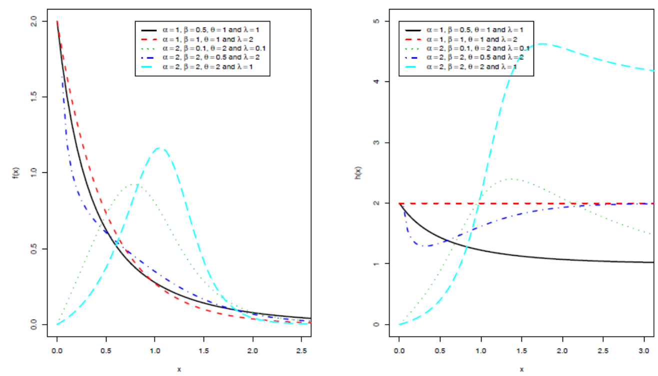
$$h(x;\alpha,\beta,\theta,\lambda,a) = \frac{\left(\frac{\alpha\theta\lambda a x^{a-1} e^{\lambda x^a}}{\beta}\right)\left(\frac{e^{\lambda x^a}-1}{\beta}\right)^{\theta-1}\left[\alpha + \left(\frac{e^{\lambda x^a}-1}{\beta}\right)^{\alpha\theta}\right]^{-1}}{\left[\alpha + \left(\frac{e^{\lambda x^a}-1}{\beta}\right)^{\alpha\theta}\right]^{\left(\frac{1}{\alpha}\right)} - \left(\frac{e^{\lambda x^a}-1}{\beta}\right)^{\theta}}, \tag{3.9}$$

where $x > 0$ and $\alpha, \beta, \theta, \lambda, a > 0$.

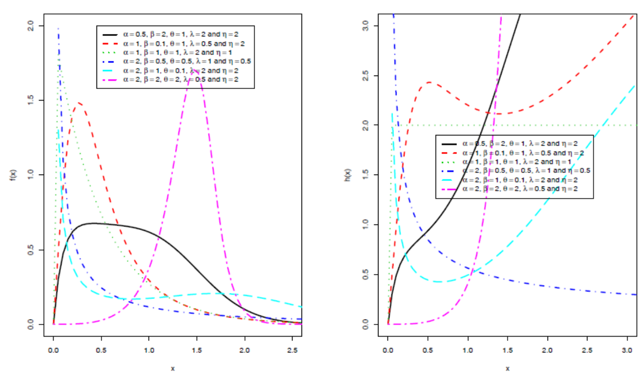
Plots of the pdf and hrf of the Odd Kappa-Weibull (OK-W) distribution are displayed in Figure 1-(c).



(a) pdf and hrf for Odd Kappa-Uniform (OK-U) distribution.



(b) pdf and hrf for Odd Kappa-Exponential (OK-E) distribution.



(c) pdf and hrf for Odd Kappa-Weibull (OK-W) distribution.

Figure 1. Plots of the pdfs and hrfs of the (a) OK-U, (b) OK-E and (c) OK-W distributions, respectively, for the selected parameter values.

3.4. Odd Kappa-Pareto (OK-P)

Let the baseline distribution be Pareto with the corresponding cdf and pdf, respectively, as follows: $G(t; \lambda, \delta) = 1 - \left(\frac{t}{\lambda}\right)^{-\delta}$ and $g(t;\ \lambda,\ \delta) = \left(\frac{\delta}{\lambda}\right)\left(\frac{t}{\lambda}\right)^{-(\delta+1)}$ for $t > \lambda$ and $\lambda,\ \delta > 0$. Hence, the cdf, pdf and hrf of the OK-P distribution, respectively, are given by

$$F(x; \alpha, \beta, \theta, \lambda, \delta) = \left[\frac{\left(\frac{(x/\lambda)^{\delta} - 1}{\beta}\right)^{\alpha\theta}}{\alpha + \left(\frac{(x/\ddot{e})^{\delta} - 1}{\beta}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)} \tag{3.10}$$

$$f(x; \alpha, \beta, \theta, \lambda, \delta) = \left(\frac{\alpha\theta\delta}{\beta\lambda}\right)\left(\frac{x}{\lambda}\right)^{\delta-1}\left(\frac{(x/\lambda)^{\delta} - 1}{\beta}\right)^{\theta-1}\left[\alpha + \left(\frac{(x/\lambda)^{\delta} - 1}{\beta}\right)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)} \tag{3.11}$$

$$h(x; \alpha, \beta, \theta, \lambda, \delta) = \frac{\left(\frac{\alpha\theta\delta}{\beta\lambda}\right)\left(\frac{x}{\lambda}\right)^{\delta-1}\left(\frac{(x/\lambda)^{\delta} - 1}{\beta}\right)^{\theta-1}\left[\alpha + \left(\frac{(x/\lambda)^{\delta} - 1}{\beta}\right)^{\alpha\theta}\right]^{-1}}{\left[\alpha + \left(\frac{(x/\lambda)^{\delta} - 1}{\beta}\right)^{\alpha\theta}\right]^{\left(\frac{1}{\alpha}\right)} - \left(\frac{(x/\lambda)^{\delta} - 1}{\beta}\right)^{\theta}}, \tag{3.12}$$

where $x > \lambda$ and $\alpha,\ \beta,\ \theta,\ \lambda,\ \delta > 0$.

In Figure 2-(a), we present plots of the pdf and hrf of the Odd Kappa-Pareto (OK-P) distribution.

3.5. Odd Kappa-Normal (OK-N)

Suppose the baseline distribution is Normal with the corresponding cdf and pdf, respectively, as following: $G(t;\ \mu,\ \sigma) = \Phi\left(\frac{t - \mu}{\sigma}\right)$ and $g(t; \mu, \sigma)$ $= \frac{e^{-\left(\frac{1}{2}\right)\left(\frac{t-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}$ for $t,\ \mu \in \Re$ and $\sigma > 0$. Therefore, the cdf, pdf and hrf of the

OK-N distribution, respectively, are given by

$$F(x;\ \alpha,\ \beta,\ \theta,\ \mu,\ \sigma) = \left[\frac{\left(\dfrac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{\beta\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]}\right)^{\alpha\theta}}{\alpha + \left(\dfrac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{\beta\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)} \tag{3.13}$$

$$f(x;\alpha,\beta,\theta,\mu,\sigma) = \left(\frac{\alpha\theta e^{-\left(\frac{1}{2}\right)\left(\frac{x-\mu}{\sigma}\right)^2}}{\beta\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^2}\right)\left(\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{\beta\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]}\right)^{\theta-1}$$

$$\times\left[\alpha + \left(\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{\beta\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]}\right)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)} \tag{3.14}$$

$h(x;\ \alpha,\ \beta,\ \theta,\ \mu,\ \sigma)$

$$= \frac{\left(\dfrac{\alpha\theta e^{-\left(\frac{1}{2}\right)\left(\frac{x-\mu}{\sigma}\right)^2}}{\beta\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^2}\right)\left(\dfrac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{\beta\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]}\right)^{\theta-1}\left[\alpha+\left(\dfrac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{\beta\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]}\right)^{\alpha\theta}\right]^{-1}}{\left[\alpha+\left(\dfrac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{\beta\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]}\right)^{\alpha\theta}\right]^{\left(\frac{1}{\alpha}\right)} - \left(\dfrac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{\beta\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]}\right)^{\theta}}, \tag{3.15}$$

where $x,\ \mu \in \mathfrak{R}$ and $\alpha,\ \beta,\ \theta,\ \sigma > 0$.

In Figure 2-(b), we display plots of the pdf and hrf of the Odd Kappa-Normal (OK-N) distribution.

3.6. Odd Kappa-Fréchet (OK-F)

Suppose the baseline distribution is Frechet with the corresponding cdf and pdf, respectively, as following: $G(t;\lambda,\delta) = e^{-\left(\frac{t}{\lambda}\right)^{-\delta}}$ and $g(t;\lambda,\delta) = \delta\lambda^{\delta}t^{-(\delta+1)}e^{-\left(\frac{t}{\lambda}\right)^{-\delta}}$ for $t > 0$ and $\lambda, \delta > 0$. Therefore, the cdf, pdf and hrf of the OK-F distribution, respectively, are given by

$$F(x;\alpha,\beta,\theta,\lambda,\delta) = \left[\frac{(\beta[e^{(x/\lambda)^{-\delta}}-1])^{-\alpha\theta}}{\alpha+(\beta[e^{(x/\lambda)^{-\delta}}-1])^{-\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)} \tag{3.16}$$

$$f(x;\alpha,\beta,\theta,\lambda,\delta) = \left(\frac{\alpha\theta\delta\lambda^{\delta}x^{-(\delta+1)}e^{-\left(\frac{x}{\lambda}\right)^{-\delta}}}{\beta\left[1-e^{-\left(\frac{x}{\lambda}\right)^{-\delta}}\right]^{2}}\right)(\beta[e^{(x/\lambda)^{-\delta}}-1])^{1-\theta}$$

$$\times[\alpha+(\beta[e^{(x/\lambda)^{-\delta}}-1])^{-\alpha\theta}]^{-\left(\frac{\alpha+1}{\alpha}\right)} \tag{3.17}$$

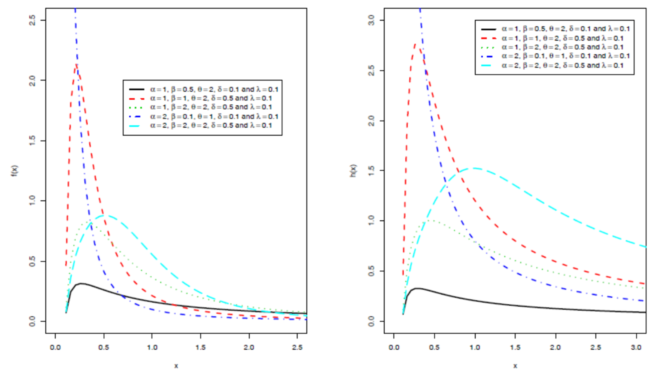
$h(x;\alpha,\beta,\theta,\lambda,\delta)$

$$= \frac{\left(\frac{\alpha\theta\delta\lambda^{\delta}x^{-(\delta+1)}e^{-\left(\frac{x}{\lambda}\right)^{-\delta}}}{\beta\left[1-e^{-\left(\frac{x}{\lambda}\right)^{-\delta}}\right]^{2}}\right)(\beta[e^{(x/\lambda)^{-\delta}}-1])^{1-\theta}[\alpha+(\beta[e^{(x/\lambda)^{-\delta}}-1])^{-\alpha\theta}]^{-1}}{[\alpha+(\beta[e^{(x/\lambda)^{-\delta}}-1])^{-\alpha\theta}]^{\left(\frac{1}{\alpha}\right)}-(\beta[e^{(x/\lambda)^{-\delta}}-1])^{-\theta}}, \tag{3.18}$$

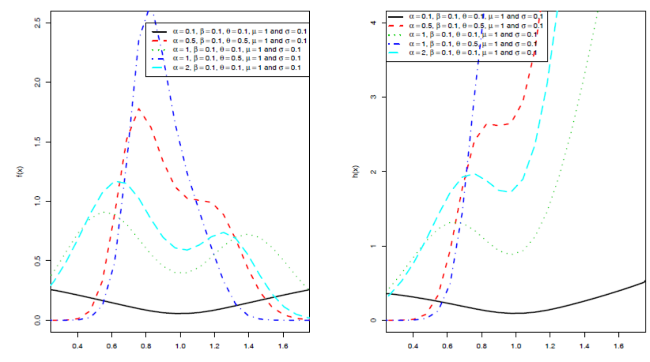
where $x > 0$ and $\alpha, \beta, \theta, \lambda, \delta > 0$.

In Figure 2-(c), plots of the pdf and hrf of the Odd Kappa-Fréchet (OK-F) distribution are presented.

(a) pdf and hrf for Odd Kappa-Pareto (OK-P) distribution



(b) pdf and hrf for Odd Kappa-Normal (OK-N) distribution



(c) pdf and hrf for Odd Kappa-Fréchet (OK-F) distribution

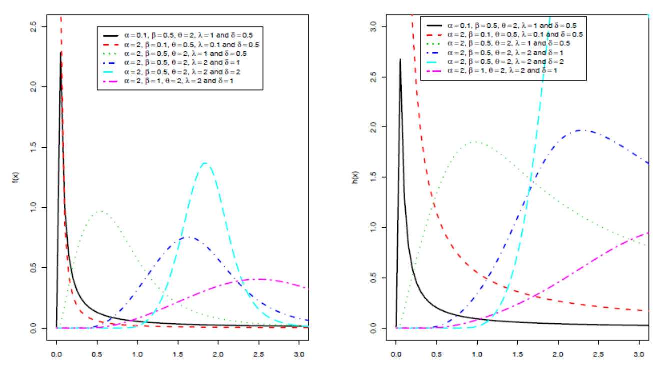


Figure 2. Plots of the pdfs and hrfs of the (a) OK-P, (b) OK-N and (c) OK-F distributions, respectively, for the selected parameter values.

3.7. Odd Kappa-Gompertz (OK-G)

Let Gompertz distribution be the baseline with the corresponding cdf and pdf, respectively, as follows: $G(t;\lambda,\delta)=1-e^{-\delta(e^{\lambda t}-1)}$ and $g(t;\lambda,\delta)=\lambda\delta e^{\lambda t}e^{-\delta(e^{\lambda t}-1)}$ for $t>0$ and $\lambda,\delta>0$. Consequently, the cdf, pdf and hrf of the OK-G distribution, respectively, are given by

$$F(x;\alpha,\beta,\theta,\lambda,\delta)=\left[\frac{\left(\frac{e^{\delta(e^{\lambda x}-1)}-1}{\beta}\right)^{\alpha\theta}}{\alpha+\left(\frac{e^{\delta(e^{\lambda x}-1)}-1}{\beta}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)} \tag{3.19}$$

$$f(x;\alpha,\beta,\theta,\lambda,\delta)=\left(\frac{\alpha\theta\lambda\delta e^{\lambda x}e^{\delta(e^{\lambda x}-1)}}{\beta}\right)\left(\frac{e^{\delta(e^{\lambda x}-1)}-1}{\beta}\right)^{\theta-1}$$

$$\times\left[\alpha+\left(\frac{e^{\delta(e^{\lambda x}-1)}-1}{\beta}\right)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)} \tag{3.20}$$

$h(x;\alpha,\beta,\theta,\lambda,\delta)$

$$=\frac{\left(\frac{\alpha\theta\lambda\delta e^{\lambda x}e^{\delta(e^{\lambda x}-1)}}{\beta}\right)\left(\frac{e^{\delta(e^{\lambda x}-1)}-1}{\beta}\right)^{\theta-1}\left[\alpha+\left(\frac{e^{\delta(e^{\lambda x}-1)}-1}{\beta}\right)^{\alpha\theta}\right]^{-1}}{\left[\alpha+\left(\frac{e^{\delta(e^{\lambda x}-1)}-1}{\beta}\right)^{\alpha\theta}\right]^{\left(\frac{1}{\alpha}\right)}-\left(\frac{e^{\delta(e^{\lambda x}-1)}-1}{\beta}\right)^{\theta}}, \tag{3.21}$$

where $x>0$ and $\alpha,\beta,\theta,\lambda,\delta>0$.

Plots of the pdf and hrf of the Odd Kappa-Gompertz (OK-G) distribution are depicted in Figure 3-(a).

