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ON THE TRACE OF RANDOM WALKS ON RANDOM GRAPHS USING POISSON (Λ)-GALTON-WATSON TREE

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Abstract

Given a base graph and a starting vertex, we select a vertex uniformly at random from its neighbors and move to this neighbor, then independently select a vertex uniformly at random from that vertex's neighbors and move to it, and so on. The sequence of vertices this process yields is a simple random walk on that graph. The set of edges traversed by this walk is called the trace of the walk, and we consider it as a sub graph of the base graph. In this talk, we shall discuss graph-theoretic properties of the trace of a random walk on a random graph. We will show that if the random graph is dense enough to be typically connected, and the random walk is long enough to typically cover the graph, then its trace is typically Hamiltonian and highly connected. For the special case where the base graph is the complete graph, we will present a hitting time result, according to which, with high probability, exactly one step after the last vertex has been visited, the trace becomes Hamiltonian. Finally, we will present results concerning the appearance of small sub graphs in the trace. The speed of random walks and dimension of harmonic measure on a Poisson (λ)-Galton-Watson tree. Analogous consequences are specified for graphs with arranged degree sequence, where cutoff is shown both for the simple and for the non-backtracking random walk.

Introduction

The conjecture of random graphs has been a powerful investigates area in the previous few decades. It is motivated by the desire to model real-life random graphs such as the internet, digital social networks. One of its aims is also to make rigorous some of the predictions of theoretical physicists in

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statistical and quantum physics. But above all, it furnishes a lot of appealing mathematical concepts and problems related to many fields such as combinatorics, ergodic theory, group theory, percolation. It is impossible to survey the monstrous literature on the subject and the purpose of this course is only to give a very restrictive view of the numerous applications of one particular concept: the local convergence of so-called unbiased random graphs.

A random walk is a finite Markov chain that is time-reversible. Given a graph and a starting point, we select a neighbor of it at random, and move to this neighbor; then we select a neighbor of this point at random, and move to it etc. The (random) sequence of points selected this way is a random walk on the graph. In fact, there is not much difference between the theory of random walks on graphs and the theory of finite Markov chains; every Markov chain can be viewed as random walk on a directed graph, if we allow weighted edges. Similarly, time-reversible Markov chains can be viewed as random walks on undirected graphs, and symmetric Markov chains, as random walks on regular symmetric graphs. In this paper we'll formulate the results in terms of random walks, and mostly restrict our attention to the undirected case.

The time it takes random walk to approach its stationary distribution on a graph is a gauge for an array of properties of the underlying geometry: it reflects the histogram of distances between vertices, both typical and extremal (radius and diameter); it is affected by local traps (e.g., escaping from a bad starting position) as well as by global bottlenecks (sparse cuts between large sets); and it is closely related to the Cheeger constant and expansion of the graph. In this work we study random walk on the giant component C1 of the classical Erdos-Renyi random graph $G(n; p)$, and build on recent advances in our understanding of its geometry on one hand, and random walks on trees and on random regular graphs on the other, to provide sharp results on mixing on C1 and on the related model of a random graph with a prescribed degree sequences.

It is an interesting and natural concept to study for its own sake, but it has also proven to have numerous applications both in combinatorics and in computer science. Indeed, random graphs have been a subject of intensive

study during the last 50 years: thousands of papers and at least three books are devoted to the subject. The term random graph is used to refer to several quite different models, each of which is essentially a distribution over all graphs on n labelled vertices. Perhaps the two most famous models are the classical (models $G(n; m)$, obtained by choosing m edges uniformly at random among the $\binom{n}{2}$ possible edges, and $G(n; p)$, obtained by selecting each edge independently with probability p.

