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COMMON FIXED POINTS FOR A PAIR OF MAPS IN BICOMPLEX VALUED GENERALIZED METRIC SPACES

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Abstract

The purpose of this paper is to setup generalized bicomplex valued metric spaces and prove common fixed point theorems for a pair of weakly iso-tone increasing self maps satisfying more general contraction condition rendered by rational expressions. These results are supported through examples. Our results generalize the results of Jebril et al. [6] and Beg et al. [2].

1. Introduction

Recently, Choi et al. [3] introduced the notion of bicomplex valued metric space which is a generalization of complex valued metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying certain contractive condition. Motivated by the above works, researchers have been worked in this direction and obtained various results in this setting. For example, we refer [2, 3, 4, 6].

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We recall some definitions and terminologies, used to prove the main results.

Bicomplex Numbers. The set of bicomplex numbers denoted by $\mathbb{BC}$ is the first setting in an infinite sequence of multicomplex sets which are generalizations of the set of complex numbers $\mathbb{C}$ Here, we recall the set of bicomplex numbers $\mathbb{BC}$ for example, [7,8]):

$$\mathbb{BC} = \{w = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 : a_p \in \mathbb{R}, (p = 0, 1, 2, 3)\}.$$

We can also express $\mathbb{BC}$ as

$$\mathbb{BC} = \{z_1 + i_2 z_2 : z_1, z_1 \in \mathbb{C}\},$$

where $z_1 = a_0 + i_1 a_1$, $z_2 = a_2 + i_1 a_3$, i_1 and i_2 are imaginary independent units such that $i_1^2 = -1 = i_2^2$. The product of $i_1 i_2 = j$ such that $j^2 = 1$. The product of units is commutative and is defined as $i_1 j = -i_2$, $i_2 j = -i_1$, with the addition and multiplication of two bicomplex numbers defined in the obvious way.

For a bicomplex number $w = z_1 + i_2 z_2$, the norm is denoted by $\| w \|$ and is defined

$$\| w \| = \| z_1 + i_2 z_2 \| = (| z_1 |^2 + | z_2 |^2)^{\frac{1}{2}}.$$

By choosing, $w = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3$, $a_p \in R$, $(p = 0, 1, 2, 3)$ then

$$\| w \| = (a_1^2 + a_2^2 + a_3^2 + a_4^2)^{\frac{1}{2}}.$$

A bicomplex number $w = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3$ is degenerated [8] if the matrix $\begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}$ is degenerated.

Further, for any two bicomplex numbers $u, v \in \mathbb{BC}$, we can show that

(i) $0 \prec_{i_2} u \prec_{i_2} v \Rightarrow \| u \| \leq \| v \|$

(ii) $\| u + v \| \leq \| u \| + \| v \|$

(iii) $\| \alpha u \| \leq \alpha \| v \|$

Also, for any two complex numbers $u, v \in \mathbb{BC}$, we have

(i) $\| uv \| \leq \sqrt{2} \| u \| \| v \|$.

(ii) $\| uv \| = \| u \| \| v \|$ whenever at least one of u and v is degenerated [8].

(iii) $\| u^{-1} \| = \| u \|^{-1}$ holds for any degenerated bicomplex number.

The partial order relation on $\preceq_{i_2}$ defined in [4] as follows.

Let $u = u_1 + i_2 u_2 \in \mathbb{BC}$ and $v = v_i + i_2 v_2 \in \mathbb{BC}$, we define a partial order relation on $\mathbb{BC}$ as $u \preceq_{i_2} v$ if and only if $u_1 \preceq_{i_1} v_1$ and $u_2 \preceq_{i_2} v_2$, where $\preceq_{i_1}$ is a partial order relation in $\mathbb{C}$. Then

(1) "$\mathrm{Re}(u_1) = \mathrm{Re}(v_1)$ and $\mathrm{Im}(u_1) = \mathrm{Im}(v_1)$"

