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ANTI FUZZY PRIME IDEALS IN NEAR-SUBTRACTION SEMIGROUPS

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Abstract

Conceptualisation and characterization of fuzzy prime ideals in near-subtraction semigroups is already carried out. We, in our paper, introduce the concept of anti-fuzzy prime ideals in near-subtraction semigroups. Further, we explore some of its properties.

Introduction

The concept of fuzzy set was introduced by Zadeh [2]. Since then, these ideas have been applied to other algebraic structures such as semigroups, rings, near-rings, subtraction semigroup etc. In [3], Dheena and Mohanraj applied the concept of fuzzy sets to prime ideals in subtraction algebra. They proved various interesting results. In [4], Nagaiah Thandu and Narasiman

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Swamy introduced the concept of anti-fuzzy ideals of near-subtraction semigroups and obtained useful results on it. In this paper, we introduce the concept of anti-fuzzy prime ideals in near-subtraction semigroups and explore some of its characteristics.

Preliminaries

Definition 2.1. A fuzzy subset is the mapping μ from the non-empty set X into the unit interval $[0, 1]$.

Definition 2.2. A fuzzy subset μ of X is called an anti-fuzzy ideal of X if

(i) $\mu(x - y) \leq \max\{\mu(x), \mu(y)\}$.

(ii) $\mu(xy) \leq \mu(y)$,

(iii) $\mu(xy) \leq \mu(x)$, for every $x, y \in X$.

A fuzzy subset with (i) and (ii) is called an anti-fuzzy left ideal of X, whereas a fuzzy subset with (i) and (iii) is called an anti-fuzzy right ideal of X.

Definition 2.3. Let μ and λ be any two fuzzy subsets of X. Then its anti-product $\mu \cdot \lambda$ is defined by, $\mu \cdot \lambda(x) = \begin{cases} \inf\limits_{x=yz} \{\max\{\mu(y), \lambda(z)\}\} & \text{if } x = yz \\ 0 & \text{otherwise.} \end{cases}$

Definition 2.4. For any fuzzy subset μ in X and $t \in [0, 1]$. We define an lower t-level cut (anti-level cut) of μ is defined by, $L(\mu, t) = \{x / x \in X, \mu(x) \leq t\}$.

Definition 2.5. Let I be a subset of X. Define an anti-characteristic function $\chi_{A^c} : A \to [0, 1]$ by, $\chi_{A^c}(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{otherwise} \end{cases}$, for every $x \in X$.

Anti-fuzzy Prime Ideals in Near-subtraction Semigroups

Definition 3.1. A anti-fuzzy ideal μ is called a anti-fuzzy prime ideal of X if for any two anti-fuzzy ideals σ and δ of X such that $\sigma \cdot \delta \geq \mu \Rightarrow \sigma \geq \mu$ (or) $\delta \geq \mu$.

Example 3.1.1. Let $X = \{0, x, y, z\}$ with "–" and "·" are defined as,

–	0	x	y	z
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
z	z	y	x	0

–	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	y	y
z	0	x	y	z

Let μ, σ and δ be fuzzy subsets of X such that,

$$\mu(0) = 0.1, \ \mu(x) = 0.4, \ \mu(y) = 0.5, \ \mu(z) = 1$$

$$\sigma(0) = 0.3, \ \sigma(x) = 0.6, \ \sigma(y) = 0.8, \ \sigma(z) = 1$$

$$\delta(0) = 0.2, \delta(x) = 0.5, \delta(y) = 0.7, \delta(z) = 1.$$

Clearly, μ is an anti-fuzzy prime ideal of X.

Example 3.1.2. Let $X = \{0, 1, 2, 3\}$ with "–" and "·" are defined as,

–	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	2	1	3

–	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	0	0	0
3	0	1	2	3

Let μ, σ and δ be fuzzy subsets of X such that,

$$\mu(0) = 0,\ \mu(1) = 0.4,\ \mu(2) = 0.4,\ \mu(3) = 1$$

$$\sigma(0) = 0,\ \sigma(1) = 0.8,\ \sigma(2) = 0,\ \sigma(3) = 0.8$$

$$\delta(0) = 0,\ \delta(1) = 0.8,\ \delta(2) = 0,\ \delta(3) = 0.8.$$

Clearly, μ is not an anti-fuzzy prime ideal of X.

