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EXPRESSION OF RATIOS OF POLYGONAL NUMBERS AS CONTINUED FRACTIONS

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Abstract

In number theory study of polygonal numbers vary in richness and variety. Also the study of continued fractions is a fast developing field. Here in this study an attempt has been made to represent ratios of polygonal numbers with triangular number, square number, pentagonal number, hexagonal number as basis.

Notations:

1. $\langle p_0, p_1, p_2, p_3, \ldots, p_n \rangle$- continued fraction expansion
2. $p(3, n)$- triangular number
3. $p(4, n)$- square number
4. $p(5, n)$- pentagonal number
5. $p(6, n)$- hexagonal number
6. $p(m, n)$- polygonal number of order and rank 'm' and rang 'n'

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1. Introduction

The theory of numbers has always involved an extraordinary position in the world of mathematics. This is because of the unchallenged verifiable significance of the subject: it is one of only a handful couple of orders having certifiable outcomes which originate before the general thought of a university or an academy. Almost consistently since old style relic has seen new and captivating revelations identifying with the properties of numbers. Number theory is a wide subject with many strong connections with different parts of science. Our craving is to show a fair perspective on the zone.

Each subspecialty has a character interestingly its own, which we have tried to depict precisely.

The Pythagoreans additionally connected numbers with geometry. They presented the possibility of polygonal numbers: triangular numbers, square numbers, pentagonal numbers, and so on. The purpose behind this geometrical terminology is clear when the numbers are spoken to by dots arranged in the form of triangles, squares, pentagons, hexagons and so on. Here in this study an endeavor has been made to present to ratios of polygonal numbers with triangular number, square number, pentagonal number, hexagonal number as basis. If is the number of sides in a polygon, the formula for '*m*'-gonal number with rank '*n*' is

$$p(m, n) = \frac{1}{2}(n^2(m-2) - n(m-4)).$$

1.1. Continued Fraction:

1.2. An expression of the form

$$\frac{a}{b} = p_0 + \cfrac{q_0}{p_1 + \cfrac{q_1}{p_2 + \cfrac{q_2}{p_3 + \cfrac{q_3}{\ddots}}}},$$

where p_i, q_i are real or complex numbers is called a continued fraction.

Theorem 1.2. *The continued fraction of* $\frac{p(3, n)}{p(m, n)} = \langle 0, k, 1, x \rangle$, *where* $x = \frac{n - 2k + 1}{2k}$.

Proof.

$$\frac{p(3,\ n)}{p(m,\ n)} = \frac{(n^2 + n)/2}{(n^2(m-2) - n(m-4))/2} = \frac{n+1}{n(m-2) - (m-4)}.$$

Take $m = k$

$$\frac{p(3,\ n)}{p(m,\ n)} = \frac{n+1}{n(k+1) - (k-1)} = \frac{n+1}{nk + n - k + 1}$$

$$= k + \frac{n - 2k + 1}{n+1}$$

$$= 1 + \frac{2k}{n - 2k + 1}$$

$$= \frac{1}{\dfrac{n - 2k + 1}{2k}}$$

$$= 1 + \frac{1}{x}$$

where

$$x = \frac{n - 2k + 1}{2k}.$$

Theorem 1.3. *The continued fraction of*

$$\frac{p(4,\ n)}{p(m,\ n)} = \begin{cases} \langle 0,\ k,\ 2,\ x\rangle,\ if\ x = \dfrac{n - 2k + 1}{4k - 2} \\ \langle 0,\ k,\ 1,\ x\rangle,\ if\ x = \dfrac{n - k}{k} \end{cases}.$$

Proof.

$$\frac{p(4,\ n)}{p(m,\ n)} = \frac{2n^2}{(n^2(m-2) - n(m-4))/2} = \frac{2n}{n(m-2) - (m-4)}.$$

