Advances and Applications in Mathematical Sciences
Volume 22, Issue 1, November 2022, Pages 149-157

CERTAIN SEQUELS ON ALMOST EQUILATERAL TRIANGLES

M. MAHALAKSHMI[1], J. KANNAN[2] and G. NARASIMMAN[3]

[1]Full Time Ph. D. Research Scholar
[2,3]Assistant Professor
Department of Mathematics
Ayya Nadar Janaki Ammal College
(Autonomous, affiliated
to Madurai Kamaraj University)
Sivakasi, Tamilnadu, India
E-mail: maha1607laksmi@gmail.com
jayram.kannan@gmail.com

Abstract

An almost equilateral triangle is one with two equal sides and the third differs by no more than one unit. That is, triangles with sides $n, n, n+1$ and $n, n, n-1$ comes under this name. This paper discusses the method to obtain all such triangles with certain conditions imposed on their area, perimeter and so on. Also relations connecting them was presented.

1. Introduction

In this emerging world, mathematics becomes a basic need for everyone. Especially from the starting days of counting to the recent cryptography, number theory masters this world. One of the important and well known branches in number theory is Diophantine Equation. A Diophantine equation is an equation of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ where $a_i \in \mathbb{Z}, 0 \le i \le n, n \in \mathbb{N}$ for which what we need is $x \in \mathbb{Z}$ such that $f(x) = 0$ [1]. Pell's equation is a kind of Diophantine equation, which takes the form $x^2 - dy^2 = 1$ where $d > 0, \sqrt{d} \notin \mathbb{Z}$. [9] discusses the solutions of certain Pell

2020 Mathematics Subject Classification: 11A55, 11B37, 11D09, 11Y65.
Keywords: Almost Equilateral Triangle, Area, Perimeter, Altitude, Pell's Equation.
Received June 29, 2022; Accepted August 30, 2022

Equations.

To solve a Diophantine equation of the form $x^2 - dy^2 = N$ where $n \in \mathbb{Z}$, one can use the solutions of $x^2 - dy^2 = 1$ [7]. By employing suitable linear transformations, many Diophantine equations are reduced to the form $x^2 - dy^2 = 1$ So Pell's equation is a key factor for solving certain kinds of Diophantine equation. This paper also makes use of all these things for its development. [2] discusses more collection of papers relating geometrical shapes. Some of such shapes are Heron Triangles, Heron Parallelograms and Rational Rectangles. Evidences in [2] show us that geometry and number theory are well connected. The key idea behind the development of this paper arises from such sources. One such example is [8] which deals with rational triangles having equal area. In this paper, we define a kind of triangle called almost equilateral triangle, which is somewhat a special case of an isosceles triangle. An almost equilateral triangle is one whose sides are of the form n, n, n, $n \pm 1$ where $n \in \mathbb{N}$. In this paper, some observations on such triangles are presented with the help of Pell's equation, recurrence relations, congruence modulo [9]. Displays the general solutions for some Pell equations. Here we make use of it and find the solution of the equation $x^2 - 3y^2 = 1$. From these solutions, the solutions of the generalized Pell equation $x^2 - 3y^2 = 4$. are developed by employing technique used in [7]. Having solved these equations, we move on to the nature of two terms $3n \pm 1$ and $n \pm 1$. These works are treated as preliminaries. In the main results section, initially, the form of n for which an almost equilateral triangle having integer area is obtained with the help of the lemmas provided in the previous section. After that, almost equilateral triangles with area equal to altitude, square of the area equal to perimeter are shown. Furthermore, height of such triangles are found by fixing the base as $n \pm 1$. Also it is proved that there are no considered type triangles exist with integer inradius and circumradius.

2. Preliminaries

In [9], Tekcan provided general solutions of the Pell equations

$x^2 - Dy^2 = 1$ for some specific values of D. Considering the value $3 = 2^2 - 1$, the following lemma (2.1) arises and lemma (2.2) is a generalization of the earlier one. Lemmas (2.3) and (2.4) discuss the hidden root behind the development of this paper.

