Advances and Applications in Mathematical Sciences
Volume 21, Issue 8, June 2022, Pages 4515-4522

GENERALIZED BI-PERIODIC BALANCING NUMBERS

S. SRIRAM[1] and P. VEERAMALLAN[2]

[1]Assistant Professor, National College
(Affiliated to Bharathidasan University)
Tiruchirappalli-620 001, India
E-mail: sriram.priya02@yahoo.com

[2]P. G. Assistant in Mathematics
GHSS, Yethapur, Salem-636 117
Tamilnadu, India
E-mail: veeramallan1986@gmail.com

Abstract

In this paper, we introduce a new generalization of the balancing numbers which we call generalized bi-periodic balancing numbers as

$$B_n^{(a,\,b,\,c)} = \begin{cases} 6aB_{n-1}^{(a,\,b,\,c)} - cB_{n-2}^{(a,\,b,\,c)} & \text{if } n \text{ is even} \\ 6bB_{n-1}^{(a,\,b,\,c)} - cB_{n-2}^{(a,\,b,\,c)} & \text{if } n \text{ is odd} \end{cases}, \; n \geq 2$$

with initial conditions $B_0^{(a,\,b,\,c)} = 0$, $B_1^{(a,\,b,\,c)} = 1$. We find the Generating function for the sequence and produce a Binet's formula.

1. Foreword

Balancing numbers is a natural number n, which satisfies the Diophantine equation $1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r)$ where r is called balancer. A. Behara and G. K. Panda [1] introduced the concept of balancing numbers in 1998. The concept of balancing numbers is closely related to triangular numbers. The positive integer n is called balancing number if and only if n^2 is a triangular number or $[8n]^{2+1}$ is a

2020 Mathematics Subject Classification: 11B99.

Keywords: Balancing Numbers, k-Balancing Numbers, Bi-Periodic Balancing Numbers, Binet formula, Generating function, Catalan identity, Cassini identity.

Received March 24, 2022; Accepted April 19, 2020

perfect square. G. K. Panda [8] generalized balancing numbers by introducing sequence balancing numbers, in which the sequence of numbers used in the definition of balancing numbers is replaced by an arbitrary sequence of real numbers. Thus, if $\{b_n\}$ be a sequence of real numbers, then a_n is called a sequence of balancing numbers if $b_1 + b_2 + \ldots + b_{n-1} = b_{n+1} + b_{n+2} + \ldots + b_{n+r}$ for some natural number r.

As is well known, the balancing sequence $\{b_n\}$ is generated from the recurrence relation $B_{n+2} = 6B_{n+1} - B_n$, $n \geq 0$ with initial states $B_0 = 0$, $B_1 = 1$. Many authors generalized the integer sequence in different ways. As a generalization of the Fibonacci sequence, Edson and Yayenie [3] introduced a bi-periodic Fibonacci sequence $\{p_n\}$ defined by $b_n = \begin{cases} 6cb_{n-1} - b_{n-2}, & \text{if } n \text{ is even} \\ 6db_{n-1} - b_{n-2}, & \text{if } n \text{ is odd} \end{cases}$, $n \geq 2$ with initial conditions $b_0 = 0$, $b_1 = 1$. They also found the generating function and Binet's formula for this sequence.

2. Main Results

Every three real numbers except zero a, b and c, the generalized bi-periodic balancing numbers $\{B_n^{(a, b, c)}\}_{n=0}^{\infty}$ is defined recursively by

$$B_n^{(a, b, c)} = \begin{cases} 6aB_{n-1}^{(a, b, c)} - cB_{n-2}^{(a, b, c)} & \text{if } n \text{ is even} \\ 6bB_{n-1}^{(a, b, c)} - cB_{n-2}^{(a, b, c)} & \text{if } n \text{ is odd} \end{cases}, \; n \geq 2$$

with initial conditions $B_0^{(a, b, c)} = 0$, $B_1^{(a, b, c)} = 1$.

When $a = b = c = 1$, we have the classic balancing numbers. If we ask $a = b = k$ and $c = 1$, for any positive integer, we get the k-balancing number. The first six elements of the generalized bi-periodic balancing numbers are

$$B_0^{(a, b, c)} = 0,\; B_1^{(a, b, c)} = 1,\; B_2^{(a, b, c)} = 6a,\; B_3^{(a, b, c)} = 36ab - c,$$

$$B_4^{(a, b, c)} = 216a^2b - 12ac \text{ and } B_5^{(a, b, c)} = 1296a^2b^2 - 108abc + c^2.$$

The square equation for the generalized bi-periodic balancing numbers is defined as

$$x^2 - 36abx + 36abc = 0$$

with the roots,

$$\alpha = 18ab + 6\sqrt{9a^2b^2 - abc} \text{ and } \beta = 18ab - 6\sqrt{9a^2b^2 - abc} \quad (1)$$

