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SEPARATION AXIOMS VIA OPERATION ON αg-OPEN SETS IN TOPOLOGICAL SPACES

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Abstract

The main objective of this paper is to introduce new separation axioms by using αg_{γ}-open sets. In this paper, we define αg_{γ}-separation axioms namely $\alpha g_{\gamma} - T_i (i = 0, 1, 2)$, $\alpha g_{\gamma} - T_i' (i = 0, 1, 2)$ and investigate the relations between them and its characterizations.

1. Introduction

The concept of α-open sets was introduced by Njastad [10] in 1965. Levine [6] introduced the notion of generalized closed sets in topological spaces. Following this, the notion of α-generalized closed sets in topological spaces was introduced by Maki et al. [7] in 1994. The concept of an operation on topological spaces was introduced by Kasahara [5] and also he introduced the concept of α-closed graphs of functions in topological spaces. Jankovic [4] analyzed the functions with α-closed graphs. Following his work, Ogata [11] renamed the operation α as a γ-operation and introduced γ-open sets in

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topological spaces. Later, Basu et al. [3] studied separation axioms with respect to an operation. Al-Swidi and Mohammed [1] examined Separation axioms using kernel set. Asaad [2] defined operation on g-open sets and investigated its applications. Recently, Mershia Rabuni and Balamani [8] defined the operation γ on $\tau_{\alpha g}$ and introduced αg_{γ}- open sets in topological spaces.

In the present paper, new types of separation axioms are defined such as $\alpha g_{\gamma} - T_i (i = 0, 1, 2, 1/2)$ and $\alpha g_{\gamma} - T_i' (i = 0, 1, 2)$ spaces by utilizing the concept of αg_{γ}- open sets and their inter-relations are derived.

2. Preliminaries

Throughout this paper (X, τ) signifies a topological space on which no separation axiom is assumed unless otherwise mentioned. Also closure and interior of a subset A are denoted by $cl(A)$ and $\operatorname{int}(A)$ respectively.

Definition 2.1 [8]**.** Let (X, τ) be a topological space. An operation γ on $\tau_{\alpha g}$ is a mapping from $\tau_{\alpha g}$ into the power set $P(X)$ of $X \ni V \subseteq \gamma(V)\ \forall V \in \tau_{\alpha g}$, the value of V under the operation γ is denoted by $\gamma(V)$.

Definition 2.2 [8]**.** A non-empty subset A of (X, τ) with an operation γ on $\tau_{\alpha g}$ is called an αg_{γ}- open set if $\forall\ x \in A, \exists$ an αg- open set $U \ni x \in U$ and $\gamma(U) \subseteq A$. The collection of all αg_{γ}- open sets in (X, τ) is denoted by $\tau_{\alpha g \gamma}$. The complement of an αg_{γ}- open set is called αg_{γ}- closed.

Definition 2.3 [8]**.** An operation γ on $\tau_{\alpha g}$ is said to be αg- open if $\forall\, \alpha g$- open set U containing $x \in X, \exists$ an αg_{γ}- open set $V \ni x \in V$ and $V \subseteq \gamma(U)$.

Definition 2.4 [8]**.** Let γ be an operation on $\tau_{\alpha g}$. A point $x \in X$ is said to be an αg_{γ}- closure point of a set A if $\gamma(U) \cap A \neq \phi\ \forall\, \alpha g$- open set U containing x.

Definition 2.5 [8]. Let γ be an operation on $\tau_{\alpha g}$. Then $\alpha g_\gamma cl(A)$ is defined as the intersection of all αg_γ-closed sets containing A.

Definition 2.6 [9]. Let A be a subset of a topological space (X, τ) and γ be an operation on $\tau_{\alpha g}$. A point $x \in A$ is said to be an αg_γ-interior point of A if $\exists$ an αg-open set V of X containing $x \ni \gamma(V) \subseteq A$. $\alpha g\ \mathrm{int}_\gamma(A)$ denotes the set of all such αg_γ-interior points of A.

Definition 2.7 [9]. Let A be a subset of a topological space (X, τ) and γ be an operation on $\tau_{\alpha g}$. Then αg_γ-interior of A is the union of all αg_γ-open sets contained in A and it is denoted by $\alpha g_\gamma\ \mathrm{int}(A)$.

Definition 2.8 [9]. Let A be a subset of a topological space (X, τ) and γ be an operation on $\tau_{\alpha g}$. Then αg_γ-kernel of A is defined as the intersection of all αg_γ-open sets containing A. It is denoted by αg_γ- ker(A).