Related Works

In [1] Miklös Ajtai, János Komlös and Endre Szemerédi et al. presents Let us produce a random graph on n vertices by selecting edges one at a time at random and stopping at the first moment that the minimum degree achieves 2. Then with probability 1-0(1), the resulting graph is Hamiltonian. We have proved that a random graph with n vertices and s is hamiltonian with probability e 2 from the empty graph with n labelled vertices, drawing edges one at a time where at each step a completely random selection is made from the remaining possible edges, and stopping at the first moment that all vertices have degree at least two, then we obtain a random graph G, that is, a probability measure on the set of all labelled graphs with n vertices. (Throughout the rest of this paper we shall refer to graphs rather than labelled graphs). This measure is concentrated on the subset of all graphs with minimum degree two, but it is not at all uniform on this set. Permutations of the possible edges and then assigning to each permutation the graph G which is the first such segment of this permutation that has the above mentioned property (that is, the minimum degree is two). This way the measure becomes a counting measure but different graphs will have different multiplicities when counting. In this paper, we shall work with this measure and when saying 'almost all graphs', we mean 1-0(1) with respect to this measure where O (1) is meant as $n + a$. It will be assumed throughout the paper that n is large enough so that all approximations are valid.

In [2] Ben Barber, Eoin Long et al. presents a random walk can be viewed as a random homomorphism of a path; a natural generalization is to consider a random homomorphism of some other tree T (sometimes called a tree-indexed random walk). Just as we traversed a path in one direction, our trees

will be rooted and we think of them as directed "downwards", away from the root. In this section we will explore to what extent the methods of Section 4 can be applied in this more general setting. We seek conditions on T such that we can apply the approach taken with minimal changes. The condition we give here imposes an upper bound on the maximum degree of T. We need an analogue of the second model for the construction of a random walk. Instead of breaking our path into many short paths, we break our tree into many small edge-disjoint sub trees. Suppose that we are trying to pack many trees into a copy of Kn. One approach is to embed some of the trees randomly. If we succeed in packing a small number of trees, then it would be good to know that the sub graph consisting of unused edges has nice enough properties that we can iterate the argument and therefore pack a much larger number of trees.

In [3] Itai Benjamini, Ori Gurel-Gurevich and Russell Lyons et al presents this fact is already known when G is a Euclidean lattice and $c \equiv 1$ since a.s. the paths there have infinitely many cut-times, a time when the past of the path is disjoint from its future. From this, recurrence follows by the criterion of Nash-Williams. By contrast, Lyons and Peres constructed an example of a transient birth-and-death chain which a.s. has only finitely many cut-times. A result of similar spirit to ours was proved by Morris who showed that the components of the wired uniform spanning forest are recurrent. We expect that a Brownian analogue of the theorem is true, that is, a.s. parabolicity of the Wiener sausage, with reflected boundary conditions. For background on recurrence in the Riemannian context, see, for example. It would be interesting to prove similar theorems for other processes. For example, consider the trace of a branching random walk on a graph G.

In [4] Colin Cooper, Alan Frieze et al. presents We consider random walks on numerous lessons of graphs and discover the probable structure of the vacant set, i.e. the place of unvisited vertices. Let $\Gamma(t)$ is the sub graph induced by the vacant set of the walk at step t. We demonstrate that for unsystematic graphs G_n, p (above the connectivity threshold) and for random standard graphs Gr, $r \geq 3$, the graph $\Gamma(t)$ undergo a chapter transition in the intelligence of the well-known Erdos-Renyi phase transition.

Thus for $t \le (1- \in)t^*$, there is a exceptional giant component, plus components of size $O(\log n)$ and for $t \ge (1+ \in)t^*$ all components are of size $O(\log n)$. For Gn, p and Gr we give the value of t^*, and the size of $\Gamma(t)$. For Gr, we also give the degree sequence of $\Gamma(t)$, the size of the giant component (if any) of $\Gamma(t)$ and the number of tree components of $\Gamma(t)$ of a given size $k = O(\log n)$. We also show that for random digraphs t^*, above the strong connectivity threshold, there is a similar directed phase transition. Thus for $t \ge (1- \in)t^*$ there is a exclusive robustly associated massive constituent, plus strongly related mechanism of size $O(\log n)$. and for $t \ge (1+ \in)t^*$ all strongly connected components are of size $O(\log n)$.