Lemma 2.1. *The generalized bi-periodic balancing numbers* $\{B_n^{(a,b,c)}\}_{n=0}^{\infty}$ *satisfies the following properties*

$$B_{2n}^{(a,b,c)} = (36ab - c)B_{2n-2}^{(a,b,c)} - c^2 B_{2n-4}^{(a,b,c)}$$

$$B_{2n+1}^{(a,b,c)} = (36ab - c)B_{2n-1}^{(a,b,c)} - c^2 B_{2n-3}^{(a,b,c)}$$

Proof of Lemma 2.1. Using the recurrence relation for the generalized bi-periodic balancing numbers we can obtain,

$$\begin{aligned} B_{2n}^{(a,b,c)} &= 6aB_{2n-1}^{(a,b,c)} - cB_{2n}^{(a,b,c)} \\ &= (36ab - c)B_{2n-2}^{(a,b,c)} - 6acB_{2n-3}^{(a,b,c)} \\ &= (36ab - 2c)B_{2n-2}^{(a,b,c)} - c^2 B_{2n-4}^{(a,b,c)} \end{aligned}$$

$$\begin{aligned} B_{2n+1}^{(a,b,c)} &= 6aB_{2n}^{(a,b,c)} - cB_{2n-1}^{(a,b,c)} \\ &= (36ab - c)B_{2n-1}^{(a,b,c)} - 6[B_{2n-1}^{(a,b,c)} + cB_{2n-3}^{(a,b,c)}] \\ &= (36ab - 2c)B_{2n-1}^{(a,b,c)} - c^2 B_{2n-3}^{(a,b,c)}. \end{aligned}$$

Lemma 2.2. *The roots* α *and* β *defined in* (1) *satisfy the following properties.*

(i) $(\alpha - c)(\beta - c) = c^2$

(ii) $\alpha + \beta = 36ab$, $\alpha\beta = 36abc$

(iii) $\alpha - c = \dfrac{\alpha^2}{36ab}$, $\beta - c = \dfrac{\beta^2}{36ab}$

(iv) $(\alpha - c)\beta = \alpha c,\ (\beta - c)\alpha = \beta c$

Proof of Lemma 2.2. By using the definitions of α and β defined in (1), the properties can be easily proved.

Theorem 2.1. *The generating function for the generalized bi-periodic balancing numbers* $\{B_n^{(a, b, c)}\}_{n=0}^{\infty}$ *is*

$$\mathbb{B}(x) = \frac{x[1 + 6ax + cx^2]}{1 - (36ab - 2c)x^2 + c^2x^4}$$

Proof of Theorem 2.1. The formal power series representation of the generating function for $\{B_n^{(a, b, c)}\}_{n=0}^{\infty}$ is

$$\mathbb{B}(x) = B_0^{(a, b, c)} + B_1^{(a, b, c)}x + B_2^{(a, b, c)}x^2 + \ldots + B_r^{(a, b, c)}x^r + \ldots$$

$$= \sum_{m=0}^{\infty} B_m^{(a, b, c)}x^m$$

By multiplying this series by $6bx$ and a^2 respectively, we can get the following series,

$$6bx\mathbb{B}(x) = 6b(B_0^{(a, b, c)}x + B_1^{(a, b, c)}x^2 + \ldots + B_{m-1}^{(a, b, c)}x^m + \ldots)$$

$$6x^2\mathbb{B}(x) = c(B_0^{(a, b, c)}x + B_1^{(a, b, c)}x^2 + \ldots + B_{m-2}^{(a, b, c)}x^m + \ldots)$$

Therefore, we can write,

$$(1 - 6bx + cx^2)B(x) = B_0^{(a, b, c)} + B_1^{(a, b, c)}x - B6b_0^{(a, b, c)}$$

$$+ \sum_{m=2}^{\infty}(B_m^{(a, b, c)} - 6bB_{m-1}^{(a, b, c)} + cB_{m-2}^{(a, b, c)})x^m$$

Since $B_{2m+1}^{(a, b, c)} = 6bB_{2m}^{(a, b, c)} - cB_{2m-1}^{(a, b, c)},\ B_0^{(a, b, c)} = 0,\ B_1^{(a, b, c)} = 1,$

$B_{2m}^{(a, b, c)} = 6aB_{2m-1}^{(a, b, c)} - cB_{2m-2}^{(a, b, c)}$

$$(1-6bx+cx^2)\mathbb{B}(x)=x+6(a-b)x\sum_{m=1}^{\infty}B_{2m-1}^{(a,\,b,\,c)}x^{2m-1}$$