Definition 2.9 [9]. In a topological space (X, τ), $A \subseteq X$ is an αg_γ-generalized closed (concisely αg_γ- g. closed) set if $\alpha gcl_\gamma(A) \subseteq U$ whenever $A \subseteq U$ and U is αg_γ-open in (X, τ).

3. $\alpha g_\gamma - T_i (i = 0, 1, 2)$ and $\alpha g_\gamma - T_i' (i = 0, 1, 2)$ spaces

Definition 3.1. A topological space (X, τ) is called

(a) αg_γ-T_0 if for any two distinct points $x, y \in X$, there exists an αg-open set U such that either $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$.

(b) αg_γ-T_0' if for any two distinct points $x, y \in X$, there exists an αg_γ-open set U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

Theorem 3.2. *If γ is an αg-open operation on $\tau_{\alpha g}$. Then the space (X, τ) is αg_γ-T_0 iff for every pair of distinct points $x, y \in X$, $\alpha gcl_\gamma(\{x\}) \neq \alpha gcl_\gamma(\{y\})$.*

Proof. (Necessity). Consider (X, τ) to be αg_γ-T_0 with two distinct points x and y. Then there exists an αg-open set U such that either $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$. Since γ is αg-open, $\forall\, \alpha g$-open set U containing $x \in X$, $\exists$ an αg_γ-open set V such that $x \in V$ and $V \subseteq \gamma(U)$. Then $y \in X \setminus \gamma(U) \subseteq X \setminus V$ implies $\alpha gcl_\gamma(\{y\}) \subseteq \alpha gcl_\gamma(X \setminus V)$. Since $X \setminus V$ is αg_γ-closed, $\alpha gcl_\gamma(\{y\}) \subseteq X \setminus V$. Now $x \in \alpha gcl_\gamma(\{x\})$ and $x \in V$, $x \notin X \setminus V$ which implies $\alpha gcl_\gamma(\{x\}) \neq \alpha gcl_\gamma(\{y\})$.

(Sufficiency). Suppose that for any pair of distinct points $x, y \in X$, $\alpha gcl_\gamma(\{x\}) \neq \alpha gcl_\gamma(\{y\})$. Then there exists some $z \in \alpha gcl_\gamma(\{x\})$ and $z \notin \alpha gcl_\gamma(\{y\})$ (or $z \in \alpha gcl_\gamma(\{y\})$ and $z \notin \alpha gcl_\gamma(\{x\})$. Suppose if $x \in \alpha gcl_\gamma(\{y\})$, then $\alpha gcl_\gamma(\{x\}) \subseteq \alpha gcl_\gamma(\{y\})$. This implies $z \in \alpha gcl_\gamma(\{y\})$ which gives rise to a contradiction. Therefore $x \notin \alpha gcl_\gamma(\{y\})$. Then there exists an αg-open set U containing x such that $\gamma(U) \cap \{y\} = \phi$. Therefore $x \in U$ and $y \notin \gamma(U)$. Hence the space (X, τ) is αg_γ-T_0.

Corollary 3.3. *If γ is an αg-open operation on $\tau_{\alpha g}$. Then the space (X, τ) is αg_γ-T_0' iff for every pair of distinct points $x, y \in X$, $\alpha g_\gamma cl(\{x\}) \neq \alpha g_\gamma cl(\{y\})$.*

Proof. It follows from the fact that for any subset A of X, $\alpha gcl_\gamma(A) = \alpha g_\gamma cl(A)$ under the αg-open operation on $\tau_{\alpha g}$ [8] and by Theorem 3.2.

Corollary 3.4. *If γ is an αg-open operation on $\tau_{\alpha g}$. Then the space (X, τ) is αg_γ-T_0 iff αg_γ-T_0'.*

Proof. Follows from Theorem 3.2 and Corollary 3.3.

