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PREPARATION OF PROMINENT FORMS OF DIOPHANTINE 3-TUPLES RELATING PECULIAR POLYNOMIALS WITH PERTINENT PROPERTIES

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Abstract

In this script, the prominent forms of 3-tuples $\{(x_1, y_1, z_{11}), (x_2, y_2, z_{31}), (x_3, y_3, z_{51}), (x_4, y_4, z_{71})\}$, $\{(x_n, z_{n1}, z_{n2}), (x_n, z_{n2}, z_{n3}), (x_n, z_{n3}, z_{n4}), (x_n, z_{n4}, z_{n5}), \ldots\}$ for $n = 2r-1$, $r \in N$ and $\{(y_n, z_{n1}, z_{n2}), (y_n, z_{n2}, z_{n3}), (y_n, z_{n3}, z_{n4}), (y_n, z_{n4}, z_{n5}), \ldots\}$ for $n = 2r$, $r \in N$ where the elements are Chebyshev polynomials of the second kind and Rook polynomials of a $m \times n$ board such that the product of two figures added by particular quantity stands for a perfect square are evaluated. Also, the MATLAB program for the confirmation of all the forms of 3-tuples supporting unrelated properties is demonstrated.

1. Introduction

"A set of positive integers $(x_1, x_2, \ldots, x_m)$ is called a Diophantine m-tuple if $x_i x_j + n$ is a perfect square for all $1 \le i \le j \le m$ with property $D(n)$, $n \in Z - \{0\}$." In [2, 5], the authors constructed the Diophantine triples connecting polygonal numbers and Cheldhiya companion sequence. In [4], the authors developed Diophantine 3-tuples using Pentatope number. For widespread analysis, one can refer [1, 3, 6]. In this script, the protruding forms of 3-tuples in which the basic data's are Chebyshev polynomials of the second kind and Rook polynomials of a $m \times n$ board such that the product of

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two data's added by certain number stays a perfect square are assessed. Also, the MATLAB program for the validation of all 3-tuples satisfying distinct properties is established.

2. Basic Definitions

2.1. a. Chebyshev polynomials of the second kind is presented by

$$U_n(t) = \binom{n+1}{1} t^n - \binom{n+1}{3} t^{n-2}(1-t^2) + \binom{n+1}{5} x^{n-4}(1-t^2)^2 - \ldots$$

The first few Chebyshev polynomials of the second kind are given by

$$U_0(t) = 1,\ U_1(t) = 2t,\ U_2(t) = 4t^2 - 1,\ U_3(t) = 8t^3 - 4t,\ \ldots$$

b. Rook polynomials of a $m \times n$ board is given by

$$R_{m,n}(t) = \sum_{k=0}^{\min(m,\,n)} \frac{n!\,m!}{k!\,(n-k)(m-k)} t^k.$$

The primary Rook polynomials are assumed by

$$R_{1,1}(t) = t + 1,\ R_{2,2}(t) = 2t^2 + 4t + 1,\ R_{3,3}(t) = 6t^3 + 18t^2 + 9t + 1,\ \ldots$$

3. Preparation of Prominent Forms Diophantine 3-Tuples

3.1. Let us instigate with 2-tuples as $(x_1, y_1) = (U_3(t), R_{1,1}(t)) = (8t^3 - 4t, t+1)$ where $x_1 y_1 + (-4t^4 + 8t^2 + 4t) = (2t^2 + 2t)^2$. Thus, the pair (x_1, y_1) is so-called a Diophantine 2-tuples with the condition $D_1(-4t^4 + 8t^2 + 4t)$. To extend this 2-tuples into 3-tuples, let us élite the fresh non-zero polynomial as z_{11} filling the following double equations.