Case (i) Take $m = 2k$,

$$\frac{p(4,\ n)}{p(m,\ n)} = \frac{2n}{n(2k + 3 - 2) - (2k + 3 - 4)}$$

$$= \frac{2n}{n(2k+1) - (2k-1)}$$

$$= \frac{2n}{2nk + n - 2k + 1}$$

$$= 0 + \frac{1}{\frac{2nk + n - 2k + 1}{2n}}$$

$$= k + \frac{n - 2k + 1}{2n}$$

$$= 2 + \frac{4k - 2}{n - 2k + 1}$$

$$= 2 + \frac{1}{x},$$

where

$$x = \frac{n - 2k + 1}{4k - 2}.$$

Case (ii) Take $m = 2k + 4$

$$\frac{p(4, n)}{p(m, n)} = \frac{2n}{n(2k + 4 - 2) - (2k + 4 - 4)}$$

$$= \frac{2n}{n(2k + 2) - 2k}$$

$$= 0 + \frac{1}{\frac{2nk + 2n - 2k}{2n}}$$

$$= k + \frac{2n - 2k}{2n}$$

$$= k + \frac{1}{\frac{n}{n - k}}$$

$$= 1 + \frac{k}{n - k}$$

$$= 1 + \cfrac{1}{\cfrac{n-k}{k}}$$

$$= 1 + \frac{1}{x},$$

where

$$x = \frac{n-k}{k}.$$

Theorem 1.4. *The continued fraction of*

$$\frac{p(5,\, n)}{p(m,\, n)} = \begin{cases} \langle 0,\, k,\, 3,\, x\rangle,\ if\ x = \dfrac{n-2k+1}{6k-4} \\ \langle 0,\, k,\, 1,\, 1,\, 1,\, x\rangle,\ if\ x = \dfrac{n-4k+1}{6k-2} \\ \langle 0,\, k,\, 1,\, x\rangle,\ if\ x = \dfrac{3n-2k-1}{2k} \end{cases}.$$

Proof.

$$\frac{p(5,\, n)}{p(m,\, n)} = \frac{3n-1}{n(m-2)-(m-4)}.$$

Case (i) Take $m = 3k+3.$

$$\frac{p(5,\, n)}{p(m,\, n)} = \frac{3n-1}{n(3k+3-2)-(3k+3-4)}$$

$$= \frac{3n-1}{n(3k+1)-(3k-1)}$$

$$= \frac{3n-1}{nnk+n-3k+1}$$

$$= k + \frac{n-2k+1}{3n-1}$$

$$= 3 + \frac{6k-4}{n-2k+1}$$

$$= 3 + \cfrac{1}{\cfrac{n-2k+1}{6k-4}}$$

$$= 3 + \frac{1}{x},$$

where $x = \dfrac{n - 2k + 1}{6k - 4}$.

Case (ii) Take $m = 3k + 4$

$$\frac{p(5, n)}{p(m, n)} = \frac{3n - 1}{n(3k + 4 - 2) - (3k + 4 - 4)}$$

$$= \frac{3n - 1}{n(3k + 2) - 3k}$$

$$= \frac{3n - 1}{3nk + 2n - 3k}$$

$$= k + \frac{2n - 2k}{3n - 1}$$

$$= 1 + \frac{n + 2k - 1}{2n - 2k}$$

$$= 1 + \frac{n - 4k + 1}{n + 2k - 1}$$

$$= 1 + \frac{6k - 2}{n - 4k + 1}$$

$$= 1 + \frac{1}{x},$$

where $x = \dfrac{n - 4k + 1}{6k - 2}$.

Case (iii) Take $m = 3k + 5$

$$\frac{p(5, n)}{p(m, n)} = \frac{3n - 1}{n(3k + 5 - 2) - (3k + 5 - 4)}$$

$$= \frac{3n - 1}{n(3k + 3) - (3k + 1)}$$

$$= \frac{3n - 1}{3nk + 3n - 3k - 1}$$

$$= k + \frac{3n - 2k - 1}{3n - 1}$$

$$= 1 + \frac{2k}{3n - 2k - 1}$$

$$= 1 + \frac{1}{x},$$

where $x = \dfrac{3n - 2k - 1}{2k}$.