$\mathrm{Re}(u_2) = \mathrm{Re}(v_2)$ and $\mathrm{Im}(u_2) = \mathrm{Im}(v_2)$

(2) $\mathrm{Re}(u_1) < \mathrm{Re}(v_1)$ and $\mathrm{Im}(u_1) < \mathrm{Im}(v_1)$

$\mathrm{Re}(u_2) = \mathrm{Re}(v_2)$ and $\mathrm{Im}(u_2) = \mathrm{Im}(v_2)$

(3) $\mathrm{Re}(u_1) = \mathrm{Re}(v_1)$ and $\mathrm{Im}(u_1) = \mathrm{Im}(v_1)$

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(4) $\mathrm{Re}(u_1) < \mathrm{Re}(v_1)$ and $\mathrm{Im}(u_1) < \mathrm{Im}(v_1)$

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We write $u \precnsim_{i_2} v$ if $u \preceq_{i_2} v$ and $u \neq v$ if any one of (1), (2) and (3) is satisfied and $u \prec_{i_2} v$ if condition (4) is satisfied.

The definition of the bicomplex metric space is introduced in [3] as follows.

Definition 1.1. Let X be a nonempty set. A function $d_{\mathbb{BC}} : X \times X \to \mathbb{C}_2$ is called a bicomplex valued metric on X if for all $x, y, z \in X$, the following conditions are satisfied:

$(BGCM1) 0 \preceq_{i_2} d_{\mathbb{BC}}(x, y)$

$(BGCM2) d_{\mathbb{BC}}(x, y) = 0$ iff $x = y$;

$(BGCM3) d_{\mathbb{BC}}(x, y) = d_{\mathbb{BC}}(x, y)$;

$(BGCM4) d_{\mathbb{BC}}(x, y) \preceq_{i_2} d_{\mathbb{BC}}(x, z) + d_{\mathbb{BC}}(y, z)$

The pair $(X, d_{\mathbb{BC}})$ is called a bicomplex valued metric space.

We now extend the definition of generalized bicomplex valued metric space as follows.

Definition 1.2. Let X be a nonempty set. A function $d_{\mathbb{BC}} : X \times X \to \mathbb{C}_2$ is called a generalized bicomplex valued metric on X if for all $x, y \in X$, and for all distinct $u, v \in X$, each one is different from x and y, the following conditions are satisfied:

$(BGCM1) 0 \preceq_{i_2} d_{\mathbb{BC}}(x, y)$

$(BGCM2) d_{\mathbb{BC}}(x, y) = 0$ iff $x = y$;

$(BGCM3) d_{\mathbb{BC}}(x, y) = d_{\mathbb{BC}}(x, y)$;

$(BGCM4) d_{\mathbb{BC}}(x, y) \preceq_{i_2} d_{\mathbb{BC}}(x, u) + d_{\mathbb{BC}}(u, v) + d_{\mathbb{BC}}(u, z)$

The pair $(X, d_{\mathbb{BC}})$ is called a generalized bicomplex valued metric space.

Example 1.3. Let $X = [-1 - 2] \cup [0, 1]$, we define $d_{\mathbb{BC}} : X \times X \to \mathbb{C}_2$ by

$d_{\mathbb{BC}}(0, 1) = d_{\mathbb{BC}}(1, 0) = 3i_1 i_2$, $d_{\mathbb{BC}}(0 - 1) = d_{\mathbb{BC}}(-1, 0) = i_2 + i_1 i_2$,

$d_{\mathbb{BC}}(-1, 1) = d_{\mathbb{BC}}(1, -1) = i_1 i_2$, $d_{\mathbb{BC}}(1 - 2) = d_{\mathbb{BC}}(-2, 1) = 2i_1 i_2$,

$d_{\mathbb{BC}}(-1, -2) = d_{\mathbb{BC}}(-2, -1) = i_2 + 3i_1 i_2$ and

$d_{\mathbb{BC}}(x, y) = (3 + 6i_1 + i_2 + 2i_1 i_2) | x - y |$, otherwise.