$$x_1 z_{11} + (-4t^4 + 8t^2 + 4t) = m^2 \quad (1)$$

$$y_1 z_{11} + (-4t^4 + 8t^2 + 4t) = n^2 \quad (2)$$

On finding this particular polynomial z_{11}, let us remark from (1) and (2) as

$$y_1 m^2 - x_1 n^2 = (y_1 - x_1)(-4t^4 + 8t^2 + 4t) \quad (3)$$

Manipulate the succeeding linear conversions in (3)

$$m + S + x_1 M,\ n = S + y_1 M \quad (4)$$

These alterations diminish (3) to the second-degree equation in terms of S and M as

$$S^2 = x_1 y_1 M^2 + (-4t^4 + 8t^2 + 4t) \quad (5)$$

Exploitation of the least positive roots $S = 2(t^2 + t),\ M = 1$ of (5) in (4) affords that $z_{11} = 8t^3 + 4t^2 + t + 1$. Hence, (x_1, y_1, z_{11}) is a triple where as the product of two numbers added with $-4t^4 + 8t^2 + 4t$ is a square number. To find an alternative triple by fixing the pair as (x_1, z_{11}), take z_{12} is an additional non-zero polynomial. Applying the same process as mentioned above, the chance of z_{12} is assessed by $z_{12} = 32t^3 + 8t^2 - 7t + 1$. Therefore, (x_1, z_{11}, z_{12}) is entitled as a Diophantine triple with the property $D_1(-4t^4 + 8t^2 + 4t)$. Similarly, if $(x_1, z_{12}), (x_1, z_{13}), (x_1, z_{14}), (x_1, z_{15})$ etc are 2-tuples, then following the similar procedure as elucidated earlier for each pair they can be prolonged into 3-tuples $(x_1, z_{12}, z_{13}), (x_1, z_{13}, z_{14})$, (x_1, z_{14}, z_{15}) etc with the property $D_1(-4t^4 + 8t^2 + 4t)$. Here $z_{13} = 72t^3 + 12t^2 - 23t + 1; z_{14} = 128t^3 + 16t^2 - 47t + 1; z_{15} = 200t^3 + 20t^2 - 79t + 1$.

Numerical samples of the above triples for few values of *t* are deliberated in table 3.1.1.

Table 3.1.1.

t	D_1	(x_1, z_{11}, z_{12})	(x_1, z_{12}, z_{13})	(x_1, z_{13}, z_{14})	(x_1, z_{14}, z_{15})
1	8	(4,14,34)	(4,34,62)	(4,62,98)	(4,98,142)
2	−24	(56,83,275)	(56,275,579)	(56,579,995)	(56,995,1523)
3	−240	(204,256,916)	(204,916,1984)	(204,1984,3460)	(204,3460,5344)

Next, consider the sequence of 2-tuples as $(y_1, z_{21}), (y_1, z_{22}), (y_1, z_{23})$, etc. Then, by means of the same techniques as enlightened above, each of the 2-tuples can be stretched into 3-tuples $(y_1, z_{21}, z_{22}), (y_1, z_{22}, z_{23})$, etc with the property $D_1(-4t^4 + 8t^2 + 4t)$ where $z_{21} = 8t^3 + 4t^2 + t + 1; z_{22} = 8t^3 + 8t + 4; z_{23} = 8t^3 + 12t^2 + 17t + 9; z_{24} = 8t^3 + 16t^2 + 28t + 16; z_{25} = 8t^3 + 20t^2 + 41t + 25$.

Numerical illustrations of all the triples for some t are pondered in

Table 3.1.2.

t	D_1	(y_1, z_{21}, z_{22})	(y_1, z_{22}, z_{23})	(y_1, z_{23}, z_{24})	(y_1, z_{24}, z_{25})
1	8	(2,14,28)	(2,28,46)	(2,46,68)	(2,68,94)
2	−24	(3,83,116)	(3,116,155)	(3,155,200)	(3,200,251)
3	−240	(4,256,316)	(4,316,384)	(4,384,460)	(4,460,544)

3.2. Consider the pair (x_2, y_2) by $(2U_0(t), R_{3,3}, (t)) = (2, 2t^2 + 4t + 1)$. As in section 3.1, $(x_2, y_2, z_{31}) = (2, 2t^2 + 4t + 1, 2t^2 + 8t + 5)$ and the patterns of 3-tuples $\{(x_2, z_{31}, z_{32}), (x_2, z_{32}, z_{33}), \ldots\}$ are exposed nourishing the property $D_2(-4t - 1)$ where $z_{32} = 2t^2 + 12t + 13; z_{33} = 2t^2 + 16t + 25; z_{34} = 2t^2 + 20t + 41; z_{35} = 2t^2 + 24t + 61$.