3.8. Odd Kappa-Burr XII (OK-B)

Considering Burr XII distribution (Burr [18]) as the baseline with the corresponding cdf and pdf, respectively, as follows:

$G(t; \lambda, \eta, \delta) = 1 - \left[1 + \left(\frac{t}{\lambda}\right)^{\eta}\right]^{-\delta}$ and $g(t;\ \lambda,\ \eta,\ \delta) = \left(\frac{\eta\delta}{\lambda}\right)\left(\frac{t}{\lambda}\right)^{\eta-1} \times \left[1 + \left(\frac{t}{\lambda}\right)^{\eta}\right]^{-(\delta+1)}$

for $t > 0$ and $\lambda,\ \eta,\ \delta > 0$. As a result, the cdf, pdf and hrf of the OK-B distribution, respectively, are written as

$$F(x;\ \alpha,\ \beta,\ \theta,\ \lambda,\ \eta,\ \delta) = \left[\frac{\left(\frac{(1 + (x/\lambda)^{\eta})^{\delta} - 1}{\beta}\right)^{\alpha\theta}}{\alpha + \left(\frac{(1 + (x/\lambda)^{\eta})^{\delta} - 1}{\beta}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)} \tag{3.22}$$

$$f(x; \alpha, \beta, \theta, \lambda, \eta, \delta) = \left(\frac{\alpha\theta\eta\delta}{\beta\lambda}\right)\left(\frac{x}{\lambda}\right)^{\eta-1}\left[1 + \left(\frac{x}{\lambda}\right)^{\eta}\right]^{\delta-1}\left(\frac{(1 + (x/\lambda)^{\eta})^{\delta} - 1}{\beta}\right)^{\theta-1}$$

$$\times\left[\alpha + \left(\frac{(1 + (x/\lambda)^{\eta})^{\delta} - 1}{\beta}\right)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)} \tag{3.23}$$

$h(x; \alpha, \beta, \theta, \lambda, \eta, \delta)$

$$= \frac{\left(\frac{\alpha\theta\eta\delta}{\beta\lambda}\right)\left(\frac{t}{\lambda}\right)^{\eta-1}\left[1 + \left(\frac{x}{\lambda}\right)^{\eta}\right]^{\delta-1}\left(\frac{(1 + (x/\lambda)^{\eta})^{\delta} - 1}{\beta}\right)^{\theta-1}\left[\alpha + \left(\frac{(1 + (x/\lambda)^{\eta})^{\delta} - 1}{\beta}\right)^{\alpha\theta}\right]^{-1}}{\left[\alpha + \left(\frac{(1 + (x/\lambda)^{\eta})^{\delta} - 1}{\beta}\right)^{\alpha\theta}\right]^{\left(\frac{1}{\alpha}\right)} - \left(\frac{(1 + (x/\lambda)^{\eta})^{\delta} - 1}{\beta}\right)^{\theta}} \tag{3.24}$$

where $x > 0$ and $\alpha,\ \beta,\ \theta,\ \lambda,\ \eta,\ \delta > 0$.

Remark 3.1. When $\eta = 1$ or $\delta = 1$, this distribution reduces to Odd Kappa-Lomax (OK-L) or Odd Kappa-Log-Logistic (OK-LL), respectively.

In Figure 3-(b), plots of the pdf and hrf of the Odd Kappa-Burr XII (OK-B) distribution are presented.

3.9. Odd Kappa-Dagum (OK-D)

Let the baseline distribution be Dagum distribution (Type I) (Dagum [23]), also called the Burr III or the Inverse Burr XII distribution (Kleiber and Kotz [46]), with the corresponding cdf and pdf, respectively, as follows:

$$G(t; \lambda, \eta, \delta) = \left[1 + \left(\frac{t}{\lambda}\right)^{-\eta}\right]^{-\delta} \quad \text{and} \quad g(t;\ \lambda,\ \eta,\ \delta) = \left(\frac{\eta\delta}{\lambda}\right)\left(\frac{t}{\lambda}\right)^{-(\eta+1)}\left[1 + \left(\frac{t}{\lambda}\right)^{-\eta}\right]^{-(\delta+1)}$$

for $t > 0$ and $\lambda,\ \eta,\ \delta > 0$. Hence, the cdf, pdf and hrf of the OK-D distribution, respectively, are given by

$$F(x;\ \alpha,\ \beta,\ \theta,\ \lambda,\ \eta,\ \delta) = \left[\frac{(\beta[1 + (x/\lambda)^{-\eta})^{\delta} - 1])^{-\alpha\theta}}{\alpha + (\beta[1 + (x/\lambda)^{-\eta})^{\delta} - 1])^{-\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)} \tag{3.25}$$

$$f(x; \alpha, \beta, \theta, \lambda, \eta, \delta) = \left(\frac{\alpha\theta\eta\delta}{\beta\lambda}\right)\left(\frac{x}{\lambda}\right)^{-(\eta+1)}\left(1 + \left(\frac{x}{\lambda}\right)^{-\eta}\right)^{-(\delta+1)}\left[1 - \left(1 + \left(\frac{x}{\lambda}\right)^{-\eta}\right)^{-\delta}\right]^{-2}$$

$$\times(\beta[1 + (x/\lambda)^{-\eta})^{\delta} - 1])^{1-\theta}$$

$$[\alpha + (\beta[1 + (x/\lambda)^{-\eta})^{\delta} - 1])^{-\alpha\theta}]^{-\left(\frac{\alpha+1}{\alpha}\right)} \tag{3.26}$$

$$h(x; \alpha, \beta, \theta, \lambda, \eta, \delta) = \left(\frac{\alpha\theta\eta\delta}{\beta\lambda}\right)\left(\frac{x}{\lambda}\right)^{-(\eta+1)}\left(1 + \left(\frac{x}{\lambda}\right)^{-\eta}\right)^{-(\delta+1)}\left[1 - \left(1 + \left(\frac{x}{\lambda}\right)^{-\eta}\right)^{-\delta}\right]^{-2}$$

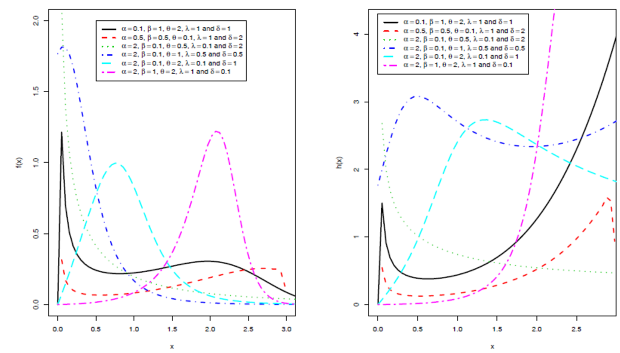
$$\times \frac{(\beta[1 + (x/\lambda)^{-\eta})^{\delta} - 1])^{1-\theta}[\alpha + (\beta[1 + (x/\lambda)^{-\eta})^{\delta} - 1])^{-\alpha\theta}]^{-1}}{[\alpha + (\beta[(1 + (x/\lambda)^{-\eta})^{\delta} - 1])^{-\alpha\theta}]^{\left(\frac{1}{\alpha}\right)} - (\beta[1 + (x/\lambda)^{-\eta})^{\delta} - 1])^{-\theta}}, \tag{3.27}$$

where $x > 0$ and $\alpha,\ \beta,\ \theta,\ \lambda,\ \eta,\ \delta > 0$.

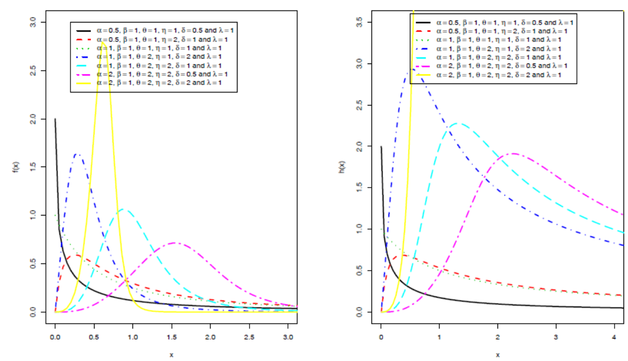
Remark 3.2. When $\eta = 1$ or $\delta = 1$, this distribution reduces to Odd Kappa-Inverse Lomax (OK-IL) or Odd Kappa-Inverse Log-Logistic (OK-ILL), respectively. However, the inverse log-logistic distribution may not be needed to be identified as separate distribution since it has also log-logistic distribution.

In Figure 3-(c), we display plots of the pdf and hrf of the Odd Kappa-Dagum (OK-D) distribution.

(a) pdf and hrf for Odd Kappa-Gompertz (OK-G) distribution.



(b) pdf and hrf for Odd Kappa-Burr XII (OK-B) distribution.



(c) pdf and hrf for Odd Kappa-Dagum (OK-D) distribution.

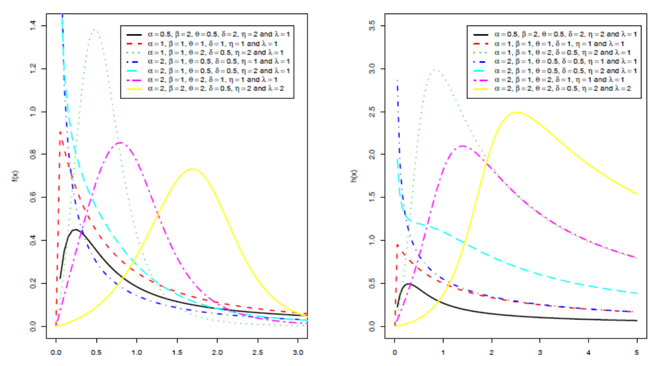


Figure 3. Plots of the pdfs and hrfs of the (a) OK-G, (b) OK-B and (c) OK-D distributions, respectively, for the selected parameter values.

3.10. Odd Kappa-Kappa (OK-K)

Let the baseline distribution be Kappa distribution (Mielke and Johnson [56]) with the corresponding cdf and pdf, respectively, as follows:

$$G(t; a, \lambda, b) = \left\{ \frac{\left(\frac{t}{\lambda}\right)^{ab}}{\left[a + \left(\frac{t}{\lambda}\right)^{ab}\right]} \right\}^{\frac{1}{a}} \quad \text{and} \quad g(t; a, \lambda, b) = \frac{ab}{\lambda}\left(\frac{t}{\lambda}\right)^{b-1}\left[a + \left(\frac{t}{\lambda}\right)^{ab}\right]^{-\left(\frac{a+1}{a}\right)}$$

for $t > 0$ and $a, \lambda, b > 0$. Hence, the cdf, pdf and hrf of the OK-D distribution, respectively, are given by

$$F(x; \alpha, \beta, \theta, a, \lambda, b) = \left[\frac{(\beta[1 + a(\lambda/x)^{ab})^{1/a} - 1])^{-\alpha\theta}}{\alpha + (\beta[1 + a(\lambda/x)^{ab})^{1/a} - 1])^{-\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)} \tag{3.28}$$

$$f(x; \alpha, \beta, \theta, a, \lambda, b) = \left(\frac{\alpha\theta ab}{\beta\lambda}\right)\left(\frac{x}{\lambda}\right)^{b-1}\left(a + \left(\frac{x}{\lambda}\right)^{ab}\right)^{-\left(\frac{a+1}{a}\right)}\left[1 - \left(\frac{\left(\frac{x}{\lambda}\right)^{ab}}{\left(a + \left(\frac{x}{\lambda}\right)^{ab}\right)}\right)^{\frac{1}{a}}\right]^{-2}$$

$$\times(\beta[(1 + a(\lambda/x)^{ab})^{1/a} - 1])^{1-\theta}$$

$$[\alpha + (\beta[(1 + a(\lambda/x)^{ab})^{1/a} - 1])^{-\alpha\theta}]^{-\left(\frac{\alpha+1}{\alpha}\right)} \tag{3.29}$$

$$h(x; \alpha, \beta, \theta, a, \lambda, b) = \left(\frac{\alpha\theta ab}{\beta\lambda}\right)\left(\frac{x}{\lambda}\right)^{b-1}\left(a + \left(\frac{x}{\lambda}\right)^{ab}\right)^{-\left(\frac{a+1}{a}\right)}\left[1 - \left(\frac{\left(\frac{x}{\lambda}\right)^{ab}}{\left(a + \left(\frac{x}{\lambda}\right)^{ab}\right)}\right)^{\frac{1}{a}}\right]^{-2}$$

$$\times \frac{(\beta[1 + a(\lambda/x)^{ab})^{1/a} - 1])^{1-\theta}[\alpha + (\beta[1 + a(\lambda/x)^{ab})^{1/a} - 1])^{-\alpha\theta}]^{-1}}{[\alpha + (\beta[(1 + a(\lambda/x)^{ab})^{1/a} - 1])^{-\alpha\theta}]^{\left(\frac{1}{\alpha}\right)} - (\beta[(1 + a(\lambda/x)^{ab})^{1/a} - 1])^{-\theta}}, \tag{3.30}$$

where $x > 0$ and $\alpha, \beta, \theta, a, \lambda, b > 0$.

In Figure 4, we display plots of the pdf and hrf of the Odd Kappa-Kappa (OK-K) distribution.

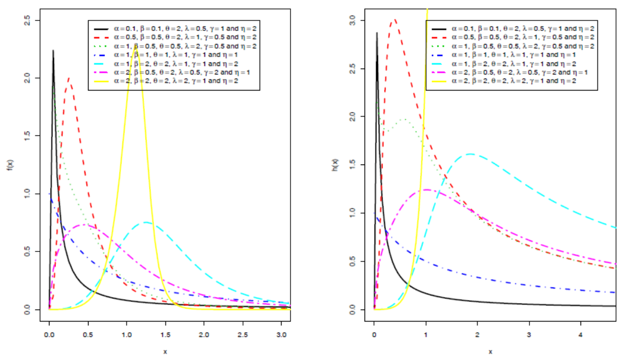


Figure 4. Plots of the pdf and hrf of the Odd Kappa-Kappa (OK-K) distribution for the selected parameter values.

Figures 1-4 indicate that the Odd Kappa-G family generates pdfs with various shapes such as unimodal, bimodal, monotonically decreasing, symmetrical, right-skewed and left-skewed. Furthermore, this family can produce flexible hazard rate shapes such as monotonically decreasing, increasing, bathtub, upside-down bathtub, *J*, reversed-*J* and *S*. This fact implies that the Odd Kappa-G family can be very useful to fit different data sets with various shapes.

4. Asymptotics and Shapes

Proposition 4.1. *Let* $a = \inf\{x : G(x) > 0\}$. *The asymptotics of cdf, pdf and hrf when* $x \to a$ *are given by*

$$F(x) \sim \frac{[G(x)]^{\theta}}{\beta^{\theta} \alpha^{\left(\frac{1}{\alpha}\right)}}, \tag{4.1}$$

$$f(x) \sim \frac{\theta g(x)[G(x)]^{(\theta-1)}}{\beta^{\theta} \alpha^{\left(\frac{1}{\alpha}\right)}}, \tag{4.2}$$

$$h(x) \sim \frac{\theta g(x)[G(x)]^{(\theta-1)}}{\beta^{\theta} \alpha^{\left(\frac{1}{\alpha}\right)}}. \tag{4.3}$$

Proposition 4.2. *The asymptotics of cdf, pdf and hrf when* $x \to \infty$ *are given by*

$$1 - F(x) \sim \left[\beta \bar{G}(x)\right]^{(\alpha\theta)}, \tag{4.4}$$

$$f(x) \sim \alpha\beta\theta g(x)\left[\beta \bar{G}(x)\right]^{(\alpha\theta-1)}, \tag{4.5}$$

$$h(x) \sim \frac{\alpha\theta g(x)}{\bar{G}(x)}. \tag{4.6}$$

Analytically, the pdf and hrf may be described. The critical points of the Odd Kappa-G pdf are the roots of the equation:

$$\frac{d\log[f(x)]}{dx} = \frac{h_g'(x; \vartheta)}{h_g(x; \vartheta)} - h_g(x; \vartheta)\left(\frac{\alpha\theta + \bar{G}(x; \vartheta)}{G(x; \vartheta)}\right)$$

$$+\alpha\beta\theta(\alpha+1)\left(\frac{g(x; \vartheta)}{G(x; \vartheta)}\right)\frac{(\beta\bar{G}(x; \vartheta))^{(\alpha\theta-1)}}{[\alpha(\beta\bar{G}(x; \vartheta))^{(\alpha\theta)} + (G(x; \vartheta))^{(\alpha\theta)}]} = 0, \tag{4.7}$$

that corresponds to points where $f'(x) = 0.$ There may be more than one root to this equation.