In [5] Paul Erdös and Alfréd Rényi et al. presents The Russian version of A collection of problems in probability theory contains a chapter devoted to statistics. That chapter has been omitted in this translation because, in the opinion of the editor, its content deviates somewhat from that which is suggested by the title: problems in probability theory. The original Russian version contains some errors; an attempt was made to correct all errors found, but perhaps a few stiII remain. An index has been further for the expediency of the reader who might be penetrating for a description, a classical trouble, or whatever. The index lists page as well as troubles where the indexed words appear. The manuscript has been translate and abridged with the anticipate of departure as much "Russian flavor" in the transcript and harms as possible. This Collection of problems in probability theory is primarily intended for university students in physics and mathematics departments. Its objective is to assist the student of probability conjecture to master the theory more pro foundly and to acquaint him with the submission of probability theory methods to the resolution of realistic roubles.

Configuration model

The configuration model is designed so that the degree sequences of the vertices are fixed a priori. More precisely, let $d = (d1, \ldots dn)$ be a sequence of integers so that $D = P$ di is even. We will then consider a random (multi) graph on n vertices $\{1, 2, \ldots, n\}$ such that the vertex i has degree di. One way

to sample such a graph is to start with those n vertices to which we attach di half-edges or stubs which are labeled from 1 to D. It have couple those half-edges in the majority natural way: we couple the half-edge number 1 with a uniformly chosen half-edge among those numbered 2, 3, …, D. We then merge the two stubs involved and create a true edge. We then iterate the procedure with the remaining stubs. We indicate by $G(d)$ the random graph obtain. This graph might not be effortless (it may enclose multiple edges or loops).

Sampling by random walks

Probably the most important applications of random walks in algorithm design make use of the fact that (for connected, non-bipartite graphs) the distribution of vt tends to the stationary distribution Π as $t \to \infty$. In most (though not all) cases, G is regular of some degree d, so Π is the uniform distribution. A node of the random walk after sufficiently many steps is therefore essentially uniformly distributed. It is perhaps surprising that there is any need for a non-trivial way of generating an element from such a simple distribution as the uniform. But think of the first application of random walk techniques in real world, namely shuffling a deck of cards, as generating a random permutation of 52 elements from the uniform distribution over all permutations. The problem is that the set we want a random element from is exponentially large (with respect to the natural size of the problem). In many applications, it has in addition a complicated structure; say, we consider the set of lattice points in a convex body or the set of linear extensions of a partial order.

Random graphs with a given degree sequence

The random graphs with a given (deterministic) degree sequence are well studied. A rather surprising fact, due to Molloy and Reed is that the phase transition in its behavior is characterized by a single real parameter computed from a degree sequence.

The symmetric group SV of V acts naturally on the network: the image of an edge being the pair of the image of its adjacent vertices. The (vertex)-auto orphism group of a network $G, Aut(G)$, is the subgroup of SV that leaves the graph invariant. More generally, a bijective map from V to V" defines a

network isomorphism. Then if $G = (V, E)$ and $G' = (V', E')$ are two networks with common mark space Ω, we say that G" and G are isomorphic if G" is the image of G by a network isomorphism. Network isomorphisms define an equivalence relation denoted by"$\simeq$". In graph theory, an equivalence class of simple graphs is called an unlabeled graph. Note that if $G \simeq G''$ then $| Aut(G) | Aut(G'') |$.

Graph terminology

We start with elementary definitions that will be used throughout these notes. Let V be a countable set, and let E be a set of distinct pairs of elements in V. We call an element in V a vertex and an element in the image of E an edge. The sets V and E define a graph $G = (V, E)$. In graph theory, this would rather be called a labeled simple graph but we will stick here to "graph". If E is a multi-set of non-necessarily separate pair of rudiments of V, the pair $\{V, E\}$ is calling a multi-graph. In a multi-graph a loop is an edge $e \in E$ such that for some vertex $v \in V$, $e = \{v, v\}$. An edge $e \notin$ is said to be multiple if e has cardinality larger than 1 in E. Note that a graph is a multigraph with no loop nor multiple edge.