Now we define $\mathfrak{B}(x)$ as

$$\mathfrak{B}(x)=\sum_{m=1}^{\infty}B_{2m-1}^{(a,\,b,\,c)}x^{2m-1}$$

By applying the same way as above, we get

$$[1-(36ab-2c)x^2+c^2x^4]\mathfrak{B}(x)$$

$$=\sum_{m=1}^{\infty}B_{2m-1}^{(a,\,b,\,c)}x^{2m-1}-(36ab-2c)\sum_{m=2}^{\infty}B_{2m-3}^{(a,\,b,\,c)}x^{2m-1}$$

$$+\sum_{m=3}^{\infty}c^2B_{2m-5}^{(a,\,b,\,c)}x^{2m-1}$$

$$=B_1^{(a,\,b,\,c)}x+B_3^{(a,\,b,\,c)}x^3-(36ab-2c)B_1^{(a,\,b,\,c)}x^3$$

$$+\sum_{m=3}^{\infty}(B_{2m-1}^{(a,\,b,\,c)}-(36ab-2c)B_{2m-3}^{(a,\,b,\,c)}+c^2B_{2m-5}^{(a,\,b,\,c)})x^{2m-1}$$

Lemma (2.1) implies that

$$B_{2m-1}^{(a,\,b,\,c)}-(36ab-2c)B_{2m-3}^{(a,\,b,\,c)}+c^2B_{2m-5}^{(a,\,b,\,c)}=0$$

So replacing this in the above expansion gives

$$\mathfrak{B}(x)=\frac{x+cx^3}{1-(36ab-2c)x^2+c^2x^4}$$

Substituting $\mathfrak{B}(x)$ in $\mathbb{B}(x)$ we obtain

$$(1-6bx+cx^2)B(x)=x+6(a-b)x\left[\frac{x+cx^3}{1-(36ab-2c)x^2+c^2x^4}\right]$$

Simplifying this, we have the generating function for the generalized bi-periodic balancing numbers as

$$\mathbb{B}(x) = \frac{x[1 + 6ax + cx^2]}{1 - (36ab - 2c)x^2 + c^2x^4}$$

Theorem 2.2 (Binet Formula). *For every m belong to many natural numbers, the Binet formula for the generalized bi-periodic balancing sequence of numbers is given by*

$$B_m = \frac{(6a)^{1-\zeta(m)}\alpha^m - \beta^m}{(36ab)^{\left\lfloor \frac{m}{2} \right\rfloor}\alpha - \beta}$$

Where $\lfloor m \rfloor$ *is the floor function of m and* $\zeta(m) = m - 2\left\lfloor \frac{m}{2} \right\rfloor$ *is the parity function.*

Proof of Theorem 2.2. We know that the generating function for the generalized bi-periodic numbers $\{B_n^{(a, b, c)}\}_{n=0}^{\infty}$ is given by

$$\mathbb{B}(x) = \frac{x[1 + 6ax + cx^2]}{1 - (36ab - 2c)x^2 + c^2x^4}$$

Using fraction expansion, $\mathbb{B}(x)$ can be written as

$$\mathbb{B}(x) = \frac{1}{c^2}\left[\frac{c\alpha x + 6a(\alpha - c)}{x^2 - \left(\frac{\alpha - c}{c^2}\right)} - \frac{c\beta x + 6a(\beta - c)}{x^2 - \left(\frac{\beta - c}{c^2}\right)}\right]$$

Since the Maclaurin series of the function $\frac{A - BZ}{Z^2 - C}$ is expressed as

$$\frac{A - BZ}{Z^2 - C} = \sum_{n=0}^{\infty}\frac{BZ^{2n+1}}{C^{n+1}} - \sum_{n=0}^{\infty}\frac{aZ^{2n}}{C^{n+1}}$$

The generating function $\mathbb{B}(x)$ can be written as

$$\mathbb{B}(x) = \frac{1}{c^2(\alpha - \beta)}\sum_{m=0}^{\infty}\left[\frac{c\beta}{\left(\frac{\beta - c}{c^2}\right)^{m+1}} - \frac{c\alpha}{\left(\frac{\alpha - c}{c^2}\right)^{m+1}}\right]x^{2m+1}$$

$$+\frac{1}{c^2(\alpha-\beta)}\sum_{m=0}^{\infty}\left[\frac{c\beta}{\left(\frac{\beta-c}{c^2}\right)^{m+1}}-\frac{c\alpha}{\left(\frac{\alpha-c}{c^2}\right)^{m+1}}\right]x^{2m}$$