Let $\tau(x) = \dfrac{d^2 \log[f(x)]}{dx^2}$. We get

$$\tau(x) = \frac{h_g''(x; \vartheta)}{h_g(x; \vartheta)} - \left[\frac{h_g'(x; \vartheta)}{h_g(x; \vartheta)}\right]^2 + \frac{(\alpha\theta+1)h_g(x; \vartheta)g(x; \vartheta)}{[G(x; \vartheta)]^2}$$

$$- h_g'(x; \vartheta)\left(\frac{\alpha\theta + \bar{G}(x; \vartheta)}{G(x; \vartheta)}\right)$$

$$+ \alpha\beta\theta(\alpha+1)\left\{\frac{(\beta\bar{G}(x; \vartheta))^{(\alpha\theta-1)}}{[\alpha(\beta\bar{G}(x; \vartheta))^{(\alpha\theta)} + (G(x; \vartheta))^{(\alpha\theta)}]}\right.$$

$$\left(\frac{g'(x; \vartheta)}{G(x; \vartheta)} - \left[\frac{g(x; \vartheta)}{G(x; \vartheta)}\right]^2\right) - \beta\left(\frac{[g(x; \vartheta)]^2}{G(x; \vartheta)}\right)(\beta\bar{G}(x; \vartheta))^{(\alpha\theta-2)}$$

$$\times\left\{\frac{(\alpha\theta-1)[\alpha(\beta\bar{G}(x;\vartheta))^{(\alpha\theta)}+(G(x;\vartheta))^{(\alpha\theta)}]}{[\alpha(\beta\bar{G}(x;\vartheta))^{(\alpha\theta)}+(G(x;\vartheta))^{(\alpha\theta)}]^2}\right.$$

$$\left.+\frac{\alpha\theta\bar{G}(x;\vartheta)[(G(x;\vartheta))^{(\alpha\theta-1)}-\alpha\beta(\beta\bar{G}(x;\vartheta))^{(\alpha\theta-1)}]}{[\alpha(\beta\bar{G}(x;\vartheta))^{(\alpha\theta)}+(G(x;\vartheta))^{(\alpha\theta)}]^2}\right\}\Bigg\}. \tag{4.8}$$

If $x = a$ is a root of the equation (4.7) then it refers to a local maximum if $\tau(x) > 0$ for all $x < a$ and $\tau(x) < 0$ for all $x > a$. In addition, it corresponds to a local minimum if $\tau(x) < 0$ for all $x < a$ and $\tau(x) > 0$ for all $x > a$. Moreover, it gives an inflexion point if either $\tau(x) > 0$ for all $x \neq a$ or $\tau(x) < 0$ for all $x \neq a$.

Similarly, the critical points of the Odd Kappa-G hrfs are the roots of the equation:

$$\frac{d\log[h(x)]}{dx}=\frac{h_g'(x;\vartheta)}{h_g(x;\vartheta)}-h_g(x;\vartheta)\left(\frac{\alpha\theta+\bar{G}(x;\vartheta)}{G(x;\vartheta)}\right)$$

$$+\alpha\theta(\alpha+1)\left(\frac{h_g(x;\vartheta)}{G(x;\vartheta)}\right)\frac{(\beta\bar{G}(x;\vartheta))^{(\alpha\theta)}}{[\alpha(\beta\bar{G}(x;\vartheta))^{(\alpha\theta)}+(G(x;\vartheta))^{(\alpha\theta)}]}$$

$$+\alpha\theta\left(\frac{h_g(x;\vartheta)}{G(x;\vartheta)}\right)[G(x;\vartheta)(\beta\bar{G}(x;\vartheta))^{\alpha}]^{\theta}\div$$

$$\{[\alpha(\beta\bar{G}(x;\vartheta))^{(\alpha\theta)}+(G(x;\vartheta))^{(\alpha\theta)}]^{\left(\frac{\alpha+1}{\alpha}\right)}$$

$$-\alpha[G(x;\vartheta)(\beta\bar{G}(x;\vartheta))^{\alpha}]^{\theta}-(G(x;\vartheta))^{\theta(\alpha+1)}\}=0, \tag{4.9}$$

that corresponds to points where $h'(x) = 0$. There may be more than one root to this equation.

Let $\varphi(x) = \dfrac{d^2\log[h(x)]}{dx^2}$. We get

$$\phi(x)=\frac{h_g''(x;\vartheta)}{h_g(x;\vartheta)}-\left[\frac{h_g'(x;\vartheta)}{h_g(x;\vartheta)}\right]^2+\frac{(\alpha\theta+1)h_g(x;\vartheta)g(x;\vartheta)}{[G(x;\vartheta)]^2}-h_g'(x;\vartheta)\left(\frac{\alpha\theta+\bar{G}(x;\vartheta)}{G(x;\vartheta)}\right)$$

$$-\frac{\alpha\beta\theta(\alpha+1)g(x;\vartheta)h_g(x;\vartheta)(G(x;\vartheta))^{(\alpha\theta-2)}(\beta\bar{G}(x;\vartheta))^{(\alpha\theta-1)}}{[\alpha(\beta\bar{G}(x;\vartheta))^{(\alpha\theta)}+(G(x;\vartheta))^{(\alpha\theta)}]^2}$$

$$\left\{\alpha\theta(2G(x;\vartheta)-1)-\left[\alpha\left(\frac{\beta\bar{G}(x;\vartheta)}{G(x;\vartheta)}\right)^{(\alpha\theta)}+1\right]\left[\frac{G(x;\vartheta)h_g'(x;\vartheta)}{[h_g(x;\vartheta)]^2}-\bar{G}(x;\vartheta)\right]\right\}$$

$$+\frac{\alpha\beta\theta g(x;\vartheta)\left(\frac{h_g(x;\vartheta)}{G(x;\vartheta)}\right)(\beta\bar{G}(x;\vartheta))^{\alpha-1}[G(x;\vartheta)(\beta\bar{G}(x;\vartheta))^{\alpha}]^{\theta-1}}{[[\alpha(\beta\bar{G}(x;\vartheta))^{(\alpha\theta)}+(G(x;\vartheta))^{(\alpha\theta)}]^{\left(\frac{\alpha+1}{\alpha}\right)}-\alpha[G(x;\vartheta)(\beta\bar{G}(x;\vartheta))^{\alpha}]^{\theta} - (G(x;\vartheta))^{\theta(\alpha+1)}]^2}$$

$$\times\left\{\left(\frac{G(x;\vartheta)h_g'(x;\vartheta)}{(h_g(x;\vartheta))^2}-\bar{G}(x;\vartheta)+\theta(1-(\alpha-1)G(x;\vartheta))\right)\right.$$

$$[[\alpha(\beta\bar{G}(x;\vartheta))^{(\alpha\theta)}+(G(x;\vartheta))^{(\alpha\theta)}]^{\left(\frac{\alpha+1}{\alpha}\right)}$$

$$-\alpha[G(x;\vartheta)(\beta\bar{G}(x;\vartheta))^{\alpha}]^{\theta}-(G(x;\vartheta))^{\theta(\alpha+1)}]$$

$$-\theta G(x;\vartheta)\bar{G}(x;\vartheta)[(\alpha+1)((G(x;\vartheta))^{(\alpha\theta-1)}$$

$$-\alpha\beta(\beta\bar{G}(x;\vartheta))^{(\alpha\theta-1)})[\alpha(\beta\bar{G}(x;\vartheta))^{(\alpha\theta)}$$

$$+(G(x;\vartheta))^{(\alpha\theta)}]^{\left(\frac{1}{\alpha}\right)}-\alpha\beta(\beta\bar{G}(x;\vartheta))^{\alpha-1}(1-(\alpha+1)G(x;\vartheta))$$

$$[G(x;\vartheta)(\beta\bar{G}(x;\vartheta))^{\alpha}]^{\theta-1}$$

$$\left.-(\alpha+1)(G(x;\vartheta))^{\theta(\alpha+1)-1}]\right\}. \tag{4.10}$$

If $x = b$ is a root of the equation (4.9) then it corresponds to a local maximum if $\varphi(x) > 0$ for all $x < b$ and $\varphi(x) < 0$ for all $x > b$. Moreover, it corresponds to a local minimum if $\varphi(x) < 0$ for all $x < b$ and $\varphi(x) > 0$ for all $x > b$. Furthermore, it refers to an inflexion point if either $\varphi(x) > 0$ for all $x \neq b$ or $\varphi(x) < 0$ for all $x \neq b$.

These equations can be examined by using most computation software platforms such as Mathematica, Maple, Matlab, sageMath, Maxima and others to determine the local minimums and maximums and inflexion points.

5. Expansions of both cdf and pdf

Some mathematical properties of the Odd Kappa-G family may be difficult to be obtained by numerical integration due to, for example, rounding off errors. Therefore, established algebraic expansions to determine these properties can be more efficient. For most practical purposes, we can substitute a large positive integer such as 10 or more instead of the infinity limit in the sums.

Proposition 5.1. *The cdf and pdf of the Odd Kappa-G family can be expressed in terms of the cdf and pdf of the exponentiated-G with power parameter* (j), *respectively as*

$$F(x; \alpha, \beta, \theta, \vartheta) = \sum_{j=0}^{\infty} w_j Q_j(x; \vartheta) \tag{5.1}$$

and

$$f(x; \alpha, \beta, \theta, \vartheta) = \sum_{j=0}^{\infty} w_j q_j(x; \vartheta), \tag{5.2}$$

where $w_j, Q_j(x; \vartheta)$ *and* $q_j(x; \vartheta)$ *are defined in* (B.6), (B.7) *and* (B.8), *respectively*:

Proof. See appendix B.

Consequently, proposition 5.1 contains the main results of this paper. Therefore, the CDF and PDF of the Odd Kappa-G family can be expressed as an infinite linear combination of exponentiated-G ("exp-G" for short) cdfs and pdfs, respectively. As a result, using exp-G characteristics, certain mathematical properties of the proposed family may be deduced from proposition 5.1. For example, the ordinary, incomplete moments and moment generating function can be obtained from those exp-G quantities. Some properties of the exp-*G* distributions are discussed by Mudholkar and Srivastava [59], Gupta et al. [33], and Nadarajah and Kotz [61] among others.

6. Quantile Function

Proposition 6.1. *The quantile function* (*qf*) *of the Odd Kappa-G family of* *distributions is given by*

$$x_\tau = Q_X(\tau) = G^{-1}\left\{\frac{\beta\left[\dfrac{\alpha\,\tau^\alpha}{1-\tau^\alpha}\right]^{\left(\frac{1}{\alpha\theta}\right)}}{1+\beta\left[\dfrac{\alpha\,\tau^\alpha}{1-\tau^\alpha}\right]^{\left(\frac{1}{\alpha\theta}\right)}}\right\}, \tag{6.1}$$

where

$$\tau \sim uniform\ (0,\ 1),$$

or

$$x_v = Q_G\left(\frac{V}{1+V}\right) = G^{-1}\left(\frac{V}{1+V}\right), \tag{6.2}$$

where

$$V \sim Kappa(\alpha,\ \beta,\ \theta).$$

Proof. See appendix C.

7. Moments and Generating Function

We obtain in this section some properties of the Odd Kappa-G family including the ordinary and incomplete moments, generating function, skewness, kurtosis, mean deviations and (reversed) residual life.

7.1. Ordinary Moments

Proposition 7.10. *The* r^{th} $(r = 1,\ 2,\ \ldots)$ *non-central moments of the Odd Kappa-G family is given by*

$$\mu'_r = \sum_{j=0}^{\infty} w_j E(\,Y_j^r\,), \tag{7.1}$$

or

$$\mu'_r = \sum_{j=0}^{\infty} w_j \int_0^1 \left[G^{-1}\left(u^{\left(\frac{1}{j}\right)}\right)\right]^r du, \tag{7.2}$$

where Y_j *is a random variable having the exp-G distribution with power parameter j and* w_j *as defined in* (B.6).

Proof. For the first formula (7.1), we have from equation (5.2)

$$\mu'_r = \int_{-\infty}^{\infty} x^r f(x)dx = \sum_{j=0}^{\infty} jw_j \int_{-\infty}^{\infty} x^r g(x)[G(x)]^{j-1} dx = \sum_{j=0}^{\infty} w_j E(Y_j^r),$$

where Y_j is a random variable having the exp-G distribution with power parameter *j*. For the second formula (7.2), we let

$$u = [G(x)]^j,$$

then

$$du = jg(x)[G(x)]^{j-1} dx.$$

Hence, the result is obtained in terms of the baseline quantile function for the second formula.

Remark 7.1. Expressions for moments of various exp-G distributions may be used to calculate $E(Y_j^r)$ given by Nadarajah and Kotz [61].

Remark 7.2. By using the relationship between the non-central moments and central moments, the s^{th} central moments of the Odd Kappa-G family, say μ_s, can be obtained as follows:

$$\mu_s = E(X - \mu)^s = \sum_{r=0}^{s} \binom{s}{r} (-\mu'_1)^{s-r} \mu'_r \text{ for } s = 1, 2, \ldots \tag{7.3}$$

where μ'_r is defined in proposition 7.1. Furthermore, $\mu_0 = \mu'_0 = 1$, $\mu_1 = 0, \mu_2 = \mu'_2 - (\mu'_1)^2 = \sigma^2, \mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3, \mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2$ $- 3(\mu'_1)^4$, etc.

7.2. Moment Generating Function

Proposition 7.2. *The Odd Kappa-G family's moment generating function* (*mgf*) *is provided by*:

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r = \sum_{j,\, r=0}^{\infty} \frac{w_j t^r}{r!} E(Y_j^r)$$

or

$$M_X(t) = \sum_{j=0}^{\infty} j w_j \int_{-\infty}^{\infty} e^{tX} g(x) [G(x)]^{j-1} dx = \sum_{j=0}^{\infty} w_j \int_0^1 e^{\left[tG^{-1}\left(u^{\left(\frac{1}{j}\right)}\right)\right]} du$$

Remark 7.3. The coefficients in the Taylor expansion of the cumulant generating function about the origin are known as cumulants, i.e., the logarithm of the moment generating function about the origin. Hence, the cumulants of the Odd Kappa-G family, say κ_s, can be obtained recursively from:

$$\kappa_s = \mu'_s - \sum_{r=1}^{s-1} \binom{s-1}{r-1} \kappa_r \mu'_{s-r} \text{ for } s = 1, 2, \ldots,$$

where μ'_r and μ_s are defined in equations (7.1) and (7.3), respectively. Also, $\kappa_1 = \mu'_1 = \mu_1$, $\kappa_2 = \mu'_2 - (\mu'_1)^2 = \mu_2$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3 = \mu_3$, $\kappa_4 = \mu'_4 - 3(\mu'_2)^2 - 4\mu'_3\mu'_1 + 12\mu'_2(\mu'_1)^2 - 6(\mu'_1)^4 = \mu_4 - 3(\mu'_2)^2$, etc. Note that higher-order cumulants are not the same as moments about the mean.

7.3. Skewness and Kurtosis. In order to make the skewness and kurtosis as measures independent of the units of data, it is customarily to divide μ_3 and μ_4 by $(\mu_2)^{(3/2)}$ and $(\mu_2)^2$, respectively. Hence, skewness $= \mu_3/(\mu_2)^{(3/2)}$ and kurtosis $= \mu_4/(\mu_2)^2$, see Casella and Berger [19]. However, it is not true that if $\mu_3 = 0$ then the distribution is necessarily symmetrical, see Elkin [26].

The three quartiles of the Odd Kappa-G family are defined by $Q_1 = Q_X(1/4; \alpha, \beta, \theta, \vartheta)$, $Q_2 = Q_X(1/2; \alpha, \beta, \theta, \vartheta) = Median(M)$ and $Q_3 = Q_X(3/4; \alpha, \beta, \theta, \vartheta)$ where $Q_X(\cdot)$ is defined in equation (6.1). In addition, the inter-quartile range is given by $IQR = Q_3 - Q_1$. Therefore, the robust statistics Galton or Bowley coefficient of skewness and the Moors coefficient of kurtosis are, respectively, given by

$$S = \frac{(Q_3 - Q_2) - (Q_2 - Q_1)}{IQR},$$

and

$$K = \frac{Q_X(7/8; \alpha, \beta, \theta, \vartheta) - Q_X(5/8; \alpha, \beta, \theta, \vartheta) + Q_X(3/8; \alpha, \beta, \theta, \vartheta) - Q_X(1/8; \alpha, \beta, \theta, \vartheta)}{IQR}.$$

See Bowley [17] and Moors [58] for more details on these coefficients, respectively.