Lemma 2.1. *Consider the equation* $x^2 - 3y^2 = 1$. *Let for* $k \geq 1$, (x_k, y_k) *be the solutions of this equation. Then*

i. *The continued fraction of* $\sqrt{3}$ is $[1; \overline{1, 2}]$.

ii. *The fundamental* (*least*) *solution is* (2,1).

iii. *For* $k \geq 1$, *we have*

$$x_{k+1} = 2x_k + 3y_k$$

$$y_{k+1} = x_k + 2y_k$$

iv. *For* $k \geq 4$, *the recurrence relations for* x_k *and* y_k *are*

$$x_k = 3(x_{k-1} + x_{k-2}) - x_{k-3}$$

$$y_k = 3(y_{k-1} + y_{k-2}) - y_{k-3}.$$

Lemma 2.2. *Consider the equation* $m^2 - 3l^2 = 4$.

i. *The fundamental solution is* (4,2)

ii. *Define* (m_k, l_k) *by* $(m_1, l_1) = (4, 2)$ *and for* $k \geq 2$,

where (x_k, y_k) *are solutions of* $x^2 - 3y^2 = 1$. *Then* (m_k, l_k) *is a solution for the given equation.*

iii. *For* $k \geq 2$, *we have*

$$m_{k+1} = 2m_k + 3l_k$$

$$l_{k+1} = m_k + 2l_k$$

iv. For $k \geq 4$, we have

$$m_k = 3(m_{k-1} + m_{k-2}) - m_{k-3} \tag{1}$$

$$l_k = 3(l_{k-1} + l_{k-2}) - l_{k-3}. \tag{2}$$

Proof. i. Since $4^2 - 3(2^2) = 16 - 3(4) = 16 - 12 = 4$, $(4, 2)$ is a solution of $x^2 - 3y^2 = 4$.

ii. It is clear that $m_k^2 - 3l_k^2 = (4x_{k-1} + 6y_{k-1})^2 - 3(2x_{k-1} + 4y_{k-1})^2 = 4$.

iii. From the fact that $m_{k+1} + l_{k+1}\sqrt{3} = (x_1 + y_1\sqrt{3})(m_k + l_k\sqrt{3})$ one can get $m_{k+1} = x_1 m_k + 3y_1 l_k$ and $l_{k+1} = y_1 m_k + x_1 l_k$. The result follows if x_1 is replaced by 2 and y_1 by 1.

iv. Let us prove the result by induction on k. We have $(m_1, l_1) = (4, 2); (m_2, l_2) = (14, 8); (m_3, l_3) = (52, 30); (m_4, l_4) = (194, 112)$. This proves that the equation (1) is true for $k = 4$. Assuming the induction hypothesis and from $m_k = 2m_{k-1} + 3l_{k-1}$, we get the desired result. Similar proof holds for l_k also.

Lemma 2.3. *One cannot find* $n \in \mathbb{N}$ *such that the product of* $3n + 1$ *and* $n - 1$ *is a perfect square, but each of them is not so.*

Proof. Suppose the statement not holds. Then $3n + 1$ and $n - 1$ can be written as

$$3n + 1 = 3^{\alpha} p_1^{\alpha_1} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i} q_1^{\beta_1} q_2^{\beta_2} \ldots q_j^{\beta_j} \tag{3}$$

$$n - 1 = 3^{\beta} p_1^{\gamma_1} p_2^{\gamma_2} \ldots p_i^{\gamma_i} q_1^{\delta_1} q_2^{\delta_2} \ldots q_j^{\delta_j} \tag{4}$$

where each of the pairs $(\alpha, \beta), (\alpha_i, \gamma_i)$ and (β_j, δ_j) are both even or both odd and for each i, j, p_i and q_i are primes congruent to 1 and 2 modulo 3 respectively. From the equation (4), for some $i, n \equiv 1 \pmod{p_i}$ which implies $p_i \equiv 3 \pmod{p_i}$, whereas the equation (3) shows that $3n \equiv p_i - 1 \pmod{p_i}$. Combining these, one can get $p_i \equiv 4 \pmod{p_i}$ which leads to a contradiction. Similar case also holds for q_j for some *j*.

Lemma 2.4. *One cannot find* $n \in \mathbb{N}$ *such that the product of* $3n + 1$ *and* $n - 1$ *is a perfect square, but each of them is not so.*

3. Main Results

The following theorems (3.1) and (3.2) collect all almost equilateral triangles with integer area. In order to find all of them, it is enough to find all n such that the sides $n, n, n+1$ or $n, n, n-1$ gives integer area.

