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$2n$-ORDER SQUARE COMMUTATIVE GROUPS WITH n-ORDER ELEMENTS

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Abstract

We classify $2n$-order groups and $2n$-order square commutative groups with n-order elements.

1. Introduction

The question of finite-dimensionality of Nichols algebras dominates an important part of the recent developments in the theory of (pointed) Hopf algebras. Heckenberger and Schneider [1] prove that if the Nichols algebra is finite-dimensional, then some elements must satisfy the square commutative law. Therefore, we can define a square commutative group similar to an abelian group since the square commutative law is an important operation rule in the algebra field similar to the law of commutation. These results will have an important impact on Nichols algebras and provide reliable examples and approaches for further research of Group theory.

Throughout, $\mathbb{N}_0 =: \{x \mid x$ is an integer; $x \geq 0\}$. $\mathbb{N}_0 =: \{x \mid x$ is an integer; $x > 0\}$. $|G|$ and $|a|$ denote the orders of group G and element a, respectively.

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2. The Main Results

Definition 2.1 (See [3, Def. 1]). A group G is called a square commutative group if $(ab)^2 = (ba)^2$ for all $a, b \in G$, otherwise, G is called a non square commutative group.

We know an abelian group is a square commutative group.

Lemma 2.2. *The greatest common divisor* $(m+1, m^2+1) = 1$ *or* 2 *for all* $m \in \mathbb{N}_0$.

Proof. Let $a := (m+1, m^2+1)$, then there exists $k \in \mathbb{N}$ such that $m+1 = ka$, thus $m^2 + 1(ka-1)^2 + 1 = k^2a^2 - 2ka + 2$. By $a \mid m^2+1$ we have $a \mid 2$. □

Proposition 2.3. *The* $2n$*-order group with* n*-order elements is isomorphic to one of the following groups*:

(i) *Abelian group of type* $(2n)$;

(ii) $G = \langle a, b \mid a^n = e, b^2 = a^{\frac{kn}{i}}, ba = a^m b\rangle$ *for* $\forall 1 \le m \le n-1, 1 \le k \le l \le n-1$ *with* $(m, n) = 1, (k, l) = 1, \frac{n}{l} \in \mathbb{N}, \frac{(m-1)}{l} \in \mathbb{N}_0$.

Proof. Let G be the $2n$-order group with n-order elements, then $|g| \mid 2n$ for $\forall g \in G$. Let a denote the highest order element in the group G.

(1) If $|a| = 2n$, then G is a $2n$-order cyclic group, i.e. an abelian group of type $(2n)$, where $|a^2| = n$.

(2) If $|a| = n$, then $a^i \in G$ for all $i \in \{0, 1, \ldots, n-1\}$, where $a^0 = e$ is the unit of the G.

There exists $b \in G$ such that $b \ne a^i$ for $\forall i = 0, 1, \ldots, n-1$. By the cancellation law of a group, we know $a^i b \ne a^j$ for all $i, j = 0, 1, \ldots, n-1$, then $G = \{a^i, a^i b \mid i = 0, 1, \ldots, n-1\}$. On the other hand, $ba^i \in G, \forall i = 0,$

$1, \ldots, n-1$ and $ba^i \neq a^j$ for all $i = 0, 1, \ldots, n-1$ since the cancellation law of a group, then there exists m such that $ba = a^m b$ with $1 \leq m \leq n-1$. Thus $ba^i = a^{im}b$ for all $i = 0, 1, \ldots, n-1$. If $(m, n) \neq 1$, then $ba^{\frac{n}{(m,n)}} = a^{\frac{n}{(m,n)}} b = b$, it is a contradiction, so $(m, n) = 1$. It is clear $b^2 \in G$, by the cancellation law of a group, we know $b^2 \neq a^i b$ for all $i = 0, 1, \ldots, n-1$, then there exists r such that $b^2 = a^r$ with $1 \leq r \leq n$,

(2.1) If $|b| = 2l+1, l \leq 1$, we know $b^{2l} \in G, b^{-1} = b^{2l} = a^{rl}$, it is a contradiction.

(2.2) If $|b| = 2l, l \geq 1$, it is clear $\frac{n}{l} \in \mathbb{N}$. We know l and $\frac{n}{(r, n)}$ are the orders of b^2 and a^r, respectively. Thus $l = \frac{n}{(r, n)}$, i.e. $(r, n) = \frac{n}{l}$, set $r = k_{\frac{n}{l}}$, we have $1 \leq k \leq l$ and $(r, n) = \left(k\frac{n}{l}, l\frac{n}{l}\right) = \frac{n}{l}, (k, l) = 1$. Since $b^3 \in G, b^3 = a^{k\frac{n}{l}}b = ba^{k\frac{n}{l}} = a^{k\frac{n}{l}}b$, we obtain $a^{k\frac{n}{l}(m-1)} = e$, i.e. $\frac{k(m-1)}{l} \in \mathbb{N}_0$, then $\frac{(m-1)}{l} \in \mathbb{N}_0$ by $(k, l) = 1$. □

Example 2.4. Assume that G is the 12-order group (not a cyclic group) with 6-order elements, i.e., $n = 6$, then $m = 1$ or 5, $l = 1$ or 2 or 3.