7.4. Incomplete Moments

Proposition 7.3. *The* n^{th} $(n = 1, 2, \ldots)$ *incomplete moments of the Odd Kappa-G family is given by*

$$m_n(y) = \sum_{j=0}^{\infty} j\, w_j \int_0^y x^n g(x)[G(x)]^{j-1} dx$$

$$= \sum_{j=0}^{\infty} w_j \int_0^{[G(y)]^j} \left[G^{-1}\left(u^{\left(\frac{1}{j}\right)} \right) \right]^n du. \tag{7.4}$$

Proof. The proof is straightforward therefore it is omitted.

Remark 7.4. As special case, if $n = 1$ in equation (7.4) then the first incomplete moment is obtained as:

$$m_1(y) = \int_0^y x f(x) dx = \sum_{j=0}^{\infty} w_j \int_0^{[G(y)]^j} G^{-1}\left(u^{\left(\frac{1}{j}\right)} \right) du. \tag{7.5}$$

Note that for most G distributions, the last integral can be obtained numerically.

Remark 7.5. Incomplete moments can be useful for measuring inequality such as Lorenz and Bonferroni curves or describing empirically the shape of many distributions. Therefore, the most widely used inequality measures in income and wealth distribution considered to be Lorenz and Bonferroni curves, Dagum [24]. For a given probability π, the Lorenz curve as follows:

$$L_X(\pi) = \frac{\int_0^q tf(t)dt}{E(X)} = \frac{\int_0^\pi F_X^{-1}(u)du}{E(X)} = \frac{m_1(q)}{E(X)}, \tag{7.6}$$

where $q = Q_X(\pi) = F_X^{-1}(\pi)$ and $m_1(q)$ is defined in equation (7.5).

Also, the Bonferroni curve as follows:

$$B_X(\pi) = \frac{\int_0^q tf(t)dt}{\pi E(X)} = \frac{\int_0^\pi F_X^{-1}(u)dt}{\pi E(X)} = \frac{m_1(q)}{\pi E(X)} = \frac{L_X(\pi)}{\pi}. \tag{7.7}$$

In economics, if $\pi = F_X(q)$ is the proportion of units with incomes less than or equal to q, $L_X(\pi)$ gives the proportion of total income volume accumulated by the set of units with an income lower than or equal to q. In a similar manner, the Bonferroni curve $B_X(\pi)$ calculates the ratio of this group's mean income to the population's mean income.

7.5. Mean devotions

Proposition 7.4. *The mean devotions about the mean* (δ_1) *and the median* (δ_2) *for the Odd Kappa-G family can be respectively expressed as*:

$$\delta_1 = E(|X - \mu_1'|) = \int_0^\infty |x - \mu_1'| f(x)dx = 2\mu_1' F(\mu_1') - 2m_1(\mu_1')$$

$$= 2\mu_1' F(\mu_1') - 2\sum_{j=0}^{\infty} w_j \int_0^{[G(\mu_1')]^j} G^{-1}\left(u^{\left(\frac{1}{j}\right)}\right) du,$$

and

$$\delta_2 = E(|X - M|) = \int_0^\infty |x - M| f(x)dx = \mu_1' - 2m_1(M)$$

$$= \mu_1' - 2\sum_{j=0}^{\infty} w_j \int_0^{[G(M)]^j} G^{-1}\left(u^{\left(\frac{1}{j}\right)}\right) du,$$

where $\mu_1' = E(X)$ *and* $M = median(X)$.

Proof. Since the proof is not difficult to show, it is omitted to conserve space.

7.6. (Reversed) Residual life Function

Applications of (reversed) residual life random variables are extensive in reliability analysis and risk theory. Therefore, we introduce the n^{th} moments of (reversed) residual life in the following proposition. Both mean residual life (MRL) and mean waiting time (MWT) are considered special cases.

Proposition 7.5. *Let X follows the Odd Kappa-G family then the n^{th} moment of residual life of X, say $M_n(t) = E[(X-t)^n \mid X > t]$ for $t > 0$, $n = 1, 2, \ldots,$ uniquely determine $F(x)$ (see Navarro et al.* [62]) *and the n^{th} moment of reversed residual life of X, say $M_n^*(t) = E[(t-X)^n \mid X \le t]$ for $t > 0$, $n = 1, 2, \ldots,$ uniquely determine $F(x)$ (see Navarro et al.* [62]) *are expressed respectively as*:

$$M_n(t) = \frac{1}{\bar{F}(t)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{n}{j} (-t)^{n-j}\, i w_i \int_t^{\infty} x^j g(x) [G(x)]^{i-1} dx$$

$$= \frac{1}{\bar{F}(t)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{n}{j} (-t)^{n-j} w_i \int_{[G(t)]^i}^{1} \left[G^{-1}\left(u^{\left(\frac{1}{i}\right)} \right) \right]^j du, \quad (7.8)$$

and

$$M_n^*(t) = \frac{1}{F(t)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{n}{j} (-1)^{n-j}\, i w_i t^n \int_0^t x^{n-j} g(x) [G(x)]^{i-1} dx$$

$$= \frac{1}{F(t)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{n}{j} (-1)^{n-j} w_i t^n \int_0^{[G(t)]^i} \left[G^{-1}\left(u^{\left(\frac{1}{i}\right)} \right) \right]^{n-j} du, \quad (7.9)$$

where $G(x)$ and $g(x)$ are the cdf and pdf of the baseline distributions, respectively.

Proof. See appendix D.

Remark 7.6. Setting $n = 1$ in the preceding equation (7.8) as $M_1(t)$ yields the mean residual life (MRL) function. The mean residual life (MRL) is a function of t and represents the predicted extra lifespan provided that a component has lived until time t. An interpretation of the MRL is as follows: How much longer is something anticipated to survive if it has made it thus far? Mean residual time provides an answer to this question.

Remark 7.7. The mean waiting time (MWT) [also known as the mean inactivity time (MIT) or the mean reversed residual life function or the mean past lifetime] can be obtained by setting $n = 1$ in the above equation (7.9) as $M_1^*(t)$. The MWT can be interpreted as follows: Consider the case where we have discovered a device has already failed at some point in the past, say t. The MWT is a useful indicator for predicting when a device will fail. In biomedicine, the efficacy of a treatment is determined by analyzing the remission period, or disease-free survival time, in cases of recurrent diseases. Because of the significant expense and effort required in continually monitoring patients, the real remission time is frequently unclear. In such cases, a MWT function can be used to estimate the real remission period.

8. Order Statistics

The following proposition shows that the pdf of the Odd Kappa-G family order statistic is a linear combination of exponentiated-G pdfs.

Proposition 8.1. *Let* $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ *be the ordered sample from the Odd Kappa-G family of distributions. The density of the* i^{th} *order statistic, say* $X_{i,n}$ *is expressed as*

$$f_{i:n}(x) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \rho_{(k,l)} q_{(k+l)}(x). \tag{8.1}$$

Proof. See appendix E.

Remark 8.1. The pdfs $(f_{1:n}(x)$ and $f_{n:n}(x))$ of the smallest and largest order statistics $X_{1:n}$ and $X_{n:n}$ are obtained from the above equation (8.1) by substituting $i = 1$ and $i = n$, respectively, as follows:

$$f_{1:n}(x) = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} \rho_{(k,l)}\, q_{(k+l)}(x),$$

where $\rho_{(k,l)} = \frac{nk}{(k+l)} \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} w_k \varphi_l$, $\varphi_0 = [w_0]^j$, j is a natural number and for $l \geq 1$, $\varphi_l = \frac{1}{(lw_0)} \sum_{m=1}^{l} [m(j+1) - l] w_m \varphi_{l-m}$.

$$f_{n:n}(x) = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} \rho_{(k,l)} q_{(k+l)}(x),$$

where $\rho_{(k,l)} = \frac{nk}{(k+l)} w_k \phi_l$, $\phi_0 = [w_0]^{(n-1)}$, $(n-1)$ is a natural number and for $l \geq 1$, $\varphi_l = \frac{1}{(l w_0)} \sum_{m=1}^{l} [m n - l] w_m \varphi_{l-m}$.

Remark 8.2. The r^{th} moment of the i^{th} order statistic $(X_{i:n})$ for the Odd Kappa-G family is given by

$$E(X_{i:n}^r) = \int_{-\infty}^{\infty} x^r f_{i:n}(x) dx = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} \rho_{(k,l)} (k+l) \int_{-\infty}^{\infty} x^r g(x) [G(x)]^{k+l-1} dx$$

$$= \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} \rho_{(k,l)} E(Y_{k+l}^r),$$

where Y_{k+l} is a random variable with the exp-G distribution and $k+l$ as the power parameter.

Remark 8.3. The L-moments are similar to the ordinary moments but they can be estimated by linear combinations of order statistics. In addition, they are linear functions of expected order statistics defined by (see Hosking [37]):

$$\lambda_n^* = n^{-1} \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} E(X_{n-i:n}),\ n \geq 1.$$

The first four L-moments are:

$$\lambda_1^* = E(X_{1:1}),$$

$$\lambda_2^* = \frac{1}{2} E(X_{2:2} - X_{1:2}),$$

$$\lambda_3^* = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3}),$$

$$\lambda_4^* = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}).$$

Expansions for the L-moments of the Odd Kappa-G distribution can be obtained as weighted linear combinations of the means of appropriate Odd Kappa-G distribution from the expansions for the moments of the order statistics with $r = 1$ in remark 8.2 above. Even though some higher moments of the distribution may not exist but the mean of this distribution exists, the L-moments have the advantage that they exist as well and are relatively robust to the effects of outliers.

9. Mathematical Properties

9.1. Probability Weighted Moments

Although probability weighted moments (PWMs), Greenwood et al. [31], are useful to characterize a distribution, they have no particularly meaning. The primary use of PWMs (and the related L-moments, defined in remark 8.3, that are linear combinations of PWMs) is in the estimation of parameters for a probability distribution. Besides deriving estimators of the parameters, the PWMs are used to derive quantiles of generalized distributions. For a more detailed description of PWMs and L-moments, see the papers by Hoskings' (Hosking [35] and Hosking [34]). PWMs are sometimes used when maximum likelihood estimates are unavailable or difficult to compute. Moreover, they may be used as starting values for maximum likelihood estimates. Often, estimates based on PWMs are considered to be superior to central moment estimates since they are known to be more robust against outliers. Also, they have low variance and no severe bias (see Greenwood et al. [31] and Landwehr et al. [48]).

Proposition 9.1. *The probability weighted moments* (*PWMs*), *Greenwood* *et al.* [31], *of the Odd Kappa-G family of distributions, say* $\mathcal{M}_{r,s,t}$ *are given by*

$$\mathcal{M}_{r,s,t} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} j\,w_j z_k v_l \int_{-\infty}^{\infty} x^r g(x)[G(x)]^{l+k+j-1} dx, \tag{9.1}$$

this equation reveals that the PWMs of X is a linear combination of exp-G distributions. Or

$$\mathcal{M}_{r,s,t} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{j w_j z_k v_l}{(l+k+j)} \int_0^1 \left[G^{-1}\left(u^{\left(\frac{1}{l+k+j}\right)} \right) \right]^r du, \tag{9.2}$$

where $r, s, t \in \mathbb{N}, z_0 = (w_0)^s, v_0 = (w_0')^t$ *and for* $k, l \geq 1,\ z_k = \frac{1}{(k\,w_0)}$ $\sum_{m=1}^{k}[m(s+1)-k]w_m z_{k-m}, v_l = \frac{1}{(l w_0')}\sum_{m=1}^{l}[m(t+1)-l]w_m' v_{l-m}.$ *Also,* $w_0' = 1 - w_0$ *and for* $j \geq 1,\ w_j' = -w_j$.

Proof. See appendix F.

9.2. Stress-strength model

In the reliability, the stress-strength model which has many applications especially in the area of engineering defines the life of a component that has a random strength subjected to an accidental random stress. This component will operate whenever the strength is greater than the stress which is a measure of component reliability. Otherwise, it fails if the stress applied to it exceeds the strength.

Proposition 9.2. *Let strength X and stress Y be two independent random variables with Odd Kappa-G* $(\alpha_1, \beta_1, \theta_1, \vartheta_1)$ *and Odd Kappa-G* $(\alpha_2, \beta_2, \theta_2, \vartheta_2)$ *distributions, respectively. Then, the stress-strength model is given by*

$$R = P(Y < X) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} i w1_i w2_j \int_{-\infty}^{\infty} g(x;\vartheta_1)[G(x;\vartheta_1)]^{i-1}[G(x;\vartheta_2)]^j dx \tag{9.3}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} iw1_i w2_j \int_0^1 [u]^{i-1} [G(G^{-1}(u; \vartheta_1); \vartheta_2)]^j \, dx, \tag{9.4}$$

where

$$w1_i = \sum\nolimits_{k=0}^{\infty} (-1)^k (\alpha_1)^k \beta_1^{\alpha_1 \theta_1 k} \begin{pmatrix} \frac{1}{\alpha_1} \\ k \end{pmatrix} d_{i,k} \;\; and$$

$$w2_j = \sum\nolimits_{l=0}^{\infty} (-1)^l (\alpha_2)^l \beta_2^{\alpha_2 \theta_2 l} \begin{pmatrix} \frac{1}{\alpha_2} \\ k \end{pmatrix} d_{j,l}.$$

In addition, $G(x; \vartheta)$ *and* $g(x; \vartheta)$ *on the formula of R are the cdf and pdf of any baseline distribution, respectively.*

Proof. See appendix G.

Remark 9.1. From proposition 9.2, if $\vartheta_1 = \vartheta_2 = \vartheta$ then

$$R = P(Y < X) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} iw1_i w2_j \int_{-\infty}^{\infty} g(x, \vartheta)[G(x, \vartheta)]^{i+j-1} \, dx$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{iw1_i w2_j}{(i+j)}.$$

Remark 9.2. If $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$, $\theta_1 = \theta_2 = \theta$ but $\vartheta_1 \neq \vartheta_2$ in proposition 9.2, then

$$R = P(Y < X) = \sum_{i=j=0}^{\infty} i(w_i)^2 \int_{-\infty}^{\infty} \frac{g(x; \vartheta_1)}{G(x; \vartheta_1)} [G(x; \vartheta_1) G(x; \vartheta_2)]^i \, dx$$

$$+ \sum_{i<j}^{\infty} (i+j) w_i w_j \int_{-\infty}^{\infty} g(x; \vartheta_1) [G(x; \vartheta_1)]^{i-1} [G(x; \vartheta_2)]^j \, dx$$

$$= \sum_{i=j=0}^{\infty} i(w_i)^2 \int_0^1 \frac{[uG(G^{-1}(u; \vartheta_1)); \vartheta_2]^i}{u} du$$

$$+\sum_{i<j}^{\infty}(i+j)w_i w_j \int_0^1 [u]^{i-1}[G(G^{-1}(u; \vartheta_1); \vartheta_2)]^j \, du.$$

Remark 9.3. From proposition 9.2, if $\alpha_1 = \alpha_2 = \alpha, \beta_1 = \beta_2 = \beta, \theta_1 = \theta_2 = \theta$ and $\vartheta_1 = \vartheta_2 = \vartheta$, then

$$R = P(Y < X) = \sum_{i=j=0}^{\infty} i(w_i)^2 \int_{-\infty}^{\infty} g(x; \vartheta)[G(x; \vartheta)]^{2i-1} dx$$

$$+\sum_{i<j}^{\infty}(i+j)w_i w_j \int_{-\infty}^{\infty} g(x; \vartheta)[G(x; \vartheta)]^{i+j-1} dx$$

$$= \sum_{i=j=0}^{\infty} \frac{(w_i)^2}{2} + \sum_{i<j}^{\infty} w_i w_j$$

$$= \frac{1}{2}\left[\sum_{i=j=0}^{\infty}(w_i)^2 + 2\sum_{i<j}^{\infty} w_i w_j\right] = \frac{1}{2}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_i w_j\right]$$

$$= \frac{1}{2}\left[\left(\sum_{i=0}^{\infty} w_i\right)\left(\sum_{j=0}^{\infty} w_j\right)\right] = \frac{1}{2}[(1)(1)] = \frac{1}{2},$$

where $w_r = \sum_{k=0}^{\infty}(-1)^k (\alpha)^k \beta^{\alpha\theta k} \binom{\frac{1}{\alpha}}{k} d_{r,k}$.