A network or weighted graph $G = (V, E, \omega)$ is a graph (V, E) together with a complete separable metric space Ω called the mark space and a map ω from $V \bigcup E$ to Ω. Images in Ω are called marks. Note that a multigraph is a network with marks on $\Omega = N = \{1, 2, \cdot\}$. For $e = \{u, v\} \in E$, $\omega(e)$ is the number of edges between u and v while $\omega(v)$ counts the number of loops on v. The degree of a vertex $v \in V$, deg (v) or deg $(v; G)$ is the number of edges incident to v with loops counting twice. A (multi)graph is regular if all vertices have the same degree. If a (multi)graph is locally finite if the degree of each vertex is finite. A (multi) graph is finite if the sets V and E are finite. We will denote by $\mathbb{G}(V)$ and $\hat{\mathbb{G}}(V)$ the set of locally finite graphs and multigraph on the vertex set V. If the vertex set is $[n] = \{1, \ldots, n\}$ for some integer n, then we will simply write $\mathbb{G}(n)$ and $\hat{\mathbb{G}}(n)$ in place of $\mathbb{G}([n])$ and $\hat{\mathbb{G}}([n])$.

Uniform graph with given degree sequence

Let $d = (d1, \ldots dn) \in Zn+$ be a sequence of non-negative integers. We say that d is graphic if $G(d)$, the set of graphs G on $[n]$ such that for all $i \in [n]$, $\deg(i; G) = di$, is not empty. If d is explicit, we might then define $G(d)$ as the consistent probability distribution on $G(d)$. It is not entirely obvious how to characterize graphic sequences. This question has been settled by Erdös and Gallai. Here, we may just notice that if d is graphic then $P_n i = 1$ di is even (since it is equal to twice the sum of degrees). An important case is $d = (d, \ldots, d)$ for some $d \geq 2$. In this case, $G(d)$ is the set of d-regular graphs on n vertices. If d is graphic, the probability distribution $G(d)$ will be usually denoted by $G(n, d)$. This probability is called the uniform d-regular graph on n vertices

Local convergence of Erdos-Renyi

Consider the complete graph on n vertices and perform a Bernoulli bond percolation with parameter $p \in (0, 1)$. The graph obtained may have many connected components and we denote by $G \bullet (n, p)$ the random graph made of the connected component of 1 pointed at the vertex 1. For $\lambda > 0$, we also denote by $\lambda > 0$, the random graph obtained from a Galton–Watson tree with offspring distribution Poisson (λ) pointed at the ancestor vertex.

For any $\lambda > 0$ we have the following convergence in distribution for d_{loc}

$$G(n, \lambda/n) \underset{n\to\infty}{\to} T_\lambda$$

Let us sketch an intuitive proof of the last result. First, the number of vertices adjacent to the vertex 1 has a Bin $(n-1, \lambda/n)$ distribution and the latter converges towards a Poisson (λ) distribution. Hence the 1-neighborhood of 1 in $G(n, \lambda/n)$ looks likes the 1-neighborhood of the origin in a Poisson (λ)- Galton–Watson tree. We then pass to the neighbors of 1: Each of these vertices is linked to 1 by definition and the number of other vertices to which it is linked follows a Bin $(n-2, \lambda/n)$ law which is also converging towards

Poisson (λ). The point here is to remark that these variables are roughly independent and that it is very unlikely that when exploring the 2-neighborhood of 1 we discover a cycle, i.e. a vertex linked to two different neighbors of 1. Hence, the 2-neighborhood of 1 looks like the 2-neighborhood of a Poisson (λ)- Galton–Watson tree. We then proceed similarly to explore the 3, 4, 5 ...-neighborhoods of the vertex 1. Although the above sketch can be made rigorous and constitutes the basis of the analysis of the Erdos-Renyi model around criticality, we give another proof using the rigidity of plane tree structure.

The last result together with Theorem 5 is an indication that in $G(n, \lambda/n)$, a dramatic change appears at $\lambda = 1$. Indeed when $\lambda < 1$ a Galton–Watson tree with Poisson (λ) off spring distribution is almost surely finite, whereas for $\lambda > 1$ the later has a positive probability $q_\lambda > 0$ of being infinite. This phase transition in the underlying Galton–Watson reflects into a phase transition in the behavior of the Erdos-Renyi random graph: When $\lambda < 1$ then all the connected components in $G(n, \lambda/n)$ are of size $O(\log n)$ with high probability as $n \to \infty$, whereas for $\lambda < 1$ there exists a unique giant connected component $G(n, \lambda/n)$ of size roughly q_λ n and all the other components are of size O(log n) with high probability as $n \to \infty$.