Theorem 3.1. *An almost equilateral triangle with sides* $n, n, n+1$ *having integer area if and only if n must be written in both the forms* $n = 1 + \frac{1}{3}[(U)^k - (V)^k]^2$ *and* $n = \frac{1}{3}[[(U)^k - (V)^k]^2 - 1]$, $k \geq 1$, *where* $U = 2 + \sqrt{3}$ *and* $V = 2 - \sqrt{3}$.

Proof. Consider an almost equilateral triangle with sides $n, n, n+1$ where n is a natural number. Then the semi perimeter s is

$$s = \frac{n + n + n + 1}{2} = \frac{3n + 1}{2}$$

With the help of Heron's formula for area of triangle, we get the area as

$$A = \sqrt{s(s-n)^2(s-n-1)}$$

$$= \frac{n+1}{4}\sqrt{(3n-1)(n-1)}$$

A is an integer if and only if $\sqrt{(3n-1)(n-1)}$ is an integer. By lemma (2.3) when each of $3n+1$ and $n-1$ is not a perfect square, it is clear that the product of $(3n+1)$ and $(n-1)$ is not a perfect square. So both terms as individual a perfect square. Let it be m^2 and l^2 respectively. From this we get, $m^2 - 3l^2 = 4$. Solving the recurrence relations (3) and (4) separately for their general term, one can obtain the desired n. Conversely, if $n = 1 + \frac{1}{3}[(U)^k - (V)^k]^2$ then one can get $n - 1 = 4b^2$ for some integer b, while if $n = \frac{1}{3}[[(U)^k - (V)^k]^2 - 1]$, then one can see that $3n + 1 = 4a^2$ for some integer a. Thus A becomes an integer.

Theorem 3.2. *One cannot find an almost equilateral triangle with sides* $n, n, n-1$ *having integer area.*

Proof. Using Heron's formula and considering area as an integer, the Pell's equation $m^2 - 3l^2 = -4$ was obtained. Suppose there are integers m and l which satisfies this equation. Then we arrive at two cases which have to be true at the same time. One is $m^2 + 4$ leaves remainder 1 or 2 when divided by 3, whereas the another one implies 3 divides $m^2 + 4$. This is a contradiction. Hereafter we present certain theorems which collect and count all almost equilateral triangles with some conditions imposed on them. Also their heights are found with the help of fixed base.

Theorem 3.3. *There are exactly four almost equilateral triangles with area equal to altitude.*

Proof. Since altitude is two times area divided by base, area same as altitude implies base length is 2 units. Consider a triangle with sides $n, n, n+1$. There are two possibilities for the base. If base is $n+1$, then $n = 1$ and so a triangle with sides 1, 1, 2 having the required property is obtained. Similarly, if base is n, then sides of length 2, 2, 3 form a triangle with area equal to altitude. Likewise, 3, 3, 2 and 2, 2, 1 are sides of triangles with sides $n, n, n+1$ whose area equals the altitude.

Theorem 3.4. *Number of almost equilateral triangle whose area square coincides with the perimeter is exactly one.*

Proof. Case 1: Consider the triangle with sides $n, n, n+1$. Then its perimeter is $3n+1$ whereas area is $\frac{n+1}{4}\sqrt{(3n-1)(n-1)}$. Equating $3n+1$ and square of $\frac{n+1}{4}\sqrt{(3n-1)(n-1)}$, one can get $n^3 + n^2 - n - n - 17 = 0$. It is clear that this equation is not solvable over $\mathbb{Z}$. Case 2: If the sides of an almost equilateral triangle is $n, n, n-1$, then its perimeter is $3n-1$ and area is $\frac{n+1}{4}\sqrt{(3n-1)(n-1)}$. As in the first case, the cubic equation $n^3 + n^2 - n - 15 = 0$. was obtained. It has exactly one positive integer solution $n = 3$. Thus 3, 3, 2 is an almost equilateral triangle whose area equals the perimeter. Remark n in theorem (3.1) is odd.