9.3. Stochastic Ordering

The easiest way to compare two random variables X and Y is by using statistical measures such as mean, median, standard deviation, etc. However, sometime confusion arises, for example, when the mean of X is larger than the mean of Y, while the median of X is smaller than that of Y. Such case will not happened when the two variables are stochastically ordered. Stochastic ordering plays a major role in many different areas of probability and statistics such as reliability and queuing theories, survival analysis, finance and economics. It used to show the mechanism in life time distributions or to examine the comparative behaviour. According to Shaked and Shanthikumar

[69], a random variable X is said to be stochastically greater than Y, i.e. $X \geq_{st} Y$, if $F_X(x) \leq F_Y(x)$ for all x. In the similar way, X is said to be stochastically greater than $Y(X \geq_{st} Y)$ in the

1. hazard rate order $(X \geq_{hr} Y)$ if $h_X(x) \leq h_Y(x)$ for all x,

2. mean residual life order $(X \geq_{mrl} Y)$ if $m_X(x) \geq m_Y(x)$ for all x,

3. likelihood ratio order $(X \geq_{lr} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ is an increasing function of x,

4. reversed hazard rate order $(X \geq_{rhr} Y)$ if $\frac{F_X(x)}{F_Y(x)}$ is an increasing function of x.

These stochastic orders defined above are related (see Shaked and Shanthikumar [69]) as follows:

$$\begin{array}{c} X \geq_{rhr} Y \Leftarrow X \geq_{lr} Y \Rightarrow X \geq_{hr} Y \Rightarrow X \geq_{st} Y \\ \Downarrow \\ X \geq_{mrl} Y. \end{array} \tag{9.5}$$

Proposition 9.3. *Let X and Y be two independent random variables with Odd Kappa-G $(\alpha_1, \beta_1, \theta_1, \vartheta_1)$ and Odd Kappa-G $(\alpha_2, \beta_2, \theta_2, \vartheta_2)$ distributions, respectively. If $\vartheta_1 = \vartheta_2 = \vartheta$ and ((1) $\beta_1 = \beta_2 = \beta$, $\theta_1 = \theta_2 = \theta$ and $\alpha_2 > \alpha_1$, (2) $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $\theta_2 > \theta_1$ or (3) $\alpha_1 = \alpha_2 = \alpha$, $\theta_1 = \theta_2 = \theta$ and $\beta_1 > \beta_2$), then $(X \geq_{lr} Y)$ for all x. Hence, $(X \geq_{rhr} Y), (X \geq_{hr} Y), (X \geq_{mrl} Y)$ and $(X \geq_{st} Y)$ are satisfied.*

Proof. See appendix H.

9.4. Bivariate extension

In this section, we introduce the joint cdf of the Odd Kappa-G distribution, say BOK-G, as follows:

$$F_{X,Y}(x, y) = \frac{\left(\frac{G(x, y; \vartheta)}{\beta[1 - G(x, y; \vartheta)]}\right)^{\theta}}{\left[\alpha + \left(\frac{G(x, y; \vartheta)}{\beta[1 - G(x, y; \vartheta)]}\right)^{\alpha\theta}\right]^{\left(\frac{1}{\alpha}\right)}},$$

where $G(x, y; \vartheta)$ is a bivariate continuous distribution with marginal cdfs $G_1(x; \vartheta)$ and $G_2(y; \vartheta)$.

The marginal cdf's are

$$F_X(x) = \frac{\left(\frac{G_1(x; \vartheta)}{\beta[1 - G_1(x; \vartheta)]}\right)^{\theta}}{\left[\alpha + \left(\frac{G_1(x; \vartheta)}{\beta[1 - G_1(x; \vartheta)]}\right)^{\alpha\theta}\right]^{\left(\frac{1}{\alpha}\right)}}$$

and

$$F_Y(y) = \frac{\left(\frac{G_2(y; \vartheta)}{\beta[1 - G_2(y; \vartheta)]}\right)^{\theta}}{\left[\alpha + \left(\frac{G_2(y; \vartheta)}{\beta[1 - G_2(y; \vartheta)]}\right)^{\alpha\theta}\right]^{\left(\frac{1}{\alpha}\right)}}.$$

The joint pdf of (X, Y) is

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y} = \frac{\alpha\theta}{\beta} \frac{A(x, y; \alpha, \beta, \theta, \vartheta)}{[1 - G(x, y; \vartheta)]^2}$$

$$\frac{\left(\frac{G(x, y; \vartheta)}{\beta[1 - G(x, y; \vartheta)]}\right)^{\theta-1}}{\left[\alpha + \left(\frac{G(x, y; \vartheta)}{\beta[1 - G(x, y; \vartheta)]}\right)^{\alpha\theta}\right]^{\left(\frac{\alpha+1}{\alpha}\right)}},$$

where

$$A(x, y; \alpha, \beta, \theta, \vartheta) = g(x, y; \vartheta) + \frac{\partial G(x, y; \vartheta)}{\partial x}\frac{\partial G(x, y; \vartheta)}{\partial y}$$

$$\times \frac{\left\{\alpha(2G(x, y; \vartheta) + \theta - 1) + (2G(x, y; \vartheta) - \alpha\theta - 1)\left(\frac{G(x, y; \vartheta)}{\beta[1 - G(x, y; \vartheta)]}\right)^{\alpha\theta}\right\}}{\left\{G(x, y; \vartheta)[1 - G(x, y; \vartheta)]\left[\alpha + \left(\frac{G(x, y; \vartheta)}{\beta[1 - G(x, y; \vartheta)]}\right)\right]^{\alpha\theta}\right\}}.$$

The marginal pdf's are

$$f_X(x) = \frac{\alpha\theta}{\beta}\frac{\left(\frac{g_1(x; \vartheta)}{[1 - G_1(x; \vartheta)]^2}\right)\left(\frac{G_1(x; \vartheta)}{\beta[1 - G_1(x; \vartheta)]}\right)^{\theta - 1}}{\left[\alpha + \left(\frac{G_1(x; \vartheta)}{\beta[1 - G_1(x; \vartheta)]}\right)^{\alpha\theta}\right]^{\left(\frac{\alpha+1}{\alpha}\right)}}$$

and

$$f_Y(x) = \frac{\alpha\theta}{\beta}\frac{\left(\frac{g_2(y; \vartheta)}{[1 - G_2(x, \vartheta)]^2}\right)\left(\frac{G_2(y; \vartheta)}{\beta[1 - G_2(x, \vartheta)]}\right)^{\theta - 1}}{\left[\alpha + \left(\frac{G_2(y; \vartheta)}{\beta[1 - G_2(y; \vartheta)]}\right)^{\alpha\theta}\right]^{\left(\frac{\alpha+1}{\alpha}\right)}}.$$

The conditional cdf's are

$$F_{X|Y}(x|y) = \left(\frac{G(x, y; \vartheta)[1 - G_2(y; \vartheta)]}{G_2(y; \vartheta)[1 - G(x, y; \vartheta)]}\right)^{\theta}\left[\frac{\alpha + \left(\frac{G_2(y; \vartheta)}{\beta[1 - G_2(y; \vartheta)]}\right)^{\alpha\theta}}{\alpha + \left(\frac{G(x, y; \vartheta)}{\beta[1 - G(x, y; \vartheta)]}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)}$$

and

$$F_{Y|X}(y|x) = \left(\frac{G(x, y; \vartheta)[1 - G_1(y; \vartheta)]}{G_1(y; \vartheta)[1 - G(x, y; \vartheta)]}\right)^{\theta}\left[\frac{\alpha + \left(\frac{G_1(y; \vartheta)}{\beta[1 - G_1(y; \vartheta)]}\right)^{\alpha\theta}}{\alpha + \left(\frac{G(x, y; \vartheta)}{\beta[1 - G(x, y; \vartheta)]}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)}.$$

The conditional pdf's are

$$f_{X|Y}(x|y)=\left(\frac{A(x, y; \alpha, \beta, \theta, \vartheta)}{g_2(y; \vartheta)}\right)\left(\frac{1-G_2(y; \vartheta)}{1-G(x, y; \vartheta)}\right)^2$$

$$\left(\frac{G(x, y; \vartheta)[1-G_2(y; \vartheta)]}{G_2(y; \vartheta)[1-G(x, y; \vartheta)]}\right)^{\theta-1}\times\left[\frac{\alpha+\left(\frac{G_2(y; \vartheta)}{\beta[1-G_2(y; \vartheta)]}\right)^{\alpha\theta}}{\alpha+\left(\frac{G(x, y; \vartheta)}{\beta[1-G(x, y; \vartheta)]}\right)^{\alpha\theta}}\right]^{\left(\frac{\alpha+1}{\alpha}\right)}.$$

and

$$f_{Y|X}(y|x)=\left(\frac{A(x, y; \alpha, \beta, \theta, \vartheta)}{g_1(y; \vartheta)}\right)\left(\frac{1-G_1(y; \vartheta)}{1-G(x, y; \vartheta)}\right)^2$$

$$\left(\frac{G(x, y; \vartheta)[1-G_1(y; \vartheta)]}{G_1(y; \vartheta)[1-G(x, y; \vartheta)]}\right)^{\theta-1}\times\left[\frac{\alpha+\left(\frac{G_1(y; \vartheta)}{\beta[1-G_1(y; \vartheta)]}\right)^{\alpha\theta}}{\alpha+\left(\frac{G(x, y; \vartheta)}{\beta[1-G(x, y; \vartheta)]}\right)^{\alpha\theta}}\right]^{\left(\frac{\alpha+1}{\alpha}\right)}.$$

10. Entropy

Entropy is a measure of randomness or variability which has various applications in different areas like sciences, including physics, molecular imaging of tumors, hydrology, statistical mechanics, engineering and sparse kernel density estimation. Two popular entropy measures are the Shannon entropy (Shannon [70]) and Renyi entropy (Renyi et al. [66]). Derivation of an expression of Renyi entropy is important since it can be used as a measure of the shape of a distribution and compare the shapes and tails of different frequently used densities, for more details see Song [71]. Renyi entropy has some recent applications such as estimating the number of components of a multicomponent non-stationary signal (Sucic et al. [72]) and identifying cardiac autonomic neuropathy in diabetes (Jelinek et al. [41]).

Proposition 10.1. *The Renyi entropy for the Odd Kappa-G family is given by*

$$I_R(\delta) = \frac{1}{1-\delta}\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega(i,\ j)\int_{-\infty}^{\infty} g^{\delta}(x)G^{j-\delta}(x)\right.$$

$$\left.\left(\alpha + \left(\frac{G(x)}{\beta[1-G(x)]}\right)^{\alpha\theta}\right)^{-(\delta+i)} dx\right\}, \tag{10.1}$$

where $\delta > 0,\ \delta \neq 1$ *and* $\omega(i,\ j) = (-1)^{i+j}\,\theta^{\delta}\alpha^{(\delta+i)}\begin{pmatrix}\frac{\delta}{\alpha}\\ i\end{pmatrix}\begin{pmatrix}-\delta\\ j\end{pmatrix}.$

Proof. See appendix I.

The Shannon entropy (Shannon [70]) of the Odd Kappa-G family is a special case of the Renyi entropy in proposition 10.1 when $\delta \to 1.$ An application of Shannon entropy, for example, is used to classify emergent behavior in a simulation of laser dynamics (Guisado et al. [32]).

Proposition 10.2. *The Shannon entropy for the Odd Kappa-G family is given by*

$$E\{-\log[f(X)]\} = -\log[\alpha] - \log[\theta] + \theta\log[\beta]$$

$$-\sum_{j=1}^{\infty} jw_j\left\{\int_0^1 \log[g(G^{-1}(u))][u]^{j-1}\,du - \frac{(\theta-1)}{j^2} - \frac{(\theta+1)[\psi(1)-\psi(j+1)]}{j} - \left(\frac{\alpha+1}{\alpha}\right)\right.$$

$$\left.\times\int_0^1 \log\left[\alpha + \left(\frac{u}{\beta[1-u]}\right)^{\alpha\theta}\right][u]^{j-1}\,du\right\}.$$

Proof. See appendix J.

11. Estimation

The most frequently used method of parameter estimation is the method of maximum likelihood see Casella and Berger [19]. In this section, the maximum likelihood estimates (MLEs) of the unknown parameters of the Odd Kappa-G family are determined from complete sample. Let $x = (x_1, \ldots, x_n)^T$ be a random sample of size n from the Odd Kappa-G distribution with parameters α, β, θ, and ϑ. Let the $p \times 1$ parameter vector be $\boldsymbol{\xi} = (\alpha, \beta, \theta, \boldsymbol{\vartheta}^T)^T$ where $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_q)^T$ represents the parameter vector of the baseline distribution G. The total log-likelihood function for $\boldsymbol{\xi}$ is $\ell_n = \ell_n(\boldsymbol{\xi}) = \sum_{i=1}^{n} \ell^{(i)}$, where $\ell^{(i)}$ is the log likelihood for the i^{th} observation $(i = 1, \ldots, n)$. Then, ℓ_n is given by

$$\ell_n = n \log[\alpha] + n \log[\theta] - n\theta \log[\beta]$$

$$+ \sum_{i=1}^{n} \log[g(x_i, \boldsymbol{\vartheta})] + (\theta - 1) \sum_{i=1}^{n} \log[G(x_i, \boldsymbol{\vartheta})]$$

$$- (\theta + 1) \sum_{i=1}^{n} \log[1 - G(x_i; \boldsymbol{\vartheta})] - \left(\frac{\alpha + 1}{\alpha}\right)$$

$$\sum_{i=1}^{n} \log\left[\alpha + \left(\frac{G(x_i; \boldsymbol{\vartheta})}{\beta[1 - G(x_i; \boldsymbol{\vartheta})]}\right)^{\alpha\theta}\right]. \tag{11.1}$$

The maximization of ℓ_n can be done by solving the nonlinear likelihood equations obtained by differentiating equation (11.1). However, these equations cannot be solved analytically. Therefore, statistical software can be used to solve them numerically using iterative methods such as Newton-Raphson type algorithms. For example, but not limited, PROC NLMIXED Procedure in SAS/STAT (see SAS [68]), MaxBFGS in Ox program (see Doornik [25]), or optim in R language (see Team [77]).