Theorem 3.5. *The topological space (X, τ) is αg_γ-T_0' iff for every pair of distinct points $x, y \in X$, $x \notin \alpha g_\gamma \ker \{y\}$ or $y \notin \alpha g_\gamma \ker \{x\}$.*

Proof. Consider (X, τ) to be αg_γ-T_0' with two distinct points x and y. Then there exists an αg_γ-open set U such that either $x \in U$ and $y \notin U$ or

$y \in U$ and $x \notin U$. Now $x \in U$ and $y \notin U$ implies $y \notin \alpha g_{\gamma} \ker \{x\}$ or $y \in U$ and $x \notin U$ implies $x \notin \alpha g_{\gamma} \ker \{y\}$. Conversely, consider for every pair of distinct points $x, y \in X, x \notin \alpha g_{\gamma} \ker \{y\}$ or $y \notin \alpha g_{\gamma} \ker \{x\}$. Take $x \notin \alpha g_{\gamma} \ker \{y\}$, which implies $x \notin$ intersection of all αg_{γ}-open sets containing the set $\{y\}$. Therefore there exists an αg_{γ}-open set U containing y but not x. Similarly $y \notin \alpha g_{\gamma} \ker \{x\}$ implies that there exists an αg_{γ}-open set V containing x but not y. Therefore (X, τ) is αg_{γ}-T_0'.

Proposition 3.6. *Every* αg_{γ}-T_0' *space is* αg_{γ}-T_0 *but not conversely.*

Proof. Consider (X, τ) to be αg_{γ}-T_0' with two distinct points x and y. Then there exists an αg_{γ}-open set U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$ Suppose if $x \in U$ and $y \notin U$. Since U is αg_{γ}-open, there exists an αg-open set V such that $x \in V \subseteq \gamma(V) \subseteq U$ and $y \notin V \subseteq \gamma(V) \subseteq U$. Therefore $x \in V$ and $y \notin \gamma(V)$. Hence (X, τ) is αg_{γ}-T_0.

Example 3.7. Consider $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then $\tau_{\alpha g} = \tau$. Let $\gamma : \tau_{\alpha g} \to P(X)$ be an operation on $\tau_{\alpha g}$ defined by

$$\gamma(A) = \begin{cases} \{a, c\} & A = \{a\} \\ A & A = \{a, b\} \ \forall\, A \in \tau_{\alpha g} \\ X & A = \{a\ c\} \end{cases}$$

Then $\tau_{\alpha g_{\gamma}} = \{\phi, \{a, b\}, X\}$. Here (X, τ) is αg_{γ}-T_0 but not αg_{γ}-T_0'.

Definition 3.8. A topological space (X, τ) is called

(a) αg_{γ}-T_1 if for any two distinct points $x, y \in X$, there exist αg-open sets U and V containing x and y respectively such that $y \notin \gamma(U)$ and $x \notin \gamma(V)$

(b) αg_{γ}-T_1' if for any two distinct points $x, y \in X$, there exist αg_{γ}-open set U and V containing x and y respectively such that $y \notin U$ and $x \notin V$.

Theorem 3.9. *If every singleton is* αg_{γ}-*closed then the topological space* (X, τ) *with the operation* γ *is* αg_{γ}-T_1

Proof. Assume that every singleton is αg_{γ}- closed. Let x and y be two distinct points. Then by assumption, we get two αg_{γ}- open sets $X \setminus \{y\}$ and $X \setminus \{x\}$ containing x and y respectively. Since $X \setminus \{y\}$ and $X \setminus \{x\}$ are αg_{γ}- open, there exist αg-open sets U and V such that $x \in U \subseteq \gamma(U) \subseteq X \setminus \{y\}$ and $y \in V \subseteq \gamma(V) \subseteq X \setminus \{x\}$ which implies $y \notin \gamma(U)$ and $x \notin \gamma(V)$. Therefore (X, τ) is αg_{γ}- T_1.

Remark 3.10. Converse of Theorem 3.9 need not be true as illustrated in the following example.

Example 3.11. Consider $X = \{a, b, c, d\}$ with $\tau = \{\phi, \{a\}, X\}$. Then $\tau_{\alpha g} = P(X) \setminus \{b, c, d\}$. Let $\gamma : \tau_{\alpha g} \to P(X)$ be an operation on $\tau_{\alpha g}$ defined by

$$\gamma(A) = \begin{cases} \{b, c\} & A = \{b\} \\ \{c, d\} & A = \{c\} \\ \{a, d\} & A = \{d\} \ \forall \ A \in \tau_{\alpha g} \\ X & A = \{b, c\} \text{ or } \{a, d\} \text{ or } \{b, d\} \\ A & \text{otherwise} \end{cases}$$

Then $\tau_{\alpha g \gamma} = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{c, d\}, \{a, d\}, \{a, c, d\}, \{a, b, c\}, \{a, b, d\}, X\}$. Here (X, τ) is αg_{γ}-T_1 but the singleton set $\{a\}$ is not αg_{γ}- closed.