Then clearly, $(X, d_{\mathbb{BC}})$ is a generalized bicomplex valued metric space such that $d_{\mathbb{BC}}(x, y)$ is degenerated for all $x, y \in X$. Here we note that

$$3i_1 i_2 + i_2 = d_{\mathbb{BC}}(-1, -2) \succeq_{i_2} d_{\mathbb{BC}}(-1, 1) + d_{\mathbb{BC}}(1, -2) = 3i_1 i_2.$$

Hence $(X, d_{\mathbb{BC}})$ is not a bicomplex valued metric space.

Definition 1.4. Let $(X, d_{\mathbb{BC}})$ be a generalized bicomplex valued metric space and $\{x_n\}$ be a sequence in X. We say that:

(i) The sequence $\{x_n\}$ converges to $x \in X$ if for each $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$ there is a $n_0 \in N$ such that for all $n > n_0$, $d_{\mathbb{BC}}(x_n, x) \prec_{i_2} c$. We denote this by $\lim_{n \to \infty} x_n = x$.

(ii) The sequence $\{x_n\}$ is a Cauchy sequence if for each $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$ there is $n_0 \in N$ such that for all $n > n_0$, $d_{\mathbb{BC}}(x_n, x_{n+m}) \prec_{i_2} c$, where $m \in N$.

(iii) $(X, d_{\mathbb{BC}})$ is complete generalized bicomplex valued metric space if every Cauchy sequence in X is convergent to a point in X.

Lemma 1.5 [3]. *Let $(X, d_{\mathbb{BC}})$ be a generalized bicomplex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $\| d_{\mathbb{BC}}(x_n, x) \| \to 0$ as $n \to \infty$.*

Lemma 1.6 [3]. *Let $(X, d_{\mathbb{BC}})$ be a generalized bicomplex valued metric space and $\{x_n\}$ be a sequence in X.* If $\lim_{n \to \infty} \| d_{\mathbb{BC}}(x_n, x_{n+m}) \| \to 0$ *then $\{x_n\}$ is a Cauchy sequence.*

Lemma 1.7 [3]. *Let $(X, d_{\mathbb{BC}})$ be a generalized bicomplex valued metric space and let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x then for any $a \in x$, $\lim_{n \to \infty} \| d_{\mathbb{BC}}(x_n, a) \| \to \| d_{\mathbb{BC}}(x, a) \|$.*

Definition 1.8. Let $(X, \preceq)$ be a partially ordered set and $S, T : X \to X$. The pair (S, T) is said to be:

(i) weakly increasing if $Sx \preceq TSx$ and $Tx \preceq STx$ for all $x \in X$[1].

If $S = T$, we have $Sx \preceq S^2 x$, for all $x \in X$. In this case, S is weakly increasing map.

(ii) S is said to be T-weakly isotone increasing [5] if for all $x \in X$, we have

$$Sx \preceq TSx \preceq STSx.$$

Note that two weakly increasing mappings need not be nondecreasing. There exist some examples to illustrate this fact we refer [5].

If $S, T : X \to X$ are weakly increasing, then S is T-weakly isotone increasing.

If $S = T$, we say that S is weakly isotone increasing. In this case for each $x \in X$, we have $Sx \preceq SSx$.

Definition 1.9. A nonempty subset W of a partially ordered set X is said to be totally ordered if every two elements of W are comparable.

Jebril et al., [6] and Beg et al., [2] explored common fixed point theorems satisfying rational inequality in $\mathbb{C}_2$ and obtained the following fixed point theorem.