(1) $m = 1, l = 1$, then $G = \langle a, b \,|\, a^6 = e, b^2 = e, ba = ab \rangle$;

(2) $m = 1, l = 2, k = 1$, then $G = \langle a, b \,|\, a^6 = e, b^2 = a^3, ba = ab \rangle$;

(3) $m = 1, l = 3, k = 1$, then $G = \langle a, b \,|\, a^6 = e, b^2 = a^2, ba = ab \rangle$;

(4) $m = 1, l = 3, k = 2$, then $G = \langle a, b \,|\, a^6 = e, b^2 = a^4, ba = ab \rangle$;

(5) $m = 5, l = 1$, then $G = \langle a, b \,|\, a^6 = e, b^2 = e, ba = a^5 b \rangle = D_{12}$ a (the dihedral group);

(6) $m = 5, l = 2, k = 1$, then $G = \langle a, b \mid a^6 = e, b^2 = a^3, ba = a^5 b\rangle$ is isomorphic to $\langle a, b \mid a^4 = e, b^3 = e, ba = b^2 a\rangle$ (see [1, Theor. 5.1 of Section 6]).

Theorem 2.5. *The* $2n$*-order square commutative group with* n*-order elements is isomorphic to one of the following groups*:

(i) *Abelian group of type* $(2n)$;

(ii) $G = \langle a, b \mid a^n = e, b^2 = a^{\frac{kn}{l}}, ba = ab\rangle$ *for* $\forall 1 \le k \le l \le n-1$ *with* $(k, l) = 1, \frac{n}{l} \in \mathbb{N}$.

(iii) $G = \langle a, b \mid a^n = e, b^2 = a^{\frac{kn}{l}}, ba = a^{\frac{n}{2}+1} b\rangle$ *for* $\forall n \ge 4, 1 \le k \le l \le n-1$ *with* $\left(\frac{n}{2}+1, n\right) = 1, (k, l) = 1, \frac{n}{2l} \in \mathbb{N}$.

Proof. Consider Proposition 2.3 (i) and the case $m = 1$ of (ii), G is an abelian group, it is clear. Now $m > 1$. Consider $G = \{a^i, a^i b \mid i = 0, 1, \dots, n-1\}$. By $ba = a^m b$, we have $(a^k a^l b)^2 = a^{(1+m)(k+l)} b^2, (a^l b a^k)^2 = a^{(1+m)(km+l)} b^2$, $(a^k b a^l b)^2 = a^{(1+m^2)(k+lm)} b^4$, so $(a^{k(m-1)(m+1)} = e, a^{(k-l)(1+m^2)(m-1)} = e$ for $\forall k, l \in \{0, 1, \dots, n-1\}$. Thus $a^{(m-1)(m+1)} = e, a^{(m-1)(1+m^2)} e, n \mid (m-1)(m+1)$, $n \mid (m-1)(1+m^2)$, then $\frac{n}{(n, m-1)} \mid (m-1), \frac{n}{(n, m-1)} \mid (1+m^2)$, we obtain $\frac{n}{(n, m-1)} = 2$ since $m < n$ and Lemma 2.2. Then $m - 1 = \frac{n}{2}$. The other is clear by Proposition 2.3. □

Example 2.6. Assume that G is the 12-order square commutative group (not a cyclic group) with 6-order elements, i.e. $n = 6$, then $m = 1$ or 5, $l = 1$ or 2 or 3.

(1) $m = 1, l = 1$, then $G = \langle a, b \mid a^6 = e, b^2 = e, ba = ab\rangle$;

(2) $m - 1, l = 2, k = 1$, then $G = \langle a, b \mid a^6 = e, b^2 = a^3, ba = ab\rangle$;

(3) $m = 1,\ l = 3,\ k = 1,$ then $G = \langle a, b \mid a^6 = e, b^2 = a^2, ba = ab \rangle$;

(4) $m = 1,\ l = 3,\ k = 2,$ then $G = \langle a, b \mid a^6 = e, b^2 = a^4, ba = ab \rangle$.

Proof. Consider Example 2.4. $m = 1$, then G is an abelian group; $m = 5 \neq \frac{6}{2} + 1$, then G is a non square commutative group. □

Proposition 2.7. *Assumed that* $n \in \mathbb{N}$, *then the n-th alternating group* A_n *is a square commutative group if and only if* $n < 4$.

Proof. If $n < 4$, the alternating group A_n is a cyclic group, of course a square commutative group. Now $n \geq 4$, put $(12)(34), (123) \in A_n$, $((12)(34)(123))^2 = (143) \neq (234) = ((123)(12)(34))^2$, then A_n is a non square commutative group. □

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