Theorem 3.2. *Arbitrary union of an anti-fuzzy prime ideals of X is also an anti-fuzzy prime ideal of X.*

Proof. Let $\{\mu_i / i \in \Omega\}$ be the set of all anti-fuzzy prime ideals in X.

To prove: $\mu = \bigcup_{i \in \Omega} \mu_i$ is also an anti-fuzzy prime ideal. Let σ and δ be any anti-fuzzy ideals of X such that $\sigma \cdot \delta \geq \bigcup_{i \in \Omega} \mu_i \Rightarrow \sigma \cdot \delta \geq \mu_i$, for some $i \in \Omega$. Since each μ_i is an anti-fuzzy prime ideal. Therefore, $\sigma \geq \mu_i$ (or) $\delta \geq \mu_i$, for some $i \in \Omega$. (i.e.) $\sigma \geq \bigcup_{i \in \Omega} \mu_i$ (or) $\delta \geq \bigcup_{i \in \Omega} \mu_i$.

Theorem 3.3. *Arbitrary intersection of an anti-fuzzy prime ideal of X is also an anti-fuzzy prime ideal of X.*

Proof. Let $\{\mu_i / i \in \Omega\}$ be the set of all anti-fuzzy prime ideals in X.

To prove: $\mu = \bigcap_{i \in \Omega} \mu_i$ is also an anti-fuzzy prime ideal. Let σ and δ be any anti-fuzzy ideals of X such that $\sigma \cdot \delta \geq \bigcap_{i \in \Omega} \mu_i \Rightarrow \sigma \cdot \delta \geq \mu_i$, for all $i \in \Omega$.

Since each μ_i is an anti-fuzzy prime ideal. Therefore, $\sigma \geq \mu_i$ (or) $\delta \geq \mu_i$, for all $i \in \Omega$. (i.e.) $\sigma \geq \bigcap_{i \in \Omega} \mu_i$ (or) $\delta \geq \bigcap_{i \in \Omega} \mu_i$.

Theorem 3.4. *If μ is an anti-fuzzy prime ideal of X then the finitely generated set X_μ is a prime ideal of X.*

Proof. Assume that μ is an anti-fuzzy prime ideal of X.

By Theorem 2.11 in [1], X_μ is an ideal of X. To prove: X_μ is a prime ideal of X. Let A and B be any two ideals in X such that $AB \subseteq X_\mu$. We have to prove $A \subseteq X_\mu$ or $B \subseteq X_\mu$. Define the fuzzy subsets σ and δ of X as,

$\sigma(x) = \begin{cases} \mu(0) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ $\delta(x) = \begin{cases} \mu(0) & \text{if } y \in B \\ 0 & \text{if } y \notin B \end{cases}$

By Theorem 2.12 in [1], σ and δ are anti-fuzzy ideals. Next we verify that $\sigma \cdot \delta \geq \mu$. Since $\sigma \cdot \delta(a) = \begin{cases} \inf_{a=bc} \{\max\{\sigma(b), \delta(c)\}\} & \text{if } a = bc \\ 0 & \text{otherwise} \end{cases} \Rightarrow \sigma(b) = \delta(c) = \mu(0)$. So $b \in A$ and $c \in B$. Now, $a = bc \in AB \subseteq X_\mu$.