Theorem 1.10 [6]. *Let $(X, d_{\mathbb{BC}})$ be a complete bicomplex and let $S, T : X \to X$ be a mapping satisfying the condition*

$$d_{\mathbb{BC}}(Sx, Ty) \preceq_{i_2} a\, d_{\mathbb{BC}}(x, y) + b \frac{d_{\mathbb{BC}}(x, Sx) d_{\mathbb{BC}}(y, Ty)}{d_{\mathbb{BC}}(x, y) + d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(y, Ty)}$$

$$+ c[d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(y, Ty)] \tag{1.10.1}$$

for all $x, y \in X$, and $d_{\mathbb{BC}}(Sx, Ty) = 0$ if $d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(y, Ty) + d_{\mathbb{BC}}(x, y) = 0$ where a, b non-negative real numbers with $a + b + 2c < 1$. Then S and T have a unique common fixed point in X.

Theorem 1.11 [2]. *Let $(X, d_{\mathbb{BC}})$ be a complete bicomplex metric space with degenerated $1 + d_{\mathbb{BC}}(x, y), \| 1 + d_{\mathbb{BC}}(x, y) \| \neq 0$ for all $x, y \in X$ and let $S, T : X \to X$ be a mapping satisfying the condition*

$$d_{\mathbb{BC}}(Sx, Ty) \preceq_{i_2} a\; d_{\mathbb{BC}}(x, y) + b \frac{d_{\mathbb{BC}}(x, Sx) d_{\mathbb{BC}}(y, Ty)}{1 + d_{\mathbb{BC}}(x, y)} \tag{1.11.1}$$

for all $x, y \in X$, where a, b non-negative real numbers with $a + b < 1$. Then S and T have a unique common fixed point in X.

The purpose of this paper is to setup generalized bicomplex valued metric spaces and prove common fixed point theorems for a pair of weakly isotone increasing selfmaps satisfying more general contraction condition rendered by rational expressions. These results are supported through examples. Our results generalize the results of Beg et al., (Theorem 1.10 and Theorem 1.11).

2. Main results

Theorem 2.1. *Let* $(X, \preceq)$ *be partially ordered set such that there exists a complete generalized bicomplex valued metric* $d_{\mathbb{BC}}$ *on* X *with degenerated* $d_{\mathbb{BC}}(x, y) + d_{\mathbb{BC}}(u, v) + d_{\mathbb{BC}}(w, z)$ *for all* $x, y, u, v, w, z \in X$ *and* $S, T : X \to X$ *be mappings satisfying the condition: for every comparable* $x, y \in X$,

$$d_{\mathbb{BC}}(Sx, Ty) \preceq_{i_2} \frac{\alpha[p + d_{\mathbb{BC}}(x, Sx)]d_{\mathbb{BC}}(y, Ty)^r d_{\mathbb{BC}}(x, Ty)^q d_{\mathbb{BC}}(y, Sx)^s}{\mu d_{\mathbb{BC}}(y, Sx) + \lambda d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}(x, y)}$$

$$+ \beta d_{\mathbb{BC}}(x, y) + \delta[(x, Sx) + d_{\mathbb{BC}}(y, Ty)] \qquad (2.1.1)$$

where $\alpha, p, q, r, s, \mu, \lambda \in \mathbb{R}^+$ *and* $\beta, \delta \in [0, 1]$ *with* $\beta + 2\delta < 1$ *and* $d_{\mathbb{BC}}(Sx, Ty) = 0$ *when* $\mu d_{\mathbb{BC}}(y, Sx) + \lambda d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}(x, y) = 0$ *Also, suppose that S is T is weakly isotone increasing on X. If S or T is continuous or for any nondecreasing sequence* $\{x_n\}$ *with* $x_n \to z$ *in X, we have* $x_n \preceq z$ *then S and T have a unique common fixed point in X. Moreover, the set of common fixed points of S and T is totally ordered if and only if S and T have a unique common fixed point in X.*

Proof. Let $x_0 \in X$. Without loss of generality suppose that $Sx_0 \neq x_0$. Let us define a sequence $\{x_n\}$ in X as follows:

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}, \text{ for } n = 0, 1, 2, 3, \ldots \qquad (2.1.2)$$

Since S is T is weakly isotone increasing, we have

$$x_1 = Sx_0 \preceq TSx_0 = Tx_1 = x_2 \preceq STSx_0 = STx_1 \preceq x_3.$$

By repeating this process, we get

$$x_1 \preceq x_2 \preceq x_3 \preceq x_4 \preceq \ldots \preceq x_n \preceq x_{n+1} \ldots \qquad (2.1.3)$$

Assume that $d(x_{2n}, x_{2n+1}) > 0$ for all $n \in \mathbb{N}$.