Theorem 3.5. *There are no almost equilateral triangles with sides* $n, n, n+1$, *having area, inradius*(Ir) *and circumradius*(R) *in integers.*

Proof. Let T be an almost equilateral triangle with sides $n, n, n+1$. If A is the area and s is the semi perimeter of T, then $Ir = \frac{A}{s}$. From theorem (3.1), it is clear that

$$Ir = \frac{n+1}{2}\sqrt{\frac{n-1}{3n+1}}$$

Thus Ir is an integer if $\frac{n-1}{3n+1} = \alpha^2$, where α is an integer. But this is not possible since $n-1 < 3n+1$.

For any triangle with sides x, y, z the circumradius R is given by $R = \frac{xyz}{4A}$. Here this gives

$$R = \frac{n^2}{\sqrt{(3n+1)(n-1)}}$$

R is an integer if $\sqrt{(3n-1)(n-1)} \in \mathbb{Z}$ and $\sqrt{(3n-1)(n-1)}$ divides n^2. By theorem (3.1), we've $3n+1 = 4a^2$ and $n-1 = 4b^2$. This shows that $\sqrt{(3n-1)(n-1)}$ is even. But the above remark implies that, n^2 is odd. Hence $\sqrt{(3n-1)(n-1)}$ does not divides n^2.

Theorem 3.6. *If* $n+1$ *is the base of an almost equilateral triangle with sides* $n, n, n+1$, *then its height is given by* $\frac{1}{2}\sqrt{(3n-1)(n-1)}$.

Proof. Let ABC be a triangle such that $AB = BC = n$, $AC = n+1$ and BD be its altitude. Then AD is half of $n+1$ since the common side is not the base. Using Pythagoras theorem on the right angled triangle ADB, one can get

$$n^2 = \left(\frac{n+1}{2}\right)^2 + BD^2$$

$$\Rightarrow BD = \frac{\sqrt{3n^2 - 2n - 1}}{2} = \frac{\sqrt{(3n+1)(n-1)}}{2}.$$

Corollary. *If area of an almost equilateral triangle with sides* $n, n, n+1$ *is an integer in which* $n+1$ *is the base, then its height is an integer.*

Proof. If area is an integer, then from theorem (3.1) $\sqrt{(3n+1)(n-1)}$ is an integer and it is equal to $4ab$ for some integers a and b. Thus by theorem (3.6), height is $2ab$ which is an integer.

Theorem 3.7. *If* $n-1$ *is the base of an almost equilateral triangle with sides* $4ab$ *then its height is given by* $\frac{1}{2}\sqrt{(3n+1)(n-1)}$.

Proof. The result follows by considering a triangle ABC such that $AB = BC = n$, $AC = n-1$ and with altitude BD.

4. Conclusion

In this paper, all almost equilateral triangles with sides $n, n, n+1$ having integer area are collected. Also proved that no triangles having integer area exists with sides $n, n, n-1$. Furthermore, some theorems are proved which relates area and altitude as well as area and perimeter. In addition to that, results on integer inradius and circumradius are studied. Taking this as an initial idea, one may think of another relations and study their properties. In future, this thought may extend to other geometrical shapes also.

References

[1] David M Burton, Elementary Number Theory, 7th edition, McGraw- Hill lnc., (1968).

[2] L. E. Dickson, History of the theory of numbers, Volume II- Diophantine Analysis, Dover Publications, (2015).

[3] Kenneth H. Rosen, Elementary Number Theory and its Applications, Addison Wesley Publishing Company, (1984).

[4] H. W. Lenstra Jr., Solving the Pell equation, Notices of the AMS 49(2) (2002), 182-192.

[5] Manju Somanath and J. Kannan, Congruum Problem, International Journal of Pure and Applied Mathematical Sciences 9(2) (2016), 123-131.

[6] Manju Somanath, K. Raja, J. Kannan and B. Jeyashree, Non Trivial Integral Solutions of Ternary Quadratic Diophantine Equation, Advances and Applications in Mathematical Sciences 19(11) (2020), 1105-1112.

[7] Manju Somanath, K. Raja, J. Kannan and M. Mahalakshmi, On a Class of Solutions for a Quadratic Diophantine Equation, Advances and Applications in Mathematical Sciences 19(11) (2020), 1097-1103.

[8] D. J. Rusin, Rational triangles with equal area, New York J. Math 4 (1998), 1-15.

[9] A. Tekcan, Continued fractions expansion of $\sqrt{D}$ and Pell equation $x^2 - Dy^2 = 1$, Mathematica Moravica 15(2) (2011), 19-27.