(i.e.) $a \in X_\mu \Rightarrow \mu(a) = \mu(0)$. Hence, $\sigma \cdot \delta(a) \geq \mu(a), \forall a \in X$. Thus $\sigma \cdot \delta \geq \mu$. Since μ is a prime anti-fuzzy bi-ideal, so we have that $\sigma \geq \mu$ (or) $\delta \geq \mu$. Suppose $\sigma \geq \mu$. If $A \not\subseteq X_\mu$, then there exists $x \in A$ such that $x \notin X_\mu$. This means that $\mu(x) \neq \mu(0)$. Already We know that, $\mu(0) \leq \mu(x), \forall x \in X$. But $\mu(0) \neq \mu(x)$ and so $\mu(0) < \mu(x)$. Now, $\sigma(x) = \mu(0) < \mu(x)$. Which is a contradiction to $\sigma \geq \mu$. Hence $A \subseteq X_\mu$. Similarly, If $\delta \geq \mu$, then we can show that $B \subseteq X_\mu$. This shows that X_μ is a prime bi-ideal of *X*.

Theorem: 3.5. *Let I be an ideal of X and μ be a fuzzy set in X defined by,* $\mu_I(x) = \begin{cases} s & x \in I \\ 1 & otherwise \end{cases}$, *for all* $x \in X$ *and* $s \in [0, 1)$. *Then* μ_I *is an anti-fuzzy prime ideal of X iff I is a prime ideal of X.*

Proof. Suppose *I* is a prime ideal of *X*. To prove: μ_I is a anti-fuzzy prime ideal of *X*. By Theorem 2.12 in [1], μ_I is an anti-fuzzy ideal of *X*.

Let σ and δ be two anti-fuzzy ideals of *X* such that $\sigma \cdot \delta \geq \mu_I$.

To prove: $\sigma \geq \mu_I$ (or) $\delta \geq \mu_I$. Suppose not, (i.e.) $\sigma < \mu_I$ and $\delta < \mu_I$.

Then $\sigma(x) < \mu_I(x)$ and $\delta(y) < \mu_I(y)$, $\forall x, y \in X$.

Now, $\mu_I(x) \neq s$ and $\mu_I(y) \neq s \Rightarrow \mu_I(x) = \mu_I(y) = 1$ and so $x, y \notin I$.

Since *I* is a prime ideal, we have that $\langle x \rangle \langle y \rangle \not\subset I$.

Then, $1 = \mu_I(a) \leq \sigma \cdot \delta(a)$. Since $a = cd$, where $c = \langle x \rangle$ and $d = \langle y \rangle$.

Now,

$$\sigma \cdot \delta(a) = \inf_{a=cd}\{\max\{\sigma(c),\ \delta(d)\}\} \leq \max\{\sigma(c),\ \delta(d)\}$$
$$\leq \max\{\sigma(x),\ \delta(y)\}$$
$$< \max\{\mu_I(x),\ \mu_I(y)\} = 1 = \mu_I(a).$$

Therefore $\sigma \cdot \delta > \mu_I$. Which is a contradiction.

Hence, μ_I is an anti-fuzzy prime ideal of X.

Corollary 3.6. *Let* χ_{I^c} *be an anti-characteristic function of a subset* $I \subseteq X$. *Then* χ_{I^c} *is an anti-fuzzy prime ideal iff I is a prime ideal of X.*

Theorem 3.7. *If* μ *is an anti-fuzzy prime ideal of X then* $\mu(c) = 1$, *where c denotes the last element of the X.*

Proof. Suppose μ is an anti-fuzzy prime ideal of X. To prove: $\mu(c) = 1$. Suppose not, (i.e.) $\mu(c) < 1$. Define the fuzzy subsets σ and δ as, $\forall x \in X,\ \sigma(x) = \mu(0)$ and $\delta(x) = \begin{cases} 0 & \text{if } \mu(x) = \mu(0) \\ 1 & \text{otherwise} \end{cases}$. Since σ is a constant function, σ is an anti-fuzzy ideal. Note that, δ is the anti-characteristics function of X_μ. By Theorem: 2.12 in [1], μ is the anti-fuzzy ideal of X. Since $\delta(0) = 0 < \mu(c)$ and $\sigma(a) = \mu(0) < \mu(a)$. We have that, $\sigma \not\geq \mu$ and $\delta \not\geq \mu$. Let $b \in X$. WKT, $\sigma \cdot \delta(b) = \begin{cases} \inf_{b=cd}\{\max\{\sigma(c),\ \delta(d)\}\} & \text{if } b = cd \\ 0 & \text{otherwise} \end{cases}$

Now, we prove, $\max\{\sigma(c),\ \delta(d)\} \geq \mu(b)$, where $b = cd$.