Otherwise, if $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N}$ then we have $x_{2n} = Sx_{2n}$. We now show that x_{2n} is a common fixed point of S and T. Indeed, suppose that

$d(x_{2n+1}, x_{2n+2}) \geq 0$, since x_{2n}, x_{2n+1} are comparable, we have

$d(Sx_{2n}, Tx_{2n+1} \precsim_{i_2}$

$$\frac{\alpha[p + d_{\mathbb{BC}}(x_{2n}, Sx_{2n})]d_{\mathbb{BC}}(x_{2n+1}, Tx_{2n+1})^r d_{\mathbb{BC}}(x_{2n}, Tx_{2n+1})^q d_{\mathbb{BC}}(x_{2n+1}, Sx_{2n})^s}{\mu d_{\mathbb{BC}}(x_{2n+1}, Sx_{2n}) + \lambda d_{\mathbb{BC}}(x_{2n}, Tx_{2n+1}) + d_{\mathbb{BC}}(x_{2n}, x_{2n+1})}$$

$$+ \beta d_{\mathbb{BC}}(x_{2n}, x_{2n+1}) + \delta[d_{\mathbb{BC}}(x_{2n}, Sx_{2n}) + d_{\mathbb{BC}}(x_{2n+1}, Tx_{n+1})]$$

$(1 - \delta)d_{\mathbb{BC}}(x2n + 1, x_{2n+2}) \precsim_{i_2} 0$, this implies $x_{2_{n+1}} = x_{2n+2}$.

Therefore suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Again by condition (2.1.1), we have

$d(x_{2n+1}, x_{2n+2} \precsim_{i_2}$

$$\frac{\alpha[p + d_{\mathbb{BC}}(x_{2n}, x_{2n+1})]d_{\mathbb{BC}}(x_{2n+1}, x_{2n+1})^r d_{\mathbb{BC}}(x_{2n}, x_{2n+2})^q d_{\mathbb{BC}}(x_{2n+1}, x_{2n+1})^s}{\mu d_{\mathbb{BC}}(x_{2n+1}, x_{2n+1}) + \lambda d_{\mathbb{BC}}(x_{2n}, x_{2n+2}) + d_{\mathbb{BC}}(x_{2n}, x_{2n+1})}$$

$$+ \beta d_{\mathbb{BC}}(x_{2n}, x_{2n+1}) + \delta[d_{\mathbb{BC}}(x_{2n}, x_{2n+1}) + d_{\mathbb{BC}}(x_{2n+1}, x_{2n+1})],$$

which implies

$$d_{\mathbb{BC}}(x_{2n+1}, x_{2n+2}) \precsim_{i_2} \frac{\beta + \delta}{1 - \delta} d_{\mathbb{BC}}(x_{2n}, x_{2n+1}). \tag{2:1:4}$$

Similarly,

$$d_{\mathbb{BC}}(x_{2n}, x_{2n+1}) \precsim_{i_2} \frac{\beta + \delta}{1 - \delta} d_{\mathbb{BC}}(x_{2n}, x_{2n-1}). \tag{2:1:5}$$

Thus from (2.1.4) and (2.1.5), we get

$$d_{\mathbb{BC}}(x_n, x_{n+1}) \precsim_{i_2} \frac{\beta + \delta}{1 - \delta} d_{\mathbb{BC}}(x_n, x_{n-1}) \tag{2:1:6}$$

for all $n \in N$.