The components of the score function $U_n(\boldsymbol{\xi}) = (U_\alpha, U_\beta, U_\theta, U_{\boldsymbol{\vartheta}^T})^T = (\partial \ell_n / \partial \alpha, \partial \ell_n / \partial \beta, \partial \ell_n / \partial \theta, \partial \ell_n / \partial \boldsymbol{\vartheta}^T)^T$ are

$$U_\alpha = \frac{\partial \ell_n}{\partial \alpha} = \frac{n}{\alpha} + \left(\frac{1}{\alpha^2}\right) \sum_{i=1}^{n} \log\left[\alpha + \left(\frac{G(x_i; \boldsymbol{\vartheta})}{\beta[1 - G(x_i; \boldsymbol{\vartheta})]}\right)^{\alpha\theta}\right] - \left(\frac{\alpha + 1}{\alpha}\right)$$

$$\times \sum_{i=1}^{n} \frac{1 + \theta \log\left[\frac{G(x_i; \boldsymbol{\vartheta})}{\beta[1 - G(x_i; \boldsymbol{\vartheta})]}\right] \left(\frac{G(x_i; \boldsymbol{\vartheta})}{\beta[1 - G(x_i; \boldsymbol{\vartheta})]}\right)^{\alpha\theta}}{\alpha + \left(\frac{G(x_i; \boldsymbol{\vartheta})}{\beta[1 - G(x_i; \boldsymbol{\vartheta})]}\right)^{\alpha\theta}}$$

$$U_\beta = \frac{\partial \ell_n}{\partial \beta} = -\frac{n\theta}{\beta} + \left(\frac{\theta(\alpha+1)}{\beta}\right)\sum_{i=1}^{n} \frac{\left(\frac{G(x_i;\,\vartheta)}{\beta[1-G(x_i;\,\vartheta)]}\right)^{(\alpha\theta)}}{\alpha + \left(\frac{G(x_i;\,\vartheta)}{\beta[1-G(x_i;\,\vartheta)]}\right)^{(\alpha\theta)}}$$

$$U_\theta = \frac{\partial \ell_n}{\partial \theta} = \frac{n}{\theta} - n\log[\beta] + \sum_{i=1}^{n} \log\left[G(x_i;\,\vartheta)\right]$$

$$-\sum_{i=1}^{n} \log\left[1-G(x_i;\,\vartheta)\right] - \left(\frac{\alpha+1}{\alpha}\right)$$

$$\times \sum_{i=1}^{n} \frac{\alpha \log\left[\frac{G(x_i;\,\vartheta)}{\beta[1-G(x_i;\,\vartheta)]}\right]\left(\frac{G(x_i;\,\vartheta)}{\beta[1-G(x_i;\,\vartheta)]}\right)^{\alpha\theta}}{\alpha + \left(\frac{G(x_i;\,\vartheta)}{\beta[1-G(x_i;\,\vartheta)]}\right)^{\alpha\theta}}$$

$$U_{\vartheta^T} = \frac{\partial \ell_n}{\partial \vartheta} = \sum_{i=1}^{n} \frac{g_{\vartheta}^{(1)}(x_i;\,\vartheta)}{g(x_i;\,\vartheta)}$$

$$+\sum_{i=1}^{n} \frac{(\theta - 1 + 2G(x_i;\,\vartheta))G_{\vartheta}^{(1)}(x_i;\,\vartheta)}{(1-G(x_i;\,\vartheta))G(x_i;\,\vartheta)} - \left(\frac{\theta(\alpha+1)}{\beta}\right)$$

$$\times \sum_{i=1}^{n} \frac{\left(\frac{G_{\vartheta}^{(1)}(x_i;\,\vartheta)}{[1-G(x_i;\,\vartheta)]^2}\right)\left(\frac{G(x_i;\,\vartheta)}{\beta[1-G(x_i;\,\vartheta)]}\right)^{(\alpha\theta-1)}}{\alpha + \left(\frac{G(x_i;\,\vartheta)}{\beta[1-G(x_i;\,\vartheta)]}\right)^{\alpha\theta}}.$$

Solving the equations $U_n(\boldsymbol{\xi}) = (U_\alpha, U_\beta, U_\theta, U_{\vartheta^T})^T = 0$ simultaneously gives the maximum likelihood estimate (MLE) $\hat{\boldsymbol{\xi}} = (\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\vartheta}^T)^T$ of $\boldsymbol{\xi} = (\alpha, \beta, \theta, \vartheta^T)^T$.

To construct interval estimation for parameters, the Fisher information matrix $I(\boldsymbol{\xi}) = \left(-E\left[\frac{\partial^2 \ell_n}{\partial \xi_i\, \partial \xi_j}\right]\right), i, j = 1, 2, \ldots, p(= 3 + q)$ is required which can be derived using the second partial derivatives of the log likelihood function with respect to each parameter. The asymptotic variance-covariance matrix of the MLEs of parameters is obtained by inverting the Fisher information matrix. Often, the evaluation of the above expectation may be difficult. Therefore, it can be estimated in practice by the observed Fisher's information matrix $J(\hat{\boldsymbol{\xi}}) = \left(-\frac{\partial^2 \ell_n}{\partial \xi_i \partial \xi_j}|_{\xi=\hat{\xi}}\right), i, j = 1, 2, \ldots, p(=3+q)$. Under some regularity conditions (defined below) on the parameters, the asymptotic distributions of $\sqrt{n}(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})$ is $N_p(\mathbf{0}, \boldsymbol{V}_n)$ where $\boldsymbol{V}_n = (v_{jj}) = \mathbf{I}_n^{-1}(\xi)$ as $n \to \infty$ using the general theory of MLEs where $\boldsymbol{I}_n(\boldsymbol{\xi}) = \begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & \boldsymbol{I}_{\alpha\theta} & \boldsymbol{I}_{\alpha\vartheta} \\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\theta} & \boldsymbol{I}_{\beta\vartheta} \\ I_{\theta\alpha} & I_{\theta\beta} & I_{\theta\theta} & \boldsymbol{I}_{\theta\vartheta} \\ \boldsymbol{I}_{\alpha\vartheta}^T & \boldsymbol{I}_{\beta\vartheta}^T & \boldsymbol{I}_{\theta\vartheta}^T & \boldsymbol{I}_{\vartheta\vartheta} \end{bmatrix}$ is the unit expected information matrix. The asymptotic behavior remains valid if $\boldsymbol{V}_n$ is replaced by $\hat{\boldsymbol{V}}_n = \boldsymbol{J}_n^{-1}(\hat{\boldsymbol{\xi}})$ where $\boldsymbol{J}_n(\boldsymbol{\xi}) = \begin{bmatrix} U_{\alpha\alpha} & U_{\alpha\beta} & U_{\alpha\theta} & \boldsymbol{U}_{\alpha\vartheta} \\ U_{\beta\alpha} & U_{\beta\beta} & U_{\beta\theta} & \boldsymbol{U}_{\beta\vartheta} \\ U_{\theta\alpha} & U_{\theta\beta} & U_{\theta\theta} & \boldsymbol{U}_{\theta\vartheta} \\ \boldsymbol{U}_{\alpha\vartheta}^T & \boldsymbol{U}_{\beta\vartheta}^T & \boldsymbol{U}_{\theta\vartheta}^T & \boldsymbol{U}_{\vartheta\vartheta} \end{bmatrix}$ is the observed information matrix. Therefore, the approximate standard errors and $(1 - \tau/2)$ 100% confidence interval for the MLE of j^{th} parameter (ξ_j) are respectively given by $\sqrt{\hat{v}_{jj}}$ and $\hat{\xi}_j \pm Z_{\tau/2}\sqrt{\hat{v}_{jj}}$, where $Z_{\tau/2}$ is the $\tau/2$ point of the standard normal distribution.

The usual regularity conditions (see Lehmann and Casella [50]) are:

1. The support of X associated with the distribution does not depend on unknown parameters.

2. The log likelihood function $\ell_n(\boldsymbol{\xi})$ has a global maximum in the open parameter space of X, say Θ.

3. The fourth order log likelihood derivatives with respect to the model parameters are continuous in an open subset of Θ that contains the true parameter and exist for almost all.

4. The expected information matrix is finite and positive definite.

5. Expected finite functions of Θ limit the absolute values of the third order log likelihood derivatives with respect to the parameters.

When these usual regularity conditions are fulfilled and the parameters are within the interior of the parameter space but not on the boundary, the maximum likelihood estimators are consistent, normally distributed, asymptotically unbiased and efficient.

For the second partial derivatives of the log likelihood function of the Odd Kappa-G family see appendix K. When the negative log-likelihood is minimized by most optimization algorithms like optim in R-software, the negative Hessian is returned. The square roots of the diagonal elements of the inverse of the negative Hessian are the estimated standard errors. Therefore, the negative Hessian evaluated at the MLE is the same as the observed Fisher information matrix evaluated at the MLE. To emphasize, the inverse of the negative Hessian is an estimator of the asymptotic covariance matrix. As a result, the square roots of the diagonal components of the covariance matrix are standard error estimators.

12. Simulation

To assess the performance of the Odd Kappa-G distribution MLEs with regard to sample size n, the following simulation study is conducted:

1. Generate B samples of size n by using the inversion method to get either $x_v = Q_G\left(\frac{V}{1+V}\right)$ where $V \sim Kappa(\alpha, \beta, \theta)$ or $x_\tau = Q_X(\tau)$ where $Q_X(\cdot)$ is the quantile function of the Odd Kappa-G distribution and τ is uniform (0, 1) (see proposition 6.1).

2. Determine the MLEs for the B samples, say $\hat{\xi}_i = \hat{\alpha}_i, \hat{\beta}_i, \hat{\theta}_i, \hat{\vartheta}_{1_i}, \ldots, \hat{\vartheta}_{q_i}$ for $i = 1, \ldots, B$.

3. Compute the standard errors of the MLEs by inverting the observed information matrices for the B samples, say $se_{\hat{\xi}_i}$ for $i = 1, \ldots, B$.

4. Compute the biases and the mean squared errors given by

$\text{Bias}(\hat{\xi}) = \frac{1}{B}\sum_{i=1}^{B}(\hat{\xi}_i - \xi)$ and MSE $(\hat{\xi}) = \frac{1}{B}\sum_{i=1}^{B}(\hat{\xi}_i - \xi)^2$.

Also, $\bar{\hat{\xi}} = \frac{1}{B}\sum_{i=1}^{B}\hat{\xi}_i$ where $\xi = \alpha, \beta, \theta, \vartheta_1, \ldots, \vartheta_q$.

5. Repeat steps (1-4) for the combinations of n and ξ.

13. Concluding Remarks

In this research paper, we introduce a new flexible family of distributions. A number of special models are presented from this family. We provide a comprehensive mathematical treatment of this family of distributions. In future research, more new distributions from this family will be investigated for complete and censored data.

Acknowledgments

The authors would like to thank the editor and the anonymous referee for their insightful comments that helped to enhance the paper. This study is a part of the master thesis of the second author under supervision of the first author.

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A. Proof of Proposition 2.1

For simplicity let $G(X) = G(X, \vartheta)$ in equation (2.1) then

$$F_X(x) = \left[\frac{\left(\frac{G(x)}{\beta(1 - G(x))} \right)^{\alpha\theta}}{\alpha + \left(\frac{G(x)}{\beta[1 - G(x)]} \right)^{\alpha\theta}} \right]^{\left(\frac{1}{\alpha}\right)}.$$

Therefore, we have

1. $F_T(t) = P(T \le t) = P(G(X) \le t) = P(X \le G^{-1}(t)) = F_X(G^{-1}(t))$

$$= \left[\frac{\left(\frac{t}{\beta[1 - t]} \right)^{\alpha\theta}}{\alpha + \left(\frac{t}{\beta[1 - t]} \right)^{\alpha\theta}} \right]^{\left(\frac{1}{\alpha}\right)}.$$

2. $F_T(t) = P\left(\frac{G(X)}{[1 - G(X)]} \le t \right) = P\left(G(X) \le \frac{t}{[1 + t]} \right) = P\left(X \le G^{-1}\left(\frac{t}{[1 + t]} \right) \right)$

$$= \left[\frac{\left(\frac{t}{\beta} \right)^{\alpha\theta}}{\alpha + \left(\frac{t}{\beta} \right)^{\alpha\theta}} \right]^{\left(\frac{1}{\alpha}\right)}.$$

3. $F_T(t) = P\left(\frac{[1-G(X)]}{G(X)} \le t\right) = P\left(G(X) \ge \frac{1}{[1+t]}\right) = 1 - P\left(G(X) \le \frac{1}{[1+t]}\right)$

$$= 1 - P\left(X \le G^{-1}\left(\frac{1}{[1+t]}\right)\right) = 1 - \left[\frac{\left(\frac{1}{\beta t}\right)^{\alpha\theta}}{\alpha + \left(\frac{1}{\beta t}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)}$$

$$= 1 - \left[\frac{1}{1 + \alpha\left(\frac{1}{\beta t}\right)^{-\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)} = 1 - [1 + \alpha(\beta t)^{\alpha\theta}]^{\left(-\frac{1}{\alpha}\right)}.$$

B. Proof of Proposition 5.1

Some of the results below are straightforward by using the well-known identity (binomial series):

$$(1-y)^r = \sum_{k=0}^{\infty} \binom{r}{k} (-1)^k y^k \text{ for } |y| < 1, \tag{B.1}$$

when it is applicable.

$$F(x; \alpha, \beta, \vartheta) = \left\{\frac{\left(\frac{G(x;\vartheta)}{\beta[1-G(x;\vartheta)]}\right)^{\alpha\theta}}{\left[\alpha + \left(\frac{G(x;\vartheta)}{\beta[1-G(x;\vartheta)]}\right)^{\alpha\theta}\right]}\right\}^{\left(\frac{1}{\theta}\right)}.$$

By using equation (B.1),

$$F(x; \alpha, \beta, \vartheta) = \sum_{i=0}^{\infty} \binom{\frac{1}{\alpha}}{i} (-1)^i \left(1 - \frac{\left(\frac{G(x;\vartheta)}{\beta[1-G(x;\vartheta)]}\right)^{\alpha\theta}}{\left[\alpha + \left(\frac{G(x;\vartheta)}{\beta[1-G(x;\vartheta)]}\right)^{\alpha\theta}\right]}\right)^i$$

$$F(x; \alpha, \beta, \vartheta) = \sum_{i=0}^{\infty} (-1)^i (\alpha)^i \binom{\frac{1}{\alpha}}{i} \left(\alpha + \left(\frac{G(x; \vartheta)}{\beta[1 - G(x; \vartheta)]} \right)^{\alpha\theta} \right)^{-i}$$

$$F(x; \alpha, \beta, \vartheta) = \sum_{i=0}^{\infty} (-1)^i (\alpha)^i \binom{\frac{1}{\alpha}}{i} \left(\frac{\alpha\beta^{\alpha\theta} [1 - G(x; \vartheta)]^{\alpha\theta} + [G(x; \vartheta)]^{\alpha\theta}}{(\beta[1 - G(x; \vartheta)])^{\alpha\theta}} \right)^{-i}$$

$$F(x; \alpha, \beta, \vartheta) = \sum_{i=0}^{\infty} (-1)^i (\alpha)^i \beta^{\alpha\theta i} \binom{\frac{1}{\alpha}}{i} \left(\frac{[1 - G(x; \vartheta)]^{\alpha\theta}}{\alpha\beta^{\alpha\theta} [1 - G(x; \vartheta)]^{\alpha\theta} + [G(x; \vartheta)]^{\alpha\theta}} \right)^{i}. \quad \text{(B.2)}$$

From equation (B.1),

$$[1 - G(x; \vartheta)]^{\alpha\theta} = \sum_{l=0}^{\infty} \binom{\alpha\theta}{l} [-G(x; \vartheta)]^l. \quad \text{(B.3)}$$

Substituting equation (B.3) into equation (B.2), we obtain

$$F(x, \alpha, \beta, \vartheta)$$

$$= \sum_{i=0}^{\infty} (-1)^i (\alpha)^i \beta^{\alpha\theta i} \binom{\frac{1}{\alpha}}{i} \left(\frac{\sum_{l=0}^{\infty} \binom{\alpha\theta}{l} [-G(x; \vartheta)]^l}{\alpha\beta^{\alpha\theta} \sum_{l=0}^{\infty} \binom{\alpha\theta}{l} [-G(x; \vartheta)]^l + [G(x; \vartheta)]^{\alpha\theta}} \right)^{i}. \quad \text{(B.4)}$$

Also, from equation (B.1),

$$[G(x; \vartheta)]^{\alpha\theta} = [1 - (1 - G(x; \vartheta))]^{\alpha\theta} = \sum_{k=0}^{\infty} a_k [1 - G(x; \vartheta)]^k$$

$$= \sum_{l=0}^{\infty} \sum_{k=l}^{\infty} (-1)^l \binom{k}{l} a_k [G(x; \vartheta)]^l, \quad \text{(B.5)}$$

where

$$a_k = (-1)^k \binom{\alpha\theta}{k}.$$

By using equation (B.5) into equation (B.4), we get

$$F(x; \alpha, \beta, \vartheta) = \sum_{i=0}^{\infty} (-1)^i (\alpha)^i \beta^{\alpha\theta i} \binom{\frac{1}{\alpha}}{i} \left(\frac{\sum_{l=0}^{\infty} a_l [G(x; \vartheta)]^l}{\sum_{l=0}^{\infty} b_l [G(x; \vartheta)]^l} \right)^i,$$

where

$$b_l = \alpha\beta^{\alpha\theta} a_l + \sum_{k=l}^{\infty} (-1)^l \binom{k}{l} a_k.$$

Therefore,

$$F(x; \alpha, \beta, \vartheta) = \sum_{i=0}^{\infty} (-1)^i (\alpha)^i \beta^{\alpha\theta i} \binom{\frac{1}{\alpha}}{i} \left(\sum_{l=0}^{\infty} c_l [G(x; \vartheta)]^l \right)^i,$$

where

$$c_0 = \frac{a_0}{b_0} \text{ and for } l \geq 1, c_l = \left(\frac{1}{b_0}\right) \left[a_l - \sum_{r=1}^{l} b_r c_{l-r} \right].$$

See page 17 in Gradshteyn and Ryzhik [30].