For this, we consider two cases, $\delta(x) = 0$ and $\delta(x) = 1$ in the following:

Case (i) Suppose $\delta(x) = 0$.

Now, $\max\{\sigma(c),\ \delta(d)\} = \max\{\mu(c),\ 0\} = \mu(c) \geq \mu(cd) = \mu(b)$.

Case (ii) Suppose $\delta(x) = 1$. Then $\mu(x) = \mu(0)$.

Now,

$$\max\{\sigma(c),\ \delta(d)\} = \max\{\mu(c),\ 1\} = 1$$
$$\geq \mu(cd) = \mu(b).$$

From this, we conclude that, $\sigma \cdot \delta(b) = \max\{\sigma(c), \delta(d)\} \geq \mu(b)$ and so $\sigma \cdot \delta \geq \mu$. Since μ is an anti-fuzzy prime ideal, we have $\sigma \geq \mu$ (or) $\delta \geq \mu$.

Which is a contradiction to $\sigma \ngeq \mu$ and $\delta \ngeq \mu$. Hence, $\mu(c) = 1$.

Theorem 3.8. *If* μ *is an anti-fuzzy prime ideal of X then*, $|\operatorname{Im}(\mu)| = 2$. *Moreover*, $\operatorname{Im}(\mu) = \{s, 1\}$, *where* $0 \leq s < 1$.

Proof. Suppose μ is an anti-fuzzy prime ideal of X. To prove: $\operatorname{Im}(\mu)$ contains exactly two values. We know that, by previous Theorem 3.7, $\mu(c) = 1$. Let a and b be two elements of X such that, $\mu(a) < 1$ and $\mu(b) < 1$. Enough to prove: $\mu(a) = \mu(b)$.

Part (i)

Define the fuzzy subsets σ and δ as, $\forall x \in X$ and $a \in X$

$$\sigma(x) = \mu(a) \text{ and } \delta(x) = \begin{cases} 0 & \text{if } x \in \langle a \rangle \\ 1 & \text{otherwise} \end{cases}.$$

By Theorem 2.12 in [1], σ and δ are anti-fuzzy prime ideals of X.

Since $a \in \langle a \rangle$, we have $\delta(a) = 0 < \mu(a)$ and so $\delta \ngeq \mu$. Let $z \in X$. We know that, $\sigma \cdot \delta(z) = \begin{cases} \inf_{z=ab}\{\max\{\sigma(a), \delta(b)\}\} & \text{if } z = ab \\ 0 & \text{otherwise} \end{cases}$. If $x \notin \langle a \rangle$, then $\delta(x) = 1$

$$\Rightarrow \max\{\delta(x), \delta(y)\} = \max\{\mu(a), 1\} = 1 \geq \mu(ab) = \mu(z).$$

If $x \in \langle a \rangle$, then $\delta(x) = 0$.

$$\Rightarrow \max\{\sigma(x), \delta(y)\} = \max\{\mu(a), 0\} = \mu(a) \geq \mu(ab) = \mu(z).$$

From these, we conclude that $\sigma \cdot \delta \geq \mu$. Since μ is an anti-fuzzy prime ideal, we have $\sigma \geq \mu$ (or) $\delta \geq \mu$. Since $\delta \ngeq \mu$. It follows that $\sigma \geq \mu$.

Now, $\mu(b) \leq \delta(b) = \mu(a)$.

Part (ii) Now, we construct fuzzy bi-ideals ρ and θ of X, $\rho(x) = \mu(b)$ and $\theta(x) = \begin{cases} 0 & \text{if } x \in \langle b \rangle \\ 1 & \text{otherwise} \end{cases}$, $\forall x \in X$.

As in part (i), we can verify that $\mu(a) \leq \mu(b)$.