Let $h = \frac{\beta + \delta}{1 - \delta}$. Thus, from (2.1.6), it follows that

$$d_{\mathbb{BC}}(x_n, x_{n+1}) \preceq_{i_2} h d_{\mathbb{BC}}(x_n, x_{n-1}). \tag{2:1:7}$$

Hence for all $n \geq 0$, we have

$$d_{\mathbb{BC}}(x_n, x_{n+1}) \preceq_{i_2} h^n d_{\mathbb{BC}}(x_n, x_{n-1}). \tag{2:1:8}$$

We now show that $\{x_n\}$ is a Cauchy sequence.

Case(1)**:** Now, for $m - n$, is odd, let $m - n = 2k + 1$ then we have

$$\begin{aligned} d_{\mathbb{BC}}(x_n, x_m) &\preceq_{i_2} d_{\mathbb{BC}}(x_n, x_{n+1}) + d_{\mathbb{BC}}(x_{n+1}, x_{n+2}) + d_{\mathbb{BC}}(x_{n+2}, x_m) \\ &\preceq_{i_2} [h^n + h^{n+1}] d_{\mathbb{BC}}(x_0, x_1) + [h^{n+2} + h^{n+3}] d_{\mathbb{BC}}(x_0, x_1) + \ldots \\ &\quad + [h^{n+2k-2} + h^{n+2k-1}] d_{\mathbb{BC}}(x_0, x_1) + d_{\mathbb{BC}}(x_{n+2k}, x_m) \\ &\preceq_{i_2} h^n [1 + h][1 + h^2 + h^4 + \ldots] d_{\mathbb{BC}}(x_0, x_1) \\ &\preceq_{i_2} \frac{(1 + h)h^n}{1 - h^2} d_{\mathbb{BC}}(x_0, x_1) \end{aligned}$$

$$\| d_{\mathbb{BC}}(x_n, x_m) \| \leq \frac{h^n(1 + h)}{1 - h^2} \| d_{\mathbb{BC}}(x_0, x_1) \|.$$

Taking limits as $n \to \infty$, we have $\| d_{\mathbb{BC}}(x_n, x_m) \| \to 0$ i.e, $d_{\mathbb{BC}}(x_n, x_m) = 0$ as $n \to \infty$.

Case (2)**.** Now, for $m - n$, is even, let $m - n = 2k$ then we have

$$\begin{aligned} d_{\mathbb{BC}}(x_n, x_m) &\preceq_{i_2} d_{\mathbb{BC}}(x_n, x_{n+1}) + d_{\mathbb{BC}}(x_{n+1}, x_{n+2}) + d_{\mathbb{BC}}(x_{n+2}, x_m) \\ &\preceq_{i_2} [h^n + h^{n+1}] d_{\mathbb{BC}}(x_0, x_1) + [h^{n+2} + h^{n+3}] d_{\mathbb{BC}}(x_0, x_1) + \ldots \\ &\quad + [h^{n+2k-2} + h^{n+2k-1}] d_{\mathbb{BC}}(x_0, x_2) + h^{n+2k-2} d_{\mathbb{BC}}(x_0, x_2) \\ &\preceq_{i_2} h^n [1 + h][1 + h^2 + h^4 + \ldots] d_{\mathbb{BC}}(x_0, x_1) + h^{n-2} d_{\mathbb{BC}}(x_0, x_1) \end{aligned}$$

let $L = \max\{d_{\mathbb{BC}}(x_0, x_1), d_{\mathbb{BC}}(x_0, x_2)\}$, then

$$d_{\mathbb{BC}}(x_n, x_m) \preceq_{i_2} \frac{(1+h)h^n}{1-h^2} L + h^{n-2} L$$

$$\| d_{\mathbb{BC}}(x_n, x_m) \| \leq \frac{h^n(1+h)}{1-h^2} \| L \| + h^{n-2} \| L \|.$$

Taking limits as $n \to \infty$, we have $\| d_{\mathbb{BC}}(x_n, x_m) \| \to 0$ i.e, $d_{\mathbb{BC}}(x_n, x_m) = 0$ as $n \to \infty$.