The following table 3.2.1 indicates some numerical examples.

Table 3.2.1.

t	D_2	(x_2, z_{31}, z_{32})	(x_2, z_{32}, z_{33})	(x_2, z_{33}, z_{34})	(x_2, z_{34}, z_{35})
1	−5	(2,15,27)	(2,27,43)	(2,43,63)	(2,63,87)
2	−9	(2,29,45)	(2,45,65)	(2,65,89)	(2,89,117)
3	−13	(2,47,67)	(2,67,91)	(2,91,119)	(2,119,151)

Also, it is scrutinized the Diophantine triples with property

$D_2(t^4 - 12t^3 - 4t)$ in which $z_{41} = 2t^2 + 8t + 5; z_{42} = 8t^2 + 24t + 10; z_{43} = 18t^2 + 48t + 17; z_{44} = 32t^2 + 80t + 26; z_{45} = 50t^2 + 120t + 37.$

Corresponding Numerical scheme for few values of t are listed in table 3.2.2 as follows.

Table 3.2.2.

t	D_2	(y_2, z_{41}, z_{42})	(y_2, z_{42}, z_{43})	(y_2, z_{43}, z_{44})	(y_2, z_{44}, z_{45})
1	−5	(7,15,42)	(7,42,83)	(7,83,138)	(7,138,207)
2	−9	(17,29,90)	(17,90,185)	(17,185,314)	(17,314,477)
3	−13	(31,47,154)	(31,154,323)	(31,323,554)	(31,554,847)

3.3. Assume the couple (x_3, y_3) by $(2U_1(t), R_{3,3}, (t)) = (4t, 2t^2 + 4t + 1).$ As in section 3.1, $(x_3, y_3, z_{51}) = (4t, 2t^2 + 4t + 1, 6t^3 + 28t^2 + 25t + 1)$ and a new sequence of triples $\{(x_3, z_{51}, z_{52}), (x_3, z_{52}, z_{53}), \ldots\}$ are perceived with property $D_3(t^4 - 12t^3 - 4t)$. Here $z_{52} = 6t^3 + 38t^2 + 49t + 1; z_{53} = 6t^3 + 48t^2 + 81t + 1; z_{54} = 6t^3 + 58t^2 + 121t + 1; z_{55} = 6t^3 + 68t^2 169t + 1.$

Arithmetical values of all the triples for restricted number of t are charted in table 3.3.1.

Table 3.3.1.

t	D_3	(x_3, z_{51}, z_{52})	(x_3, z_{52}, z_{53})	(x_3, z_{53}, z_{54})	(x_3, z_{54}, z_{55})
1	−15	(4,60,94)	(4,94,136)	(4,136,186)	(4,186,244)
2	−88	(8,211,299)	(8,299,403)	(8,403,523)	(8,523,659)
3	−255	(12,490,652)	(12,652,838)	(12,838,1048)	(12,1048,1282)

Further, the sequence of triples $(y_3, z_{61}, z_{62}), (y_3, z_{62}, z_{63}), (y_3, z_{63}, z_{64})$, etc are created with the same property. Here $z_{61} = 6t^3 + 28t^2 + 25t + 1; z_{62} = 24t^3 + 92t^2 + 64t + 4; z_{63} = 54t^3 + 192t^2 + 121t + 9; z_{64} = 96t^3 + 328t^2 + 196t + 16; z_{65} = 150t^3 + 500t^2 + 289t9 + 25.$

Examples for verification of all the triples for limited number of t are presented in table 3.3.2.