Theorem 3.12. *The space* (X, τ) *is* αg_{γ}-T_1' *iff every singleton is* αg_{γ}- *closed.*

Proof. Consider (X, τ) to be αg_{γ}-T_1' with two distinct points x and y. Then there exist two αg_{γ}- open sets U and V containing x and y such that $y \notin U$ and $x \notin V$. Consequently, $y \in V \subseteq X \setminus \{x\}$ implies $X \setminus \{x\} = \bigcup \{V : y \in X \setminus \{x\}\}$ which is αg_{γ}- open, since the union of αg_{γ}- open sets is αg_{γ}- open. Therefore $\{x\}$ is αg_{γ}- closed. Conversely, assume that every singleton is αg_{γ}- closed. Let x and y be two distinct points. Consequently, $y \notin \{x\}$ and $x \notin \{y\}$ implies y and x belongs to the αg_{γ}- open sets $X \setminus \{x\}$

and $X \setminus \{y\}$ respectively. Therefore (X, τ) is αg_{γ}-T_1'.

Proposition 3.13. *The topological space* (X, τ) *is* αg_{γ}-T_1' *iff for every pair of distinct points* $x, y \in X$, $x \notin \alpha g_{\gamma} \ker\{y\}$ *and* $y \notin \alpha g_{\gamma} \ker\{x\}$.

Proof. Similar to the Proof of Theorem 3.5

Proposition 3.14. *A topological space is* αg_{γ}-T_1' *iff* $\alpha g_{\gamma} \ker\{x\} = \{x\}\, \forall x \in X$.

Proof. Consider X to be αg_{γ}-T_1' and a point x in X such that $\alpha g_{\gamma} \ker\{x\} \neq \{x\}$. Then there exists some $y \neq x$ such that $y \in \alpha g_{\gamma} \ker\{x\}$. By Proposition 3.13, X is not αg_{γ}-T_1', which is a contradiction. Therefore $\alpha g_{\gamma} \ker\{x\} = \{x\}$. Conversely, Consider $\alpha g_{\gamma} \ker\{x\} = \{x\}\ \forall\ x \in X$. Suppose X is not αg_{γ}-T_1', then by Proposition 3.13, for some $y \neq x$ such that $y \in \alpha g_{\gamma} \ker\{x\}$. This implies $\alpha g_{\gamma} \ker\{x\} \neq \{x\}$, which is a contradiction. Hence X is αg_{γ}-T_1'.

Proposition 3.15. *Every* αg_{γ}-T_1' *space is* αg_{γ}-T_1 *but not conversely.*

Proof. Obvious by Definition 3.8.

Example 3.16. In Example 3.11, the space (X, τ) is αg_{γ}-T_1 but not αg_{γ}-T_1'.

Definition 3.17. A space (X, τ) is called

(a) αg_{γ}-T_2 if for any two distinct points $x, y \in X$, there exist a αg-open sets U and V containing x and y such that $\gamma(U) \cap \gamma(V) = \phi$

(b) αg_{γ}-T_2' if for any two distinct points $x, y \in X$, there exist a αg_{γ}-open sets U and V containing x and y such that $U \cap V = \phi$.

Theorem 3.18. *The following are equivalent for a topological space* (X, τ) *with an operation* γ:

(a) (X, τ) *is* αg_{γ}- T_2

(b) *For each* $x \neq y$ *in X, there exists an* αg-*open set U containing x such that* $y \notin \alpha gcl_{\gamma}(\gamma(U))$

(c) *For each x in* $X, \bigcap \{\alpha gcl_{\gamma}(\gamma(U)) : U \in \tau_{\alpha g}$ *and* $x \in U\} = \{x\}$.

Proof. (a) $\Rightarrow$ (b) Consider X to be αg_{γ}- T_2. Then there exist αg-open sets U and V containing x and y such that $\gamma(U) \bigcap \gamma(V) = \phi$. This implies $\gamma(U) \subseteq X \setminus \gamma(V) \subseteq X \setminus V$. Then $\alpha gcl_{\gamma}(\gamma(U)) \subseteq \alpha gcl_{\gamma}(X \setminus V) = X \setminus V$. Since $y \notin X \setminus V$, $y \notin \alpha gcl_{\gamma}(\gamma(U))$.