Erdos-Renyi random graph

Let p be a positive real and n an positive integer, the Erdos-Renyi random graph $G(n, p)$, $0 \le p \le 1$, is a probability distribution on $G(n)$ such that each of the $n(n-1)/2$ possible edge is present independently and with probability $\min(p, 1)$. In other words, if G is a random graph with distribution $G(n, p)$, $0 \le p \le 1$, and H is a graph with n vertices and m edges then

$$\mathbb{P}(\{H\}) = \mathbb{P}(G - H) = p^m (1-p)^{\frac{n(n-1)}{2} - m}$$

In particular, $G(n, 1/2)$ is the uniform measure on $G(n)$. It is momentous to location out that random graph G is homogeneous: for any variation σ in n, $\sigma(G)$ and G contain the identical distribution (in other words G is exchangeable).

The distribution of $\deg(1; G)$ is a Binomial distribution with parameter $n-1$ and p. In particular, the average degree of vertex 1 is

$$\mathbb{E}\deg(1; G) = (n-1)p$$

In these remarks, we will mostly revise the asymptotic property of random graphs with consistently bordered standard degrees. We will thus be largely involved by the probability distribution $G(n, \lambda/n)$ with $\lambda \in R+$. In this case, $\deg(1; G)$ is a Binomial distribution with parameter $n-1$ and λ/n. It follows for all integer k

$$\mathbb{P}(\deg(1; G) = k) = \binom{n}{k}\binom{\lambda}{n}\left(1 - \frac{\lambda}{n}\right)^{n-k}$$

As n goes to infinity, this converges to $e^{-\lambda}\lambda^k/k!$. In other words, we retrieve the well known fact that the Binomial distribution with parameter n and λ/n converges to a Poisson distribution with parameter λ.

Galton–Watson tree

We are given a probability distribution $P = (P_k)_{k\leq 0}$ on $\{0, 1, 2, \ldots, \}$. Informally, a Galton–Watson tree with offspring distribution p is the genealogical tree obtained by starting from a single ancestor particle and such that each particle reproduces independently of the others by following the offspring distribution p.

Galton–Watson tree with immigration

Recall that p is the offspring distribution of T which is supposed to be critical. We denote by p the size biased distribution obtained by putting for k $k \geq 0$

$$\overline{P_k} = kP_k$$

This is indeed a probability distribution thanks to the criticality of p. Notice that p is supported by (strictly) positive integers. We at the present make a random endless tree $\tilde{\tau}$ which is the genealogical tree completed of two sorts of particles: standard and mutant particles. Initially there is only one mutant particle. All particles reproduce independently of each other, and standard particles produce a random number of standard particles

distributed as p. Mutant particles however, reproduce according to p. Among the descendant of a mutant particle, a uniform child is picked (independently of the past) and is declared "mutant" whereas the other children are standard particles. Clearly, in the construction of the plane tree $\tau^{\sim}$ there is a distinguished infinite ray corresponding to the genealogical line of the mutant particles (there is exactly one mutant particle at each level of the tree). We call this distinguished line the "spine" of $\tau^{\sim}$. By the above description, all the trees hanging off this spine are independent p-Galton–Watson trees and so are all a.s. finite.

Conclusion

To examine quite a few motivating graph property (least quantity degree, vertex-connectivity, Hamilton i city) of the trace of a long-enough random walk on a dense-enough random graphs, performance that in the relevant regimes, the trace behaves much like a random graph with a similar density. In the particular container of an absolute graph, we have exposed a beating time result, which is comparable to normal consequences about random graph processes. However, the two models are, in some aspects, very different. For example, an elementary result from random graphs states that the threshold for the appearance of a vertex of degree 2 is $n - 3/2$, whereas the expected density of the trace of the walk on K_n at the moment the maximum degree reaches 2, is of order $n - 2$ (as it typically happens after two steps). It is therefore natural to ask for which graph properties and in which regimes the two models are alike. Further natural questions inspired by our results include asking for the properties of the trace of the walk in different random environments, such as random regular graphs, or in deterministic environments, such as (n, d, λ)- graphs and other pseudo-random graphs. We have decided to leave these questions for a future research.

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