Table 3.3.2.

t	D_3	(y_3, z_{61}, z_{62})	(y_3, z_{62}, z_{63})	(y_3, z_{63}, z_{64})	(y_3, z_{64}, z_{65})
1	−15	(34,60,184)	(34,184,376)	(34,376,636)	(34,636,964)
2	−88	(139,211,692)	(139,692,1451)	(139,1451,2488)	(139,2488,3803)
3	−255	(352,490,1672)	(352,1672,3558)	(352,3558,6148)	(352,6148,9442)

3.4. Consider (x_4, y_4) as $(U_1(t), 2R_1, 1(t)) = (2t, 2t+2)$. Following the procedure as in section 3.1, $(x_4, y_4, z_{71}) = (2t, 2t+2, 8t+8)$ and the patterns of 3-tuples $\{(x_4, z_{71}, z_{72}), (x_4, z_{72}, z_{73}), (x_4, z_{73}, z_{74}), (x_4, z_{74}, z_{75}), \ldots\}$ are recognized with exact property $D_4(8t+9)$ where $z_{72} = 18t+14; z_{73} = 32t+20; z_{74} = 50t+26; z_{75} = 72t+32;$

Illustrations for authentication of the triples for limited number of *t* are arranged in table 3.4.1.

Table 3.4.1.

t	D_4	(x_4, z_{71}, z_{72})	(x_4, z_{72}, z_{73})	(x_4, z_{73}, z_{74})	(x_4, z_{74}, z_{75})
1	17	(2,16,32)	(2,32,52)	(2,52,76)	(2,76,104)
2	25	(4,24,50)	(4,50,84)	(4,84,126)	(4,126,176)
3	33	(6,32,68)	(6,68,116)	(6,116,176)	(6,176,248)

Likewise, the sequence of triples $(y_4, z_{81}, z_{82}), (y_4, z_{82}, z_{83}), (y_4, z_{83}, z_{84})$ etc are invented with property $D_4(8t+9)$ where $z_{83} = 32t+36; z_{84} = 50t+56; z_{85} = 72t+80$

Samples for confirmation of the triples for some *t* are presented in table 3.4.2.

Table 3.4.2.

t	D_4	(y_4, z_{81}, z_{82})	(y_4, z_{82}, z_{83})	(y_4, z_{83}, z_{84})	(y_4, z_{84}, z_{85})
1	17	(4,16,38)	(4,38,68)	(4,68,106)	(4,106,152)
2	25	(6,24,56)	(6,56,100)	(6,100,156)	(6,156,224)
3	33	(8,32,74)	(8,74,132)	(8,132,206)	(8,206,296)

All the above curious forms of Diophantine 3-tuples in which the product of two numbers increased by specific numbers provides a square number is validated by the subsequent MATLAB programming.

```
clc;clear all;close all;t=input('enter the value of t:');
disp('U3(t) and R1(t)');D1=4*(t^4)+8*(t^2)+(4*t);
x1=8*(t^3)-4*t; z11=8*(t^3)+4*(t^2)+t+1;
z12=32*(t^3) +8*(t^2)-(7*t) +1;
z13=72*(t^3) +12*(t^2)-(23*t)+1;
z14=128*(t^3) +16*(t^2)-(47*t)+1;
z15=200*(t^3) +20*(t^2)-(79*t)+1;
y1=t+1;z21=8**(t^3) +4*(t^2)+t+1;
z22=8*(t^3)+8*(t^2)+(8*t)+4;z23=8*(t^3) +12*(t^2)-(17*t)+9;
z24=8*(t^3)+16*(t^2)+(28*t)+16;
z25=8*(t^3)+20*(t^2)+(41*t)+25;
fprintf('D1=%d\n',D1); fprintf('x1 = %d\n',x1);
fprintf ('z11 = %d\n',z11); fprintf('z12 = %d\n',z12);
fprintf('z13 = %d\n',z13); fprintf('z14 = %d\n',z14);
fprintf('z15= %d\n',z15); fprintf('y1 = %d\n',y1);
fprintf('z21 = %d\n',z21); fprintf('z22 = %d\n',z22);
fprintf('z23 = %d\n',z23); fprintf('z24 = %d\n',z24);
fprintf('z25 = %d\n',z25);disp('2U0(t) and R2(t)');
D2=−(4∗t)−1;x2=2;z31=2∗(t2)+(8∗t)+5;
z32=2∗(t^2)+(12∗t)+13;z33=2∗(t2)+(16∗t)+25;
z34=2∗(t^2)+(20∗t)+41;z35=2∗(t2)+(24∗t)+61;
y2=2∗(t^2)+(4∗t)+1;z41=2∗(t2)+(8∗t)+5;
z42=8∗(t^2)+(24∗t)+10;z43=18∗(t2)+(48∗t)+17;
z44=32∗(t^2)+(80∗t)+26;z45=50∗(t2)+(120∗t)+37;
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```
fprintf('D2 = %d\n',D2); fprintf('x2 = %d\n',x2);
fprintf('z31 = %d\n',z31); fprintf('z32 = %d\n',z32);
fprintf ('z33 = %d\n',z33); fprintf('z34 = %d\n',z34);
fprintf ('z35= %d\n',z35); fprintf('y2 = %d\n',y2);
fprintf ('z41 = %d\n',z41); fprintf('z42 = %d\n',z42);
fprintf ('z43 = %d\n',z43); fprintf ('z44 = %d\n',z44);
fprintf ('z45 = %d\n',z45);disp('2U1(t) andR3(t)');
D3=(t^4)−12*(t^3)−(4*t);
x3=4*t;z51=6*(t^3)+28*(t^2)+(25*t)+1;
z52=6*(t^3)+38*(t^2)+(49*t)+1;
z53=6*(t^3)+48*(t^2)+(81*t)+1;
z54=6*(t^3)+58*(t^2)+(121*t)+1;
z55=6*(t^3)+68*(t^2)+(169*t)+1;
y3=6*(t^3)+18*(t^2)+(9*t)+1;
z61=6*(t^3)+28*(t^2)+(25*t)+1;
z62=24*(t^3)+92*(t^2)+(64*t)+4;
z63=54*(t^3)+192*(t^2)+(121*t)+9;
z64=96*(t^3)+328*(t^2)+(196*t)+16;
 z65=150*(t^3)+500*(t^2)+(289*t)+25;
fprintf ('D3 = %d\n',D3); fprintf ('x3 = %d\n',x3);
fprintf ('z51 = %d\n',z51); fprintf ('z52 = %d\n',z52);
fprintf ('z53 = %d\n',z53); fprintf ('z54 = %d\n',z54);
fprintf ('z55= %d\n',z55); fprintf ('y3 = %d\n',y3);
fprintf ('z61 = %d\n',z61); fprintf ('z62 = %d\n',z62);
fprintf ('z63 = %d\n',z63); fprintf ('z64 = %d\n',z64);
fprintf ('z65 = %d\n',z65);disp('U1(t) and 2R1(t)');D4=(8*t)+9;
```

$xx4$=2∗t;$z71$=(8∗t)+8;$z72$=(18∗t)+14;$z73$=(32∗t)+20;

$z74$=(50∗t)+26; $z75$=(72∗t)+32;$y4$=(2∗t)+2; $z81$=(8∗t)+8;

$z82$=(18∗ t)+20; $z83$=(32∗t)+36;$z84$=(50∗t)+56;

$z85$=(72∗ t)+80; *fprintf* ('$D4$ = %d\n',$D4$);

fprintf ('$x4$ = %d\$n$',$x4$); *fprintf* ('$z71$ = %d\$n$',$z71$);

fprintf ('$z72$ = %d\$n$',$z72$); *fprintf* ('$z73$ = %d\$n$',$z73$);

fprintf ('$z74$ = %d\$n$',$z74$); *fprintf* ('$z75$= %$d$\$n$',$z75$);

fprintf ('$y4$= %d\$n$',$y4$); *fprintf* ('$z81$ = %d\$n$',$z81$);

fprintf ('$z82$ = %d\$n$',$z82$); *fprintf* ('$z83$ = %d\$n$',$z83$);

fprintf('$z84$ = %d\$n$',$z84$); *fprintf* ('$z85$ = %d\$n$',$z85$);

4. Conclusion

In this script, the prominent forms of 3-tuples concerning the elements are Chebyshev polynomials of the second kind and Rook polynomials of a $m \times n$ board such that the multiplication of two numbers augmented by an appropriate number remains a perfect square are estimated. Also, the MATLAB program for the validation of all the forms of 3-tuples with disparate properties is displayed. In this manner, one can search variety of patterns of 4-tuples having other polynomials sustaining various properties.

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