(b) $\Rightarrow$ (c) Assume for each $x \neq y$ in X, there exists an αg-open set U containing x such that $y \notin \alpha gcl_{\gamma}(\gamma(U))$. Suppose if for each x in X, $\bigcap \{\alpha gcl_{\gamma}(\gamma(U)) : U \in \tau_{\alpha g}$ and $x \in U\} \neq \{x\}$, then there exists some $y \in \bigcap \{\alpha gcl_{\gamma}(\gamma(U)) : U \in \tau_{\alpha g}$ and $x \in U\}$. Consequently, $y \in \alpha gcl_{\gamma}(\gamma(U))$ $\forall$ αg-open set U containing x, which leads to a contradiction.

(c) $\Rightarrow$ (a) Consider that $\bigcap \{\alpha gcl_{\gamma}(\gamma(U)) : U \in \tau_{\alpha g}$ and $x \in U\} = \{x\}$, for each x in X i.e., for each $x \neq y$ in X, $y \notin \alpha gcl_{\gamma}(\gamma(U))$ $\forall$ αg-open set U containing x. This implies that $\gamma(U) \bigcap \gamma(V) = \phi$ for some αg-open set V containing y.

Theorem 3.19. *The following are equivalent for a topological space* (X, τ) *with an operation* γ:

(a) (X, τ) *is* αg_{γ}- T_2'

(b) *For each* $x \neq y$ *in X, there exists an* αg_{γ}- *open set U containing x such that* $y \notin \alpha g_{\gamma} cl(U)$

(c) *For each x in* $X, \bigcap \{\alpha g_{\gamma} cl(U) : U \in \tau_{\alpha g_{\gamma}}$ *and* $x \in U\} = \{x\}$.

Proof. (a) $\Rightarrow$ (b) Consider X to be αg_{γ}- T_2'. Then there exist αg_{γ}-open sets U and V containing x and y such that $U \bigcap V = \phi$. This implies

$U \subseteq X \setminus V$. Then $\alpha g_\gamma cl(U) \subseteq \alpha g_\gamma cl(X \setminus V) = X \setminus V$. Since $y \notin X \setminus V$, $y \notin \alpha g_\gamma cl(U)$.

(b) $\Rightarrow$ (c) Assume for each $x \neq y$ in X, there exists an αg_γ-open set U containing x such that $y \notin \alpha g_\gamma cl(U)$. Suppose if for each x in X, $\bigcap \{\alpha g_\gamma cl(U) : U \in \tau_{\alpha g_\gamma}$ and $x \in U\} \neq \{x\}$, then there exists some $y \in \bigcap \{\alpha g_\gamma cl(U) : U \in \tau_{\alpha g_\gamma}$ and $x \in U\}$. Consequently, $y \in \alpha g_\gamma cl(U)$ $\forall$ αg_γ-open set U containing x, which leads to a contradiction.

(c) $\Rightarrow$ (a) Consider that $\bigcap \{\alpha g_\gamma cl(U) : U \in \tau_{\alpha g_\gamma}$ and $x \in U\} = \{x\}$, for each x in X i.e., for each $x \neq y$ in X, $y \notin \alpha g_\gamma cl(U)$ $\forall$ αg_γ-open set U containing x. This implies that $U \cap V = \phi$ for some αg_γ-open set V containing y.

Proposition 3.20. *Every αg_γ-T_2' space is αg_γ-T_2 but not conversely.*

Proof. Obvious by Definition 3.17.

Example 3.21. Consider $X = \{a, b, c, d\}$ with $\tau = \{\phi, \{a\}, X\}$. Then $\tau_{\alpha g} = P(X) \setminus \{b, c, d\}$. Let $\gamma : \tau_{\alpha g} \to P(X)$ be an operation on $\tau_{\alpha g}$ defined by

$$\gamma(A) = \begin{cases} A \cup \{c\} & A = \{b\} \\ A \cup \{a\} & A = \{c\} \quad \forall\ A \in \tau_{\alpha g} \\ gcl(A) & A = \{a, b\} \text{ or } \{a, d\} \text{ or } \{b, d\} \text{ or } \{c, d\} \\ X & \text{otherwise} \end{cases}$$

Then $\tau_{\alpha g_\gamma} = \{\phi, \{a, b\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Here (X, τ) is αg_γ-T_2 but not αg_γ-T_2'.