Moreover,

$$F(x; \alpha, \beta, \vartheta) = \sum_{l=0}^{\infty} (-1)^i (\alpha)^i \beta^{\alpha\theta i} \binom{\frac{1}{\alpha}}{i} \sum_{j=0}^{\infty} d_{j,i} [G(x; \vartheta)]^j,$$

where

$$d_{0,i} = (c_0)^i \text{ and for } j \geq 1,\ d_{j,i} = \left(\frac{1}{jc_0}\right) \sum_{r=1}^{j} [r(i+1) - j] c_r d_{j-r,i}.$$

See page 17 in Gradshteyn and Ryzhik [30].

Hence,

$$F(x; \alpha, \beta, \vartheta) = \sum_{j=0}^{\infty} w_j Q_j(x; \vartheta)$$

and by differentiation with respect to x, we get

$$f(x; \alpha, \beta, \vartheta) = \sum_{j=0}^{\infty} w_j q_j(x; \vartheta),$$

where

$$w_j = \sum_{i=0}^{\infty} (-1)^i (\alpha)^i \beta^{\alpha\theta i} \binom{\frac{1}{\alpha}}{i} d_{j,i}, \tag{B.6}$$

$$Q_a(x; \vartheta) = [G(x; \vartheta)]^a \tag{B.7}$$

and

$$q_a(x; \vartheta) = ag(x; \vartheta)[G(x; \vartheta)]^{a-1}. \tag{B.8}$$

C. Proof of Proposition 6.1

For the first equation (6.1), the qf of the Odd Kappa-G family produces the value of x_τ such that $F(x_\tau) = P(X \leq x_\tau) = \tau$ where $\tau \sim$ uniform (0, 1). Hence,

$$\left[\frac{\left(\frac{G(x_\tau; \vartheta)}{\beta[1-G(x_\tau; \vartheta)]}\right)^{\alpha\theta}}{\alpha + \left(\frac{G(x_\tau; \vartheta)}{\beta[1-G(x_\tau; \vartheta)]}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)} = \tau. \tag{C.1}$$

Set $x_\tau = Q_X(\tau)$ in the above equation (C.1) and solve for $Q_X(\tau)$ by using inverse transformation which gives

$$x_\tau = Q_X(\tau) = G^{-1}\left\{\frac{\beta\left[\frac{\alpha - \tau^\alpha}{1 - \tau^\alpha}\right]^{\left(\frac{1}{\alpha\theta}\right)}}{1 + \beta\left[\frac{\alpha\tau^\alpha}{1 - \tau^\alpha}\right]^{\left(\frac{1}{\alpha\theta}\right)}}\right\}.$$

For the second equation (6.2), let

$$V = \frac{G(x_v;\, \vartheta)}{[1 - G(x_v;\, \vartheta)]} \to G(x_v;\, \vartheta) = \frac{V}{1+V} \Rightarrow x_v = Q_G(v) = G^{-1}\left(\frac{V}{1+V}\right),$$

where V is the value of the quantile function of Kappa $(\alpha,\, \beta,\, \theta)$ at τ.

D. Proof of Proposition 7.6

The n^{th} moment of residual life of X following the Odd Kappa-G family is defined as

$$M_n(t) = \frac{1}{\overline{F(t)}} \int_t^{\infty} (x-t)^n f(x) dx.$$

By using equation (5.2), we obtain

$$M_n(t) = \frac{1}{\overline{F(t)}} \sum_{i=0}^{\infty} i w_i \int_t^{\infty} (x-t)^n g(x) [G(x)]^{i-1} dx.$$

Now, $(x - y)^n = \sum_{j=0}^{\infty} \binom{n}{j} (-1)^{n-j} x^j y^{n-j}$. Therefore,

$$M_n(t) = \frac{1}{\overline{F(t)}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{n}{j} (-t)^{n-j} i w_i \int_t^{\infty} x^j g(x) [G(x)]^{i-1} dx.$$

Let $u = [G(x)]^i$ then $du = ig(x)[G(x)]^{i-1} dx$

$$M_n(t) = \frac{1}{\overline{F(t)}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{n}{j} (-t)^{n-j} w_i \int_{[G(t)]^i}^{1} \left[G^{-1}\left(u^{\left(\frac{1}{i}\right)} \right) \right]^j du.$$

Moreover, the n^{th} moment of reversed residual life of X following the Odd Kappa-G family is defined as:

$$M_n^*(t) = \frac{1}{F(t)} \int_0^t (t-x)^n f(x) dx = \frac{1}{F(t)} \sum_{i=0}^{\infty} i w_i \int_0^t (t-x)^n g(x) [G(x)]^{i-1} dx.$$

Now, $(x - y)^n = \sum_{j=0}^{\infty} \binom{n}{j} (-y)^{n-j} x^j$. So, we get

$$M_n^*(t) = \frac{1}{F(t)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{n}{j} (-1)^{n-j}\, i w_i t^n \int_0^t x^{n-j} g(x) [G(x)]^{i-1}\, dx.$$

Let $u = [G(x)]^i$, then $du = ig(x)[G(x)]^{i-1}\, dx$

$$M_n^*(t) = \frac{1}{F(t)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{n}{j} (-1)^{n-j}\, w_i t^n \int_0^{[G(t)]^i} \left[G^{-1}\left(u^{\left(\frac{1}{i}\right)} \right) \right]^{n-j} du.$$

E. Proof of Proposition 8.1

The density of the i^{th} order statistic, say $X_{i:n}$, is defined as (see Arnold et al. [11])

$$f_{i:n}(x) = \mathbb{C} f(x) [F(x)]^{i-1} [1 - F(x)]^{n-i},$$

where $\mathbb{C} = \dfrac{n!}{(i-1)!(n-i)!}$.

By using the binomial theorem $(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k$, we have

$$f_{i:n}(x) = \mathbb{C} \sum_{j=0}^{n-i} (-1)^j \binom{n-j}{j} f(x) [F(x)]^{i+j-1}.$$

From proposition 5.1, we get

$$f_{i:n}(x) = \mathbb{C} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left\{ \sum_{k=1}^{\infty} k w_k\, g(x) [G(x)]^{k-1} \right\} \left[\sum_{l=0}^{\infty} w_l [G(x)]^l \right]^{i+j-1}$$

$$f_{i:n}(x) = \mathbb{C} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left\{ \sum_{k=1}^{\infty} k w_k\, g(x) [G(x)]^{k-1} \right\} \sum_{l=0}^{\infty} \phi_l [G(x)]^l,$$

where $\varphi_0 = [w_0]^{i+j-1}$, $(i+j-1)$ is a natural number and for $l \geq 1$,

$$\varphi_l = \frac{1}{(l w_0)} \sum_{m=1}^{l} [m(i+j) - l] w_m \varphi_{l-m}.$$

See page 17 in Gradshteyn and Ryzhik [30]. Hence,

$$f_{i:n}(x) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \rho_{(k,\, l)}\, q_{(k+l)}(x),$$

where $\rho_{(k,l)} = \frac{k\square}{(k+l)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} w_k \phi_l$ and $q_{(k+l)}(x) = (k+l) g(x)$ $[G(x)]^{k+l-1}$. Therefore, $f_{i:n}(x)$ is a linear combination of exp-G densities.

F. Proof of Proposition 9.1

If $F(x)$ is a distribution function then it may be characterized by the probability weighted moments (PWMs), say $\mathcal{M}_{r,s,t}$ as follows:

$$\mathcal{M}_{r,s,t} = E\{X^r [F(X)]^s [1 - F(X)]^t\} = \int_{-\infty}^{\infty} x^r [F(x)]^s [1 - F(x)]^t f(x) dx.$$

By using proposition 5.1, we have

$$\mathcal{M}_{r,s,t} = \int_{-\infty}^{\infty} x^r \left[\sum_{j=0}^{\infty} w_j [G(x)]^j \right]^s \left[\sum_{j=0}^{\infty} w'_j [G(x)]^j \right]^t \left\{ \sum_{j=0}^{\infty} j w_j g(x) [G(x)]^{j-1} \right\} dx,$$

where $w'_0 = 1 - w_0$ and for $j \geq 1$, $w'_j = -w_j$. Therefore,

$$\mathcal{M}_{r,s,t} = \sum_{j=0}^{\infty} j w_j \int_{-\infty}^{\infty} x^r g(x) [G(x)]^{j-1} \left[\sum_{j=0}^{\infty} w_j [G(x)]^j \right]^s \left[\sum_{j=0}^{\infty} w'_j [G(x)]^j \right]^t dx$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} j w_j z_k \int_{-\infty}^{\infty} x^r g(x) [G(x)]^{k+j-1} \left[\sum_{j=0}^{\infty} w'_j [G(x)]^j \right]^t dx,$$

where $z_0 = (w_0)^s$ and for $k \geq 1$, $z_k = \frac{1}{(k w_0)} \sum_{m=1}^{k} [m(s+1) - k] w_m z_{k-m}$, see Gradshteyn and Ryzhik [30]. Moreover,

$$\mathcal{M}_{r,s,t} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} j w_j z_k v_l \int_{-\infty}^{\infty} x^r g(x) [G(x)]^{l+k+j-1} dx,$$

where $v_0 = (w_0')^t$ and for $l \geq 1, v_l = \frac{1}{(lw_0')} \sum_{m=1}^{l} [m(t+1)-l] w_m' v_{l-m}$, see page 17 in Gradshteyn and Ryzhik [30]. In addition, let $u = [G(x)]^{l+k+j}$, then $du = (l+k+j) g(x) [G(x)]^{l+k+j-1} dx$. Hence, the proof is completed.

G. Proof of Proposition 9.2

The stress-strength model is defined as

$$R = P(Y < X) = \int_{-\infty}^{\infty} f_X(x; \alpha_1, \beta_1, \theta_1, \vartheta_1)\, F_Y(x; \alpha_2, \beta_2, \theta_2, \vartheta_2) dx.$$

From proposition 5.1, we get

$$R = P(Y < X) = \int_{-\infty}^{\infty} \left\{ \sum_{i=0}^{\infty} iw1_i, g(x; \vartheta_1)[G(x; \vartheta_1)]^{i-1} \right\} \left\{ \sum_{j=0}^{\infty} w2_j, [G(x; \vartheta_2)]^j \right\} dx$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} iw1_i w2_j \int_{-\infty}^{\infty} g(x; \vartheta_1)[G(x; \vartheta_1)]^{i-1} [G(x; \vartheta_2)]^j \, dx.$$

Let $u = G(x; \vartheta_1)$, then $du = g(x; \vartheta_1) dx$. Hence, we complete the proof.

H. Proof of Proposition 9.3

Given that $X \sim$ Odd Kappa- $\mathrm{G}(\alpha_1, \beta_1, \theta_1, \vartheta_1)$ and $Y \sim$ Odd Kappa-$\mathrm{G}(\alpha_2, \beta_2, \theta_2, \vartheta_2)$, then the likelihood ratio is given by

$$\frac{f_X(x)}{f_Y(x)}$$

$$= \frac{\frac{\alpha_1 \theta_1}{\beta_1} \left(\frac{g(x; \vartheta_1)}{[1 - G(x; \vartheta_1)]^2} \right) \left(\frac{G(x; \vartheta_1)}{\beta_1 [1 - G(x; \vartheta_1)]} \right)^{\theta_1 - 1} \left[\alpha_1 + \left(\frac{G(x; \vartheta_1)}{\beta_1 [1 - G(x; \vartheta_1)]} \right)^{\alpha_1 \theta_1} \right]^{-\left(\frac{\alpha_1 + 1}{\alpha_1} \right)}}{\frac{\alpha_2 \theta_2}{\beta_2} \left(\frac{g(x; \vartheta_2)}{[1 - G(x; \vartheta_2)]^2} \right) \left(\frac{G(x; \vartheta_2)}{\beta_2 [1 - G(x; \vartheta_2)]} \right)^{\theta_2 - 1} \left[\alpha_2 + \left(\frac{G(x; \vartheta_2)}{\beta_2 [1 - G(x; \vartheta_2)]} \right)^{\alpha_2 \theta_2} \right]^{-\left(\frac{\alpha_2 + 1}{\alpha_2} \right)}}.$$

If $\vartheta_1 = \vartheta_2 = \vartheta$ then

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1 \theta_1 \beta_1^{\alpha_1 \theta_1} G(x; \vartheta)^{\theta_1 - \theta_2}}{\alpha_2 \theta_2 \beta_2^{\alpha_2 \theta_2} \left[1 - G(x; \vartheta)\right]^{\alpha_2 \theta_2 - \alpha_1 \theta_1}}$$

$$\times \frac{\left[\alpha_2 \beta_2^{\alpha_2 \theta_2} \left[1 - G(x; \vartheta)\right]^{\alpha_2 \theta_2} + G(x; \vartheta)^{\alpha_2 \theta_2}\right]^{\left(\frac{\alpha_2 + 1}{\alpha_2}\right)}}{\left[\alpha_1 \beta_1^{\alpha_1 \theta_1} \left[1 - G(x; \vartheta)\right]^{\alpha_1 \theta_1} + G(x; \vartheta)^{\alpha_1 \theta_1}\right]^{\left(\frac{\alpha_1 + 1}{\alpha_1}\right)}}.$$

Also,

$$\log\left(\frac{f_X(x)}{f_Y(x)}\right) = \log\left(\frac{\alpha_1}{\alpha_2}\right) + \log\left(\frac{\theta_1}{\theta_2}\right) + \alpha_1 \theta_1 \log(\beta_1) - \alpha_2 \theta_2 \log(\beta_2)$$

$$+ (\theta_1 - \theta_2) \log(G(x; \vartheta)) - (\alpha_2 \theta_2 - \alpha_1 \theta_1) \log(1 - G(x; \vartheta))$$

$$+ \left(\frac{\alpha_2 + 1}{\alpha_2}\right) \log\left(\alpha_2 \beta_2^{\alpha_2 \theta_2} (1 - G(x; \vartheta))^{\alpha_2 \theta_2} + G(x; \vartheta)^{\alpha_2 \theta_2}\right)$$

$$- \left(\frac{\alpha_1 + 1}{\alpha_1}\right) \log\left(\alpha_1 \beta_1^{\alpha_1 \theta_1} (1 - G(x; \vartheta))^{\alpha_1 \theta_1} + G(x; \vartheta)^{\alpha_1 \theta_1}\right).$$

Differentiating the log of the ratio, we obtain

$$\frac{d}{dx} \log\left(\frac{f_X(x)}{f_Y(x)}\right) = \frac{(\theta_1 - \theta_2)}{G(x; \vartheta)} g(x; \vartheta) + \frac{(\alpha_2 \theta_2 - \alpha_1 \theta_1)}{[1 - G(x; \vartheta)]} g(x; \vartheta)$$

$$+ \theta_2 (\alpha_2 + 1) g(x; \vartheta)$$

$$\times \left(\frac{(G(x; \vartheta)^{\alpha_2 \theta_2 - 1} - \alpha_2 \beta_2^{\alpha_2 \theta_2} (1 - G(x; \vartheta))^{\alpha_2 \theta_2 - 1})}{(\alpha_2 \beta_2^{\alpha_2 \theta_2} (1 - G(x; \vartheta))^{\alpha_2 \theta_2} + G(x; \vartheta)^{\alpha_2 \theta_2})}\right)$$

$$- \theta_1 (\alpha_1 + 1) g(x; \vartheta)$$

$$\left(\frac{(G(x; \vartheta)^{\alpha_1 \theta_1 - 1} - \alpha_1 \beta_1^{\alpha_1 \theta_1} (1 - G(x; \vartheta))^{\alpha_1 \theta_1 - 1})}{(\alpha_1 \beta_1^{\alpha_1 \theta_1} (1 - G(x; \vartheta))^{\alpha_1 \theta_1} + G(x; \vartheta)^{\alpha_1 \theta_1})}\right). \tag{H.1}$$

From equation (H.1) above, we observe computationally that (1) if $\beta_1 = \beta_2 = \beta$, $\theta_1 = \theta_2 = \theta$ and $\alpha_2 > \alpha_1$ then $\frac{d}{dx} \log\left(\frac{f_X(x)}{f_Y(x)}\right) > 0$.