Therefore $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete there exists $v \in X$ such that $\lim_{n\to\infty} x_n = v$.

Suppose that S is continuous then

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Sx_{2n} = S(\lim_{n\to\infty} x_{2n}) = Sv.$$

Hence $v = Sv$.

We now show that $Tv = v$.

$$\| d_{\mathbb{BC}}(Tv, v) \| = \| d_{\mathbb{BC}}(Tv, Sv) \|$$

$$\leq \frac{\| \alpha[p + d_{\mathbb{BC}}(v, Sv)] d_{\mathbb{BC}}(v, Tv)^r d_{\mathbb{BC}}(v, Tv)^q d_{\mathbb{BC}}(v, Sv)^s \|}{\| \mu d_{\mathbb{BC}}(v, Sv) + \lambda d_{\mathbb{BC}}(v, Tv) + d_{\mathbb{BC}}(v, v) \|}$$

$$+ \beta \| d_{\mathbb{BC}}(v, v) \| + \delta \| d_{\mathbb{BC}}(v, Sv) + d_{\mathbb{BC}}(v, Tv) \|$$

$$\leq \delta \| d_{\mathbb{BC}}(v, Tv) \|,$$

which is a contradiction, since $\delta < 1$. Therefore $v = Tv$.

Hence $Tv = Sv = v$. Thus S and T have a common fixed point in X.

Next, suppose neither S nor T is continuous, then we have $x_n \preceq v$ for all $n \in \mathbb{N}$.

We now claim that v is a fixed point of S.

$$d_{\mathbb{BC}}(v, Sv) = d_{\mathbb{BC}}(Sv, Tv) \preceq_{i2} d_{\mathbb{BC}}(v, x_{2n+1}) + d_{\mathbb{BC}}(x_{2n+1}, x_{2n+2}) + d_{\mathbb{BC}}(x_{2n+2}, Sv)$$

$$\preceq_{i_2} d_{\mathbb{BC}}(v, x_{2n+1}) + d_{\mathbb{BC}}(x_{2n+1}, x_{2n+2})$$

$$+ \frac{\alpha[p + d_{\mathbb{BC}}(v, Sv)]d_{\mathbb{BC}}(x_{2n+1}Tx_{2n+1})^r d_{\mathbb{BC}}(v, Tx_{2n+1})^q d_{\mathbb{BC}}(x_{2+1}, Sv)^s}{\mu d_{\mathbb{BC}}(x_{2n+1}, Sv) + \lambda d_{\mathbb{BC}}(v, Tx_{2n+1}) + d_{\mathbb{BC}}(x_{2n+1}, v)}$$

$$+ \beta d_{\mathbb{BC}}(x_{2n+1}, v) + \delta[d_{\mathbb{BC}}(v, Sv) + d_{\mathbb{BC}}(x_{2n+1}, x_{2n+2})]$$

$$\| d_{\mathbb{BC}}(v, Sv) \| \leq \| d_{\mathbb{BC}}(v, x_{2n+1}) \| + \| d_{\mathbb{BC}}(x_{2n+1}, x_{2n+2}) \|$$

$$+ 2\sqrt{2}\, \frac{\alpha\| [p + d_{\mathbb{BC}}(v, Sv)] \| \| d_{\mathbb{BC}}(x_{2n+1}, Tx_{2n+1}) \|^r \| d_{\mathbb{BC}}(v, Tx_{2n+1}) \|^q \| d_{\mathbb{BC}}(x_{2n+1}, Sv) \|^s}{\| \mu d_{\mathbb{BC}}(x_{2n+1}, Sv) + \lambda d_{\mathbb{BC}}(v, Tx_{2n+1}) + d_{\mathbb{BC}}(x_{2n+1}, v) \|}$$

$$+ \beta \| d_{\mathbb{BC}}(v, x_{2n+1}) \| + \delta[\| d_{\mathbb{BC}}(v, Sv) + d_{\mathbb{BC}}(x_{2n+1}, Sv) \|].$$

Taking limits as $n \to \infty$, using lemma 1.7, we get

$$(1 - \delta) d_{\mathbb{BC}}(v, Sv) \leq 0.$$

This implies $d_{\mathbb{BC}}(v, Sv) = 0$, which implies v is a fixed point of S.