Definition 3.22. A topological space (X, τ) is called a αg_γ-$T_{1/2}$ space if every αg_γ-g. closed set of (X, τ) is αg_γ-closed.

Remark 3.23. Every αg_γ-closed set is αg_γ-g. closed [9]. Therefore in αg_γ-$T_{1/2}$ space, the collections of αg_γ-closed sets coincide with αg_γ-g. closed sets.

Theorem 3.24. *A topological space* (X, τ) *is* αg_{γ}- $T_{1/2}$ *with a* γ*-operation iff* $\{x\}$ *is* αg_{γ}*- closed or* αg_{γ}- open $\forall\ x \in X$.

Proof. Consider $\{x\}$ to be not αg_{γ}- closed in (X, τ). Then $X \setminus \{x\}$ is αg_{γ}-g. closed by Theorem 4.11 [9]. Since (X, τ) is αg_{γ}- $T_{1/2}$ space, $X \setminus \{x\}$ is αg_{γ}- closed. Therefore $\{x\}$ is αg_{γ}- open. Conversely, Assume that $\{x\}$ is αg_{γ}- closed or αg_{γ}- open $\forall\ x \in X$. Let C be αg_{γ}-g. closed. By Theorem 3.47 [8], to prove C is αg_{γ}- closed it is enough to prove that $\alpha gcl_{\gamma}(C) = C$. Suppose if $\alpha gcl_{\gamma}(C) \neq C$, then there exists a point x such that $x \in \alpha gcl_{\gamma}(C) \setminus C$. Then by the hypothesis, $\{x\}$ is αg_{γ}- closed or αg_{γ}- open. Suppose if $\{x\}$ is αg_{γ}- closed, then the αg_{γ}- closed set $\{x\} \subseteq \alpha gcl_{\gamma}(C) \setminus C$ which contradicts the Theorem 4.6 [9]. Suppose if $\{x\}$ is αg_{γ}- open, then $\{x\} \cap C \neq \phi$, since $x \in \alpha gcl_{\gamma}(C)$ and γ is αg- open. Consequently, we have $x \in C$, which is a contradiction. Hence $\alpha gcl_{\gamma}(C) = C$. Therefore C is αg_{γ}- closed.

Proposition 3.25. *For a topological space* (X, τ) *with an operation* γ *on* $\tau_{\alpha g}$, *the following are true*:

(1) *Every* αg_{γ}- T_2 *space is* αg_{γ}- T_1

(2) *Every* αg_{γ}- T_1 *space is* αg_{γ}- $T_{1/2}$

(3) *Every* αg_{γ}- $T_{1/2}$ *space is* αg_{γ}- T_0'

(4) *Every* αg_{γ}- T_2' *space is* αg_{γ}- T_1'

Proof. Obvious by Definitions – 3.1, 3.8, 3.17 and 3.22.

Remark 3.26. The converse of Proposition 3.25 is not true.

Example 3.27. Consider $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{b, c\} X\}$. Then $\tau_{\alpha g} = P(X)$. Let $\gamma : \tau_{\alpha g} \to P(X)$ be an operation on $\tau_{\alpha g}$ defined by

$$\gamma(A) = \begin{cases} A \cup \{c\} & A = \{a\} \text{ or } \{b\} \\ gcl(A) & \text{otherwise} \end{cases} \ \forall\ A \in \tau_{\alpha g}$$

Then $\tau_{\alpha g_\gamma} = \{\phi, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$. Here (X, τ) is αg_γ- T_1 but not αg_γ- T_2.

Example 3.28. Consider $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Then $\tau_{\alpha g} = P(X) \setminus \{\{c\}, \{b, c\}\}$. Let $\gamma : \tau_{\alpha g} \to P(X)$ be an operation on $\tau_{\alpha g}$ defined by

$$\gamma(A) = \begin{cases} cl(A) & c \in A \\ A & \text{otherwise} \end{cases} \quad \forall\ A \in \tau_{\alpha g}$$

Then $\tau_{\alpha g_\gamma} = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Here (X, τ) is αg_γ- $T_{1/2}$ but not αg_γ- T_1.

Example 3.29. Consider $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. Then $\tau_{\alpha g} = P(X)$. Let $\gamma : \tau_{\alpha g} \to P(X)$ be an operation on $\tau_{\alpha g}$ defined by

$$\gamma(A) = \begin{cases} A & A = \{a, c\} \\ \alpha cl(A) & \text{otherwise} \end{cases} \quad \forall\ A \in \tau_{\alpha g}$$

Then $\tau_{\alpha g_\gamma} = \{\phi, \{a\}, \{a, c\}, \{b, c\}, X\}$. Here (X, τ) is αg_γ- T_0' but not αg_γ- $T_{1/2}$.