Thus from parts (i) and (ii), it follows that $\mu(a) = \mu(b)$.

Theorem 3.9. *Let* μ *be an anti-fuzzy ideal in X. Then* μ *is an anti-fuzzy prime ideal of X iff each anti-level subset* μ_t, $t \in \mathrm{Im}(\mu)$ *of* μ *is a prime ideal of X.*

Proof. Assume that μ is an anti-fuzzy prime ideal of X.

By Theorem 3.7, μ_t is an ideal of X. To prove: μ_t is a prime ideal of X.

Let A and B be two ideals in X such that $AB \subseteq \mu_t$. Define the fuzzy subsets σ and δ of X as, $\sigma(x) = \begin{cases} t & \text{if } x \in A \\ 1 & \text{otherwise} \end{cases}$ and $\delta(x) = \begin{cases} t & \text{if } x \in B \\ 1 & \text{otherwise} \end{cases}$.

By Theorem 2.12 in [1], σ and δ are anti-fuzzy ideals of X. Next we verify that $\sigma \cdot \delta \geq \mu$. Since $\sigma \cdot \delta(a) = \begin{cases} \inf\limits_{a=bc} \{\max \sigma(b), \delta(c)\} & \text{if } a = bc \\ 0 & \text{otherwise} \end{cases}$.

We conclude that $\sigma(b) = \delta(c) \leq t$. So $b \in A$ and $c \in B$.

Now, $a = bc \in AB \subseteq \mu_t$ (i.e.) $a \in \mu_t \Rightarrow \mu(a) \leq t$.

Hence $\sigma \cdot \delta(a) \geq \mu(a), \forall a \in X$. Thus $\sigma \cdot \delta \geq \mu$. Since μ is an anti-fuzzy prime ideal, we have $\sigma \geq \mu$ (or) $\delta \geq \mu$. Suppose $\sigma \geq \mu$. If $A \not\subseteq \mu_t$, then there exists $a \in A$ such that $a \notin \mu_t$. This means that $\mu(a) \not\leq t$ (i.e.) $\mu(a) > t$. Now, $\sigma(a) \leq t < \mu(a)$. Which is a contradiction to $\sigma \geq \mu$.

Similarly, if $\delta \leq \mu$, then we can show that $B \subseteq \mu_t$. This shows that μ_t is a prime ideal of X.

Conversely, assume that μ_t, $t \in \mathrm{Im}(\mu)$ is a prime ideal of X. To prove: μ is an anti-fuzzy prime ideal. Let μ be a fuzzy set in X which is defined by, $\mu(x) = \begin{cases} t & \text{if } x \in \mu_t \\ 1 & \text{otherwise} \end{cases}$. By Theorem 2.12 in [1], μ is an anti-fuzzy ideal of X. To prove: μ is prime. Let σ and δ be two anti-fuzzy ideals of X such that $\sigma \cdot \delta \geq \mu$. Enough To prove: $\sigma \geq \mu$ (or) $\delta \geq \mu$. Suppose $\sigma \not\geq \mu$ and $\delta \not\geq \mu$. Then $\sigma(x) < \mu(x)$ and $\delta(y) < \mu(y), \forall x \in X \Rightarrow \mu(x) = \mu(y) = 1$ and also

$x, y \notin \mu_t$. Since μ_t is a prime ideal, we have that $\langle x \rangle \langle y \rangle \not\subset \mu_t$. Then $\mu(a) = t$ and hence $\sigma \cdot \delta(a) \geq \mu(a) = t$. Since $a = cd$, $c = \langle x \rangle$ and $d = \langle y \rangle$.

Now,

$$\sigma \cdot \delta(a) = \inf_{a=cd} \{\max \{\sigma(c), \delta(d)\}\} \leq \max \{\sigma(c), \delta(d)\}$$

$$\leq \max \{\sigma(x), \delta(y)\} < \max \{\mu(x), \mu(y)\} = t.$$

Therefore, $\sigma \cdot \delta(a) < t$. Which is a contradiction.

Hence μ is an anti-fuzzy prime ideal of X.

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