By using the properties in Lemma (2.2), we get

$$\mathbb{B}(x)=\frac{1}{\alpha-\beta}\sum_{m=0}^{\infty}\frac{1}{(36ab)^m}[\alpha^{2m+1}-\beta^{2m+1}]x^{2m+1}$$

$$+\frac{6a}{\alpha-\beta}\sum_{m=0}^{\infty}\frac{1}{(36ab)^m}[\alpha^{2m}-\beta^{2m}]x^{2m}$$

Use the parity function $\zeta(m)$, and for all $m \geq 0$ the above extension simplifies to

$$\mathbb{B}(x)=\frac{(6a)^{1-\zeta(m)}\alpha^m-\beta^m}{(36ab)^{\left\lfloor\frac{m}{2}\right\rfloor}\alpha-\beta}$$

Theorem 2.3 (Catalan's Identity). *For any two non-negative integer n and r, with $r \leq n$, we have*

$$a^{\zeta(n-r)}b^{1-\zeta(n-r)}B_{n-r}B_{n+r}-a^{\zeta(n)}b^{1-\zeta(n)}B_n^2=-a^{\zeta(r)}c^{n-r}b^{1-\zeta(r)}B_r^r.$$

Proof of Theorem 2.3. Using the Binet formula, we obtain

$$a^{\zeta(n-r)}b^{1-\zeta(n-r)}B_{n-r}B_{n+r}=\frac{a}{(36ab)^{n-1}}\left(\frac{\alpha^{2n}-(\alpha\beta)^{n-r}(\alpha^{2r}+\beta^{2r})+\beta^{2n}}{(\alpha-\beta)^2}\right)$$

and

$$a^{\zeta(n)}b^{1-\zeta(n)}B_n^2=\frac{a}{(36ab)^{n-1}}\left(\frac{\alpha^{2n}-2(\alpha\beta)^n+\beta^{2n}}{(\alpha-\beta)^2}\right)$$

Therefore,

$$a^{\zeta(n-r)}b^{1-\zeta(n-r)}B_{n-r}B_{n+r}-a^{\zeta(n)}b^{1-\zeta(n)}B_n^2$$

$$=\frac{-a^{\zeta(r)}c^{n-r}b^{1-\zeta(r)}}{(36ab)^{n-1}}\frac{(36ab)^{n-r}(36ab)^{r-\zeta(r)}}{(36ab)^{1-\zeta(r)}}B_r^2$$

$$= -a^{\zeta(n)}c^{n-r}b^{1-\zeta(n)}B_r^2$$

This completes the proof.

Theorem 2.4. (Cassini's identity)**.** *For any non-negative integer n, we have*

$$a^{1-\zeta(n)}b^{\zeta(n)}B_{n-1}B_{n+1} - a^{\zeta(n)}b^{1-\zeta(n)}B_n^2 = -ac^{n-1}.$$

Proof of Theorem 2.4. In Catalan's identity, if we use $r = 1$, we get Cassini's identity.

Theorem 2.5. *The non-negative terms of the bi-periodic balancing numbers are defined in terms of the positive terms as*

$$c^m B_{-m} = -B_m$$

Proof of Theorem 2.5.

$$B_{-m} = \frac{(6a)^{1-\zeta(-m)}\alpha^{-m} - \beta^{-m}}{(36ab)^{\left\lfloor\frac{-m}{2}\right\rfloor}\alpha - \beta} = \frac{-1(6a)^{1-\zeta(m)}\alpha^{m} - \beta^{m}}{c^m(36ab)^{\left\lfloor\frac{m}{2}\right\rfloor}\alpha - \beta} = \frac{-1}{c^m}B_m.$$

References

[1] A. Behara and G. K. Panda, On the square roots of triangular numbers, Fibonacci Quarterly 37 (1999), 98.

[2] Y. Choo, Relations between Generalized Bi-periodic Fibonacci and Lucas Sequences, Mathematics (2020), 15-27.

[3] M. Edson and O. Yayenie, A new generalization of Fibonacci sequence and extended Binet's Formula, Integers 9 (2009), 639.

[4] T. Koshy, Fibonacci and Lucas Numbers with applications, Wiley, New York, 2001.

[5] K. Liptai, F. Luca and A. Pinter, Generalized balancing numbers, Indagationes Mathematicae 20(1) (2009), 87-100.

[6] A. Ozkoc and A. Tekcan, On *k*-balancing numbers, Notes on Number Theory and Discrete Mathematics 23(3) (2017), 38-52.

[7] G. K. Panda, Some fascinating properties of balancing numbers, Proc. Elev. Int. Conf. Fibonacci Numbers Their Appl. Congr. Numer 194 (2009), 185-189.

[8] G. K. Panda, Sequence balancing and cobalancing numbers, the Fibonacci Quarterly 45 (2007), 265-271.

[9] D. Tasci and E. Sevgi, Bi-periodic balancing numbers, J. Sci. Arts 50 (2020), 75-84.