Example 3.30. Consider $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. Then $\tau_{\alpha g} = P(X)$. Let $\gamma : \tau_{\alpha g} \to P(X)$ be an operation on $\tau_{\alpha g}$ defined by

$$\gamma(A) = \begin{cases} A & A = \{a, b\} \text{ or } \{b, c\} \text{ or } \{a, c\} \\ X & \text{otherwise} \end{cases} \forall\ A \in \tau_{\alpha g}$$

Then $\tau_{\alpha g_\gamma} = \{\phi, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Here (X, τ) is αg_γ- T_1' but not αg_γ- T_2'.

Theorem 3.31. *An* αg_γ- T_0'-*space* (X, τ) *with an operation* γ *on* $\tau_{\alpha g}$ *is* αg_γ- T_2' *iff* $\forall\ x \neq y$ *in* X *with* $\alpha g_\gamma \ker\{x\} \neq \alpha g_\gamma \ker\{y\}$, *there exist* αg_γ-*closed sets* C *and* D *such that* $\alpha g_\gamma \ker\{x\} \subseteq C$, $\alpha g_\gamma \ker\{x\} \cap D = \phi$, $\alpha g_\gamma \ker\{y\} \subseteq D$, $\alpha g_\gamma \ker\{y\} \cap C = \phi$ *and* $X = C \cup D$.

Proof. Consider X to be αg_γ- T_0' By Theorem 3.5, for every pair of distinct points $x, y \in X, x \notin \alpha g_\gamma \ker\{y\}$ or $y \notin \alpha g_\gamma \ker\{x\}$. This implies $\alpha g_\gamma \ker\{x\} \neq \alpha g_\gamma \ker\{y\}$. Then by the hypothesis, there exist αg_γ- closed sets C and D such that $\alpha g_\gamma \ker\{x\} \subseteq C, \alpha g_\gamma \ker\{x\} \cap D = \phi, \alpha g_\gamma \ker\{y\} \subseteq D$, $\alpha g_\gamma \ker\{y\} \subseteq D, \alpha g_\gamma \ker\{y\} \cap C = \phi$ and $X = C \cup D$. Hence $X \setminus C$ and $X \setminus D$ are disjoint αg_γ- open sets containing $\alpha g_\gamma \ker\{y\}$ and $\alpha g_\gamma \ker\{x\}$ respectively. Therefore X is αg_γ- T_2'.

Conversely, let X be αg_γ- T_2' and $\forall\ x \neq y$ in X with $\alpha g_\gamma \ker\{x\} \neq \alpha g_\gamma \ker\{y\}$. Then there exist disjoint αg_γ- open sets U and V containing x and y such that $U \cap V = \phi$. Since every αg_γ- T_2'- space is αg_γ- T_1', by Proposition 3.14, $\alpha g_\gamma \ker\{x\} = \{x\}$ and $\alpha g_\gamma \ker\{y\} = \{y\}$. Therefore, now $X \setminus U$ and $X \setminus V$ are αg_γ- closed sets containing $\alpha g_\gamma \ker\{y\}$ and $\alpha g_\gamma \ker\{x\}$ respectively such that $(X \setminus U) \cup (X \setminus V) = X$, $\alpha g_\gamma \ker\{y\} \cap (X \setminus V) = \phi$, and $\alpha g_\gamma \ker\{x\} \cap (X \setminus U) = \phi$.

Proposition 3.32. *An* αg_γ- T_1', *space* (X, τ) *with an operation* γ *on* $\tau_{\alpha g}$ *is* αg_γ- T_2', *iff* $\forall x \neq y$ *in* X *with* $\alpha g_\gamma \ker\{x\} \neq \alpha g_\gamma \ker\{y\}$, *there exist* αg_γ- *closed sets C and D such that* $\alpha g_\gamma \ker\{x\} \subseteq C, \alpha g_\gamma \ker\{x\} \cap D = \phi, \alpha g_\gamma \ker\{y\} \subseteq D$, $\alpha g_\gamma \ker\{y\} \cap C = \phi$ *and* $X = C \cup D$.

Proof. Similar to Theorem 3.31.

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