Similar to the above argument, we can show that $Tv = v$.

Thus, S and T have a common fixed point in X.

Next, we suppose that the set of all common fixed points of S and T is totally ordered.

If possible suppose that z, is another common fixed point of S and T. From condition (2.1.1), we have

$$d_{\mathbb{BC}}(v, z) = d_{\mathbb{BC}}(Sv, Tz) \preceq_{i_2} \frac{[p + d_{\mathbb{BC}}(v, Sv)]d_{\mathbb{BC}}(z, Tz)^r d_{\mathbb{BC}}(v, Tz)^q d_{\mathbb{BC}}(z, Sv)^s}{\mu d_{\mathbb{BC}}(z, Sv) + d_{\mathbb{BC}}(v, Tz) + d_{\mathbb{BC}}(v, z)}$$

$$+ \beta d_{\mathbb{BC}}(v, Tz) + \delta[d_{\mathbb{BC}}(v, Sv) + d_{\mathbb{BC}}(z, Tz)],$$

which implies $d_{\mathbb{BC}}(v, z) \preceq_{i_2} \beta d_{\mathbb{BC}}(v, z)$,

thus $(1 - \beta) d_{\mathbb{BC}}(v, z) \preceq_{i_2} 0$, hence $v = z$, since $\beta < 1$.

Hence S and T have a unique common fixed point in X.

Conversely, suppose that S and T have common fixed point, then the set of common fixed points of S and T being singleton, is totally ordered. This completes the proof of this theorem.

By choosing $S = T,$ we have the following corollary.

Corollary 2.2. *Let* $(X, \preceq)$ *be partially ordered set such that there exists a complete generalized bicomplex valued metric space on* X *with degenerated* $d_{\mathbb{BC}}(x, y) + d_{\mathbb{BC}}(u, v) + d_{\mathbb{BC}}(w, z)$ *for all* $x, y, u, v, w, z \in X$ *and* $T : X \to X$ *be a mapping satisfying the condition: for every comparable* $x, y \in X,$

$$d_{\mathbb{BC}}(Tx, Ty) \preceq_{i_2} \frac{\alpha[p + d_{\mathbb{BC}}(x, Tx)]d_{\mathbb{BC}}(y, Ty)^r d_{\mathbb{BC}}(x, Ty)^q d_{\mathbb{BC}}(y, Tx)^s}{\mu d_{\mathbb{BC}}(y, Tx) + \lambda d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}(x, y)}$$

$$+ \beta d_{\mathbb{BC}}(x, y) + \delta[d_{\mathbb{BC}}(x, Tx) + d_{\mathbb{BC}}(y, Ty)] \tag{2.2.1}$$

where $\alpha, p, q, r, s, \mu, \lambda \in \mathbb{R}^+$ *and* $\beta, \delta \in [0, 1]$ *with* $\beta + 2\delta < 1$ *and*

$d_{\mathbb{BC}}(Tx, Ty) = 0$ *when* $\mu d_{\mathbb{BC}}(y, Tx) + \lambda d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}(x, y) = 0.$

Also, suppose that T is weakly isotone increasing on X. If T is continuous or for any nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X, we have $x_n \preceq z$ then T has a unique common fixed point in X. Moreover, the set of fixed points of T is totally ordered if and only if T has a unique common fixed point in X.

By choosing $p = r = 1, q = 0, s = 0, \delta = 0, \mu d_{\mathbb{BC}}(y, Tx) + \lambda d_{\mathbb{BC}}(x, Ty) = 1,$