Theorem 1.5. *The continued fraction of*

$$\frac{p(6, n)}{p(m, n)} = \begin{cases} \langle 0, k, 4, x \rangle, \text{ if } x = \dfrac{n - 2k + 1}{8k - 6} \\ \langle 0, k, 2, x \rangle, \text{ if } x = \dfrac{n - k}{2k - 1} \\ \langle 0, k, 1, 2, 1, x \rangle, \text{ if } x = \dfrac{n - 6k + 1}{8k - 2} \\ \langle 0, k, 1, x \rangle, \text{ if } x = \dfrac{2n - k - 1}{k} \end{cases}.$$

Proof.

$$\frac{p(6, n)}{p(m, n)} = \frac{4n - 2}{n(m - 2) - (m - 4)}.$$

Case (i) Take $m = 4k + 3$

$$\frac{p(6, n)}{p(m, n)} = \frac{4n - 2}{n(4k + 3 - 2) - (4k + 3 - 4)}$$

$$= \frac{4n - 2}{n(4k + 1) - (4k - 1)}$$

$$= \frac{4n - 2}{4nk + n - 4k + 1}$$

$$= k + \frac{n - 2k + 1}{4n - 2}$$

$$= 4 + \frac{8k - 6}{n - 2k + 1}$$

$$= 4 + \frac{1}{x},$$

where $x = \dfrac{n - 2k + 1}{8k - 6}$.

Case (ii) Take $m = 4k + 4$

$$\frac{p(6,\ n)}{p(m,\ n)} = \frac{4n-2}{n(4k+4-2)-(4k+4-4)}$$

$$= \frac{4n-2}{4nk+2n-4k}$$

$$= k + \frac{2n-2k}{4n-2}$$

$$= 2 + \frac{4k-2}{2n-2k}$$

$$= 2 + \frac{1}{\frac{n-k}{2k-1}}$$

$$= 2 + \frac{1}{x},$$

where $x = \frac{n-k}{2k-1}$.

Case (iii) Take $m = 4k+5$

$$\frac{p(6,\ n)}{p(m,\ n)} = \frac{4n-2}{n(4k+5-2)-(4k+5-4)}$$

$$= \frac{4n-2}{n(4k+3)-(4k+1)}$$

$$= \frac{4n-2}{4nk+3n-4k-1}$$

$$= k + \frac{3n-2k-1}{4n-2}$$

$$= 1 + \frac{n+2k-1}{3n-2k-1}$$

$$= 2 + \frac{n-6k+1}{n+2k-1}$$

$$= 1 + \frac{8k-2}{n-6k+1}$$

$$= 1 + \frac{1}{x},$$

where $x = \frac{n-6k+1}{8k-2}$.

Case (iv) Take $m = 4k + 6$

$$\frac{p(6,\, n)}{p(m,\, n)} = \frac{4n - 2}{n(4k + 6 - 2) - (4k + 6 - 4)}$$

$$= \frac{4n - 2}{4nk + 4n - 4k - 2}$$

$$= k + \frac{4n - 2k - 2}{4n - 2}$$

$$= 1 + \frac{2k}{4n - 2k - 2}$$

$$= 1 + \frac{1}{\frac{2n - k - 1}{k}}$$

$$= 1 + \frac{1}{x},$$

where $x = \dfrac{2n - k - 1}{k}$.

2. Remark

It is worth mentioning that, depending on the base considered, the number of patterns followed is lesser by 2 to the order of the polygonal number.

3. Conclusion

In this paper, expression of ratios of polygonal numbers as continued fractions with triangular number, square number, pentagonal number, hexagonal number as basis are considered. Similarly, the other kinds of ratios of polygonal numbers and centered polygonal numbers may be studied in detail. The study can also be extended to higher dimensional figurate numbers.

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