Moreover, (2) if $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $\theta_2 > \theta_1$ then $\frac{d}{dx}\log\left(\frac{f_X(x)}{f_Y(x)}\right) > 0$. In addition, (3) if $\alpha_1 = \alpha_2 = \alpha$, $\theta_1 = \theta_2 = \theta$ and $\beta_1 > \beta_2$ then $\frac{d}{dx}\log\left(\frac{f_X(x)}{f_Y(x)}\right) > 0$. As a result, $\frac{f_X(x)}{f_Y(x)}$ is an increasing function of x. Therefore, X is stochastically larger than Y with respect to likelihood ratio. Hence, from the implication in (9.5) the results for the remaining statements follows.

I. Proof of Proposition 10.1

The proof is similar to that of proposition 5.1.

$$f(x) = \frac{\alpha\theta}{\beta}\left(\frac{g(x)}{[1-G(x)]^2}\right)\left(\frac{G(x)}{\beta[1-G(x)]}\right)^{\theta-1}\left[\alpha + \left(\frac{G(x)}{\beta[1-G(x)]}\right)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)}$$

$$= \frac{\alpha\theta g(x)F(x)}{G(x)[1-G(x)]\left[\alpha + \left(\frac{G(x)}{\beta[1-G(x)]}\right)^{\alpha\theta}\right]},$$

where

$$F(x) = \left[\frac{\left(\frac{G(x)}{\beta[1-G(x)]}\right)^{\alpha\theta}}{\alpha + \left(\frac{G(x)}{\beta[1-G(x)]}\right)^{\alpha\theta}}\right]^{\left(\frac{1}{\alpha}\right)}.$$

Therefore,

$$f^{\delta}(x) = \frac{(\alpha\theta)^{\delta} g^{\delta}(x)F^{\delta}(x)}{G^{\delta}(x)[1-G(x)]^{\delta}\left[\alpha + \left(\frac{G(x)}{\beta[1-G(x)]}\right)^{\alpha\theta}\right]^{\delta}}.$$

By using the identity (B.1) when it is applicable, we get

$$f^{\delta}(x) = \sum_{i=0}^{\infty} (-1)^{i} \theta^{\delta} \alpha^{(\delta+i)} \begin{pmatrix} \frac{\delta}{\alpha} \\ i \end{pmatrix} \left(\frac{g^{\delta}(x)}{G^{\delta}(x)[1-G(x)]^{\delta}} \right) \left(\alpha + \left(\frac{G(x)}{\beta[1-G(x)]} \right)^{\alpha\theta} \right)^{-(\delta+i)}.$$

Also,

$$f^{\delta}(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \theta^{\delta} \alpha^{(\delta+i)} \begin{pmatrix} \frac{\delta}{\alpha} \\ i \end{pmatrix} \binom{-\delta}{j} g^{\delta}(x)$$

$$G^{j-\delta}(x) \left(\alpha + \left(\frac{G(x)}{\beta[1-G(x)]} \right)^{\alpha\theta} \right)^{-(\delta+i)}$$

where

$$G^{-\delta}(x)[1 - G(x)]^{-\delta} = \sum_{j=0}^{\infty} \binom{-\delta}{j} [-G(x)]^{j-\delta}.$$

Then,

$$f^{\delta}(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega(i, j) g^{\delta}(x) G^{j-\delta}(x) \left(\alpha + \left(\frac{G(x)}{\beta[1 - G(x)]} \right)^{\alpha\theta} \right)^{-(\delta+i)}, \quad \text{(I.1)}$$

where $\omega(i, j) = (-1)^{i+j} \theta^{\delta} \alpha^{(\delta+i)} \begin{pmatrix} \frac{\delta}{\alpha} \\ i \end{pmatrix} \binom{-\delta}{j}$.

The equation (I.1) cannot be reduced further since the quantity $\left(\alpha + \left(\frac{G(x)}{\beta[1 - G(x)]} \right)^{\alpha\theta} \right)$ raised to a power $(-(\delta + i))$ that is not an integer.

The Renyi entropy (Renyi et al. [66]) is defined as

$$I_R(\delta) = \frac{1}{1-\delta} \log \left\{ \int_{-\infty}^{\infty} f^{\delta}(x) dx \right\}, \delta > 0 \text{ and } \delta \neq 1. \quad \text{(I.2)}$$

Hence, this proof completed by substituting equation (I.1) into equation (I.2) to get equation (10.1) as required.

J. Proof of Proposition 10.2

From equation (2.2), we get

$$E\{-\log[f(X)]\} = E\left\{-\log\left[\frac{\alpha\theta}{\beta}\left(\frac{g(X)}{[1-G(X)]^2}\right)\left(\frac{G(X)}{\beta[1-G(X)]}\right)^{\theta-1}\right.\right.$$

$$\left.\left.\times\left[\alpha+\left(\frac{G(X)}{\beta[1-G(X)]}\right)^{\alpha\theta}\right]^{-\left(\frac{\alpha+1}{\alpha}\right)}\right]\right\}.$$

Therefore,

$$\begin{aligned} E\{-\log[f(X)]\} &= -\log[\alpha] - \log[\theta] + \theta\log[\beta] - E\{\log[g(X)]\} \\ &\quad -(\theta-1)E\{\log[G(X)]\} + (\theta+1)E\{\log[1-G(X)]\} \\ &\quad +\left(\frac{\alpha+1}{\alpha}\right)E\left\{\log\left[\alpha+\left(\frac{G(X)}{\beta[1-G(X)]}\right)^{\alpha\theta}\right]\right\}. \end{aligned} \tag{J.1}$$

Now,

$$E\{\log[g(X)]\} = \int_{-\infty}^{\infty}\log[g(x)]f(x)dx.$$

From equation (5.2), we obtain

$$\begin{aligned} E\{\log[g(X)]\} &= \sum_{j=1}^{\infty} jw_j \int_{-\infty}^{\infty}\log[g(x)]g(x)[G(x)]^{j-1}dx \\ &= \sum_{j=1}^{\infty} jw_j \int_0^1 \log[g(G^{-1}(u))][u]^{j-1}du. \end{aligned} \tag{J.2}$$

Moreover,

$$\begin{aligned} E\{\log[G(X)]\} &= \sum_{j=1}^{\infty} jw_j \int_{-\infty}^{\infty}\log[G(x)]g(x)[G(x)]^{j-1}dx \\ &= \sum_{j=1}^{\infty} jw_j \int_0^1 \log[u][u]^{j-1}du = -\sum_{j=1}^{\infty}\frac{w_j}{j}. \end{aligned} \tag{J.3}$$

See Prudnikov [65].

Furthermore,

$$E\{\log[1-G(X)]\} = \sum_{j=1}^{\infty} jw_j \int_{-\infty}^{\infty} \log[1-G(x)] g(x) [G(x)]^{j-1} dx$$

$$= \sum_{j=1}^{\infty} jw_j \int_{0}^{1} \log[1-u][u]^{j-1} du$$

$$= \sum_{j=1}^{\infty} w_j [\psi(1) - \psi(j+1)], \tag{J.4}$$

where $\psi(\cdot)$ is the digamma function, see Prudnikov [65].

Also,

$$E\left\{\log\left[\alpha + \left(\frac{G(X)}{\beta[1-G(X)]}\right)^{\alpha\theta}\right]\right\} = \sum_{j=1}^{\infty} jw_j \int_{-\infty}^{\infty} \log\left[\alpha + \left(\frac{G(x)}{\beta[1-G(x)]}\right)^{\alpha\theta}\right]$$

$$\times g(x)[G(x)]^{j-1} dx = \sum_{j=1}^{\infty} jw_j$$

$$\times \int_0^1 \log\left[\alpha + \left(\frac{u}{\beta[1-u]}\right)^{\alpha\theta}\right][u]^{j-1} du. \tag{J.5}$$

By substituting equations (J.2), (J.3), (J.4) and (J.5) into equation (J.1), we obtain

$$E\{-\log[f(X)]\} = -\log[\alpha] - \log[\theta] + \theta\log[\beta] - \sum_{j=1}^{\infty} jw_j \int_0^1 \log[g(G^{-1}(u))][u]^{j-1} du$$

$$+(\theta-1)\sum_{j=1}^{\infty}\frac{w_j}{j} + (\theta+1)\sum_{j=1}^{\infty} w_j[\psi(1)-\psi(j+1)] + \left(\frac{\alpha+1}{\alpha}\right)$$

$$\times \sum_{j=1}^{\infty} jw_j \int_0^1 \log\left[\alpha + \left(\frac{u}{\beta[1-u]}\right)^{\alpha\theta}\right][u]^{j-1} du.$$

After getting a common factor, the proof is completed.

K. Second Partial Derivatives of the Odd Kappa-G family

The second partial derivatives of the log likelihood function with respect to each parameter of the Odd Kappa-G family as follows:

$$U_{\alpha\alpha}=\frac{\partial^2\ell_n}{\partial\alpha^2}=-\frac{n}{\alpha^2}-\left(\frac{2}{\alpha^3}\right)\sum_{i=1}^{n}\log\left[\alpha+\left(\frac{G(x_i;\vartheta)}{\beta(1-G(x_i;\vartheta)]}\right)^{\alpha\theta}\right]+\left(\frac{1}{\alpha^2}\right)$$

$$\times\sum_{i=1}^{n}\frac{1+\theta\log\left[\frac{G(x_i;\vartheta)}{\beta[(1-G(x_i;\vartheta)]}\right]\left(\frac{G(x_i;\vartheta)}{\beta[(1-G(x_i;\vartheta)]}\right)^{\alpha\theta}}{\left[\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}\right]^2}$$

$$\{\alpha(\alpha+3)+\left(\frac{G(x_i;\vartheta)}{\beta[(1-G(x_i;\vartheta)]}\right)^{\alpha\theta}$$

$$\times\left[2+\alpha\theta(\alpha+1)\log\left[\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right]\right\}-\left(\frac{\alpha+1}{\alpha}\right)$$

$$\times\sum_{i=1}^{n}\frac{\left(\theta\log\left[\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right]\right)^2\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}}{\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}}$$

$$U_{\alpha\beta}=\frac{\partial^2\ell_n}{\partial\alpha\partial\beta}=\left(\frac{\theta}{\beta}\right)\sum_{i=1}^{n}\frac{\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}}{\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}}$$

$$\left\{\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}+\alpha\theta(\alpha+1)\log\left[\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right]-1\right\}$$

$$U_{\alpha\theta}=\frac{\partial^2\ell_n}{\partial\alpha\partial\theta}=\sum_{i=1}^{n}\frac{\log\left[\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right]\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}}{\left[\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}\right]^2}$$

$$\left\{1-\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}-\alpha\theta(\alpha+1)\log\left[\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right]\right\}$$

$$U_{\alpha\vartheta} = \frac{\partial^2 \ell_n}{\partial\alpha\partial\vartheta} = \theta\sum_{i=1}^{n}\frac{\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{(\alpha\theta-1)}\left(\frac{G_{\vartheta}^{(1)}(x_i;\vartheta)}{[1-G(x_i;\vartheta)]^2}\right)}{\left[\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}\right]^2}$$

$$\left\{1-\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}-\alpha\theta(\alpha+1)\log\left[\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right]\right\}$$

$$U_{\beta\beta} = \frac{\partial^2 \ell_n}{\partial\beta^2} = \frac{n\theta}{\beta^2}-\frac{\theta(\alpha+1)}{\beta^2}\sum_{i=1}^{n}\frac{\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}}{\left[\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}\right]^2}$$

$$\left\{(1+\alpha\theta)\left(\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}\right)-\alpha\theta\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}\right\}$$

$$U_{\beta\theta} = \frac{\partial^2 \ell_n}{\partial\beta\partial\theta} = -\frac{n}{\beta}+\frac{(\alpha+1)}{\beta}\sum_{i=1}^{n}\frac{\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}}{\left[\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}\right]^2}$$

$$\left\{\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}+\alpha^2\theta\log\left[\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right]\right\}$$

$$U_{\beta\vartheta} = \frac{\partial^2 \ell_n}{\partial\beta\partial\vartheta} = (\alpha+1)\left(\frac{\alpha\theta}{\beta}\right)^2\sum_{i=1}^{n}\frac{\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{(\alpha\theta-1)}\left(\frac{G_{\vartheta}^{(1)}(x_i;\vartheta)}{[1-G(x_i;\vartheta)]^2}\right)}{\left[\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}\right]^2}$$

$$U_{\theta\theta} = \frac{\partial^2 \ell_n}{\partial\theta^2} = -\frac{n}{\theta^2}-(\alpha+1)\sum_{i=1}^{n}\frac{\left(\alpha\log\left[\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right]\right)^2\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}}{\left[\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}\right]^2}$$

$$U_{\theta\vartheta}=\frac{\partial^2\ell_n}{\partial\theta\partial\vartheta}=\sum_{i=1}^{n}\frac{G_{\vartheta}^{(1)}(x_i;\vartheta)}{(1-G(x_i;\vartheta))G(x_i;\vartheta)}-\left(\frac{(\alpha+1)}{\beta}\right)$$

$$\times\sum_{i=1}^{n}\frac{\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{(\alpha\theta-1)}\left(\frac{G_{\vartheta}^{(1)}(x_i;\vartheta)}{[1-G(x_i;\vartheta)]^2}\right)}{\left[\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}\right]^2}$$

$$\left\{\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}+\alpha^2\theta\log\left[\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right]\right\}$$

$$U_{\vartheta\vartheta}=\frac{\partial^2\ell_n}{\partial\vartheta^2}=\sum_{i=1}^{n}\frac{g(x_i;\vartheta)g_{\vartheta}^{(2)}(x_i;\vartheta)-(g_{\vartheta}^{(1)}(x_i;\vartheta))^2}{(g(x_i;\vartheta))^2}$$

$$+\sum_{i=1}^{n}\frac{1}{[G(x_i;\vartheta)(1-G(x_i;\vartheta))]^2}\ \{G(x_i;\vartheta)(1-G(x_i;\vartheta))[(\theta-1)G_{\vartheta}^{(2)}(x_i;\vartheta)$$

$$+2(G(x_i;\vartheta)G_{\vartheta}^{(2)}(x_i;\vartheta)+(G_{\vartheta}^{(1)}(x_i;\vartheta))^2)]$$

$$-(G_{\vartheta}^{(1)}(x_i;\vartheta))^2(1-2G(x_i;\vartheta))(\theta-1+2G(x_i;\vartheta))\}$$

$$-\left(\frac{\theta(\alpha+1)}{\beta^2}\right)\sum_{i=1}^{n}\frac{\left(\frac{1}{[1-G(x_i;\vartheta)]^4}\right)\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{(\alpha\theta-2)}}{\left[\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}\right]^2}$$

$$\left\{(G_{\vartheta}^{(1)}(x_i;\vartheta))^2\left[\alpha(\alpha\theta-1)+2\alpha G(x_i;\vartheta)+(2G(x_i;\vartheta)-1)\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}\right]\right.$$

$$\left.+G(x_i;\vartheta)G_{\vartheta}^{(2)}(x_i;\vartheta)\times(1-G(x_i;\vartheta))\left(\alpha+\left(\frac{G(x_i;\vartheta)}{\beta[1-G(x_i;\vartheta)]}\right)^{\alpha\theta}\right)\right\},$$

where $\mathcal{F}_{\vartheta}^{(1)}(\cdot)$ and $\mathcal{F}_{\vartheta}^{(2)}(\cdot)$ represent the first and second derivatives of the function $\mathcal{F}$ with respect to ϑ, respectively.