Advances and Applications in Mathematical Sciences
Volume 21, Issue 12, October 2022, Pages 6669-6676

POWER DOMINATION ON SHADOW GRAPH OF CERTAIN GRAPHS

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Abstract

A set X consisting of some vertices of the vertex set V of a graph H is called a power dominating set if all the vertices as well as the edges of H are monitored by the vertices of X based on the observation rules of a power monitoring system. The number of vertices of a minimal power dominating set of a graph H is called the power domination number $\gamma_p(H)$. In this paper, we investigate the power domination number γ_p of the shadow graph of certain classes of graphs.

1. Introduction

We consider only a finite, simple, undirected nontrivial graph G with vertex set V and edge set E. The cardinality of V is known as the order of the graph. For a vertex v of G, the neighbourhood $N_G(v)$ is the set of all adjacent vertices of v. A universal vertex v has the property that v along with all its neighbours constitute the vertex set. A subset S of the vertex set of a graph G is called a dominating set [3, 4] if each of the vertices of $V - S$ is adjacent to at least one vertex in S. A dominating set S of a graph G is minimum if the number of vertices of S is least when compared with any other dominating set

2020 Mathematics Subject Classification: 05C15, 05C69.
Keywords: Domination, Power Domination, Shadow graphs.
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Received February 3, 2022; Accepted March 2, 2022

S' of G. The notation $\gamma(G)$ denotes the domination number of G and is the number of elements of a minimum dominating set of G.

As a variant of domination in graphs, the notion of power domination was introduced by Haynes et al. [3], while developing a graph theoretical formulation of a problem concerned with electric power system. There has been a number of studies related to power domination number for common classes of graphs [4] and also on the relationship between domination number and power domination number. A set $S \subseteq V$ is a power dominating set [3] of G if every vertex of V can be recursively observed based on the rules given below:

(i) Initially every vertex in the neighbour set $N[S]$ is observed and (ii) a vertex v is observed by an observed vertex u if v is a neighbour of u and all the neighbours of u are observed vertices except for v.

The notation $\gamma_p(G)$ denotes the power domination number of G and is the number of elements in a minimum power dominating set of G. A power dominating set S of G with the number of elements in $S = \gamma_p(G)$ is referred to as a $\gamma_p(G)$-set.

When operations are performed on graphs, new kinds of graphs result from the initial graphs considered. The shadow is one such operation, having some applications as well [2, 5, 6, 8]. Let $P_n, C_n, W_{1,n}, K_{m,n}, K_{1,n},$ and $B(n, n)$, respectively denote, the path, cycle, graph which is complete, wheel, complete bipartite graph, star graph and bistar graph [1, 9, 10]. For details not explained here, the reader can refer to [1, 7]. Here we obtain the power domination number of shadow graph of some kinds of graphs.

2. Main Results

We recall now the shadow graph of a connected graph.

Definition 1 [2]**.** Let G be a graph which is connected and G' be a copy of G. Let v' be the vertex in G' corresponding to the vertex v of G. The shadow graph $D_2(G)$ of G is a graph obtained by the following operation: Join each vertex v in G to the neighbours of the vertex v' in G' which corresponds to v.

Bounds for the $\gamma(D_2(G))$ are stated in Theorem 2 given below.

Theorem 2. *If G is a graph which is connected and has n vertices, then*

$$\gamma(G) \leq \gamma(D_2(G)) \leq 2\gamma(G).$$

Theorem 3 [4]**.** *If G is a given graph, then* $1 \leq \gamma_p(G) \leq \gamma(G)$.

We obtain bounds for the γ_p of the shadow graph of connected graphs.

Theorem 4. *If G is a graph which is connected and has n vertices,* $n \geq 2$, *then* $1 \leq \gamma_p(G) \leq \gamma_p(D_2(G)) \leq n/2$.

Proof. Consider G, a connected graph with $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$. Let the vertex set of the graph G', which is the copy of G in the formation of the shadow graph $D_2(G)$, be $V(G') = \{v'_1, v'_2, v'_3, \ldots, v'_n\}$ Clearly, the vertex set of $D_2(G)$ is $V \cup V'$. Note that $D_2(G)$ has $2n$ vertices. If $S_{D_2(G)} = X \cup X'$ is a power dominating set for $D_2(G)$ where $X \subseteq V$ and $X' \subseteq V'$, then $S_G = X \cup Y$ is a power dominating set for G such that $X, Y \subseteq V(G)$. So $\gamma_p(G) \leq \gamma_p(D_2(G))$.

Let now S_G^1 be a dominating set of a graph G. Now we claim that, S_G^1 is a minimum power dominating set for $D_2(G)$ because each element in S_G^1 dominates and hence power dominates, the vertices v_i in V and their corresponding vertices v'_i in $V'(1 \leq i \leq n)$. By the definition of power domination, the rest of the vertices of V' are also power dominated by the elements of S_G^1. It is known that [7], for a graph without isolated vertices, $\gamma(G) \leq n/2$. Thus $|S_G^1| = |S_{D_2(G)}| \leq n/2$. Hence $\gamma_p(D_2(G)) \leq n/2$.

Example.

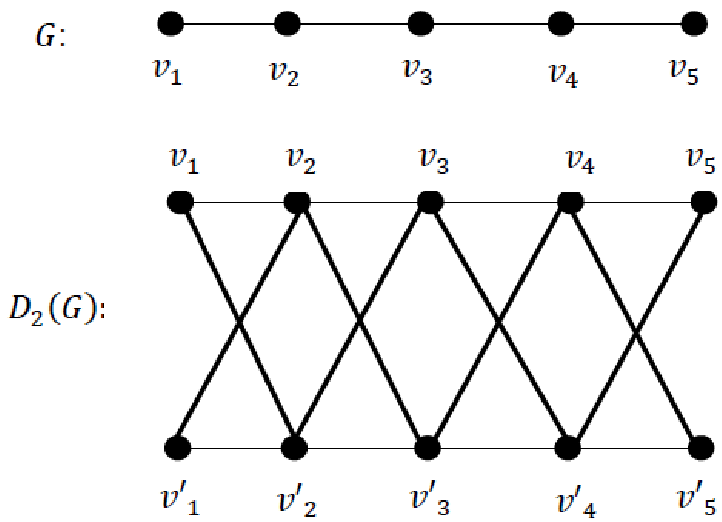

Figure 1. The shadow graph $D_2(G)$ of graph G.

Theorem 5. *For the path P_m on m vertices, $m \geq 2$, we have*

$$\gamma_p(D_2(P_m)) = \left\lceil \frac{m}{3} \right\rceil.$$

Proof. Let $\{v_1, v_2, \ldots, v_m\}$ be the vertex set of P_m. In the graph P'_m which is a copy of P_m in the formation of the shadow graph $D_2(P_m)$. Let the vertex set be $\{v'_1, v'_2, \ldots, v'_m\}$ where v'_r corresponds to v_r, $1 \leq r \leq m$. Then $V(G) = \{v_r, v'_r \mid 1 \leq r \leq m\}$ and $E(G) = \{v_r, v_{r+1} \mid 1 \leq r \leq m-1\} \cup \{v'_r, v'_{r+1} \mid 1 \leq r \leq m-1\} \cup \{v'_r, v_{r+1} \mid 1 \leq r \leq m-1\} \cup \{v'_r, v_{r+1} \mid 2 \leq r \leq m\}$. We note that $|V(D_2(P_m))| = 2m$. while $|E(D_2(P_m))| = 4m-4$. Let S be a minimum power dominating set of the graph $D_2(P_m)$. The following procedure gives the γ_p of $D_2(P_m)$. We first deal with m taking values 2 to 6. When $m = 2$, the shadow graph of path P_2 is isomorphic to C_4. By a known result $\gamma_p(C_m) = 1$. Hence $\gamma_p(D_2(P_m)) = 1$.

When $m = 3$, $S = \{v_2\}$ is a γ_p-set for $D_2(P_3)$ since v_1, v_3, v'_1, v'_3 are dominated initially and other vertices of $D_2(P_3)$ are power dominated by v_2. Hence $\gamma_p(D_2(P_3)) = 1$.

When $m = 4$, the set $S = \{v_2, v_3\}$ is a γ_p-set for $D_2(P_4)$ since v_1, v_4, v_1', v_4' are dominated initially and other vertices of $D_2(P_4)$ are power dominated by S. In this case, S with two vertices are needed.

When $m = 5$, the set $S = \{v_2, v_4\}$ is a γ_p-set for $D_2(P_5)$ since $v_1, v_3, v_5, v_1', v_3', v_5'$ are dominated initially and other vertices of $D_2(P_5)$ are power dominated by the elements of S. In this case S with two vertices are needed.

When $m = 6$, the set $S = \{v_2, v_5\}$ is a γ_p-set for $D_2(P_6)$ since $v_1, v_3, v_4, v_6, v_1', v_3', v_4', v_6'$ are dominated initially and other vertices of $D_2(P_6)$ are power dominated by the elements of S. In this case also S with two vertices are needed.

We now deal with the general case: When $m \geq 7$, we claim that $S = \left\{\{v_2\} \cup \{v_{m-1}\} \cup \left\{v_{3r+2} : 1 \leq r \leq \left\lceil \frac{m-6}{3} \right\rceil\right\}\right\}$ is a γ_p-set for $D_2(P_m)$ because the vertices v_2 and v_{m-1} power dominate the vertices $\{v_1, v_3, v_1', v_3', v_2'\}$ and $\{v_{m-2}, v_m, v_{m-2}', v_m', v_{m-1}'\}$ respectively and $\left\{v_{3r+2} : 1 \leq r \leq \left\lceil \frac{m-6}{3} \right\rceil\right\}$ power dominates the remaining of vertices of $D_2(P_m)$. This is a minimum power dominating set. Thus $\gamma_p(D_2(P_m)) = \frac{m-6}{3} + 2 = \left\lceil \frac{m}{3} \right\rceil$.

Theorem 6. *Consider $C_n, n \geq 3$, a cycle with n vertices. Then* $\gamma_p(D_2(C_m)) = \left\lceil \frac{n}{3} \right\rceil$.

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of C_n. In the graph C_n' which is a copy of C_n in the formation of the shadow graph $D_2(C_m)$, let the vertex set be $\{v_1', v_2', \ldots, v_n'\}$ where v_r' corresponds to v_r, for $1 \leq r \leq n$. Here we note that, $|V(D_2(C_m))| = 2n$ and $|E(D_2(C_m))| = 4n$. The construction of power domination set S of $D_2(C_n)$ is on lines similar to Theorem 5.

Theorem 7 [3]**.** *For the graph P_n, $n \geq 2$, $\gamma_p(G) = 1$.*

Remark 8. From Theorem 5, clearly, the power domination number is 1 only for the graphs $D_2(P_2)$ and $D_2(P_3)$ while the γ_p of all paths P_n, $n \geq 2$ is 1 as noted in Theorem 7. In general, for the graph $G = D_2(P_n)$, $n \geq 4$, $\gamma_p(G) < \gamma_p(D_2(G))$.

3. Characterization of Graphs with $\gamma_p(G) = \gamma_p(D_2(G)) = \gamma(G)$

For a given graph G, we have $\gamma_p(G) \leq \gamma(G)$. We now investigate graphs for which $\gamma_p(G) = \gamma_p(D_2(G)) = \gamma(G)$. In particular, for a wheel, graph which is complete, star, graph which is complete bipartite, and bistar graph we have $\gamma_p(G) = \gamma_p(D_2(G)) = \gamma(G)$.

Theorem 9 [3, 4]**.** *If* $G \in \{W_{1,n}, K_{1,n}, K_n\}$ *and* n *any positive integer, then* $\gamma_p(G) = \gamma(G) = 1$.

Theorem 10. *If* G *is a graph which is connected and has a universal vertex, then* $\gamma_p(D_2(G)) = 1$.

Proof. Consider a graph G with n vertices $v_1, v_2, \ldots, v_n$. Let v_r for some r, $1 \leq r \leq n$ be the universal vertex of G. Then v_r is adjacent to every other vertex of G. It is clear that $S = \{v_r\}$ is a power dominating set for $D_2(G)$, because the vertices of $V(D_2(G)) - \{v'_r\}$ are dominated initially and v'_r is dominated by power domination step.

Corollary given below, directly follows from the Theorem 10.

Corollary 11. $\gamma_p(D_2(W_{1,n})) = \gamma_p(D_2(W_{1,n})) = \gamma_p(K_n) = 1$.

Theorem 12. *For* $K_{m,n}$, $m, n \geq 3$, *a complete bipartite graph, we have*

$$\gamma_p(D_2(K_{m,n})) = 2.$$

Proof. Let $G = D_2(K_{m,n})$ be a shadow graph of $K_{m,n}$. Let $V(G) = \{u_r, v_s, u'_r, v'_s : 1 \leq r \leq m, 1 \leq s \leq n\}$ and $E(G) = \{u_r, v_s, u'_r, v'_s, u_r, v'_s, v_s, u'_r : 1 \leq r \leq m, 1 \leq s \leq n\}$. Here note that $|V(G)| = 2(m+n)$ and $|E(G)| = 8mn$. If $u_r \in \{u_1, u_2, \ldots, u_m\}$ and

$v_s \in \{v_1, v_2, \ldots, v_n\}$ then $S = \{u_r, v_s\}$ is γ_p-set of $G = D_2(K_{m,n})$ since the vertices of $V(G) - \{u'_r : 1 \le r \le m\}$ are power dominated by u_r and the vertices of $V(G) - \{v'_s : 1 \le s \le n\}$ are power dominated by v_s. Therefore $\gamma_p(D_2(K_{m,n})) = 2$.

Definition 13 [10]**.** The bistar $B_{n,n}$ is the graph resulting from two copies of $K_{1,n}$ making their center vertices adjacent.

Theorem 14. *For* $n \ge 2$, $\gamma_p(D_2(B_{n,n})) = 2$.

Proof. Let $G = D_2(B_{n,n})$ be a shadow graph of bistar.

Let $V(G) = \{u, v, u', v'\} \cup \{u_r, v_r, u'_r, v'_r : 1 \le r \le n\}$ and $E(G) = \{uu_r, uv, vv_r, u'u'_r, u'u_r, uu'_r, u'v', v'v'_r, v'v_r, vv'_r, uv', u'v : 1 \le r \le n\}$. Here note that $|V(G)| = 4n + 4$ and $|E(G)| = 8n + 4$. A power dominating set in $D_2(B_{n,n})$ is $S = \{u, v\}$. It is clear that S is a γ_p-set for $D_2(B_{n,n})$ since the vertices $\{u_r, u'_r, v_r, v'_r \mid 1 \le r \le n\}$ are dominated and hence power dominated by $\{u, v\}$. Thus $\gamma_p(D_2(B_{n,n})) = 2$.

4. Conclusion

In this work, we have computed the power domination number of the shadow graph of several standard graphs. Also, we have compared the γ_p of shadow graphs and some standard graphs and obtained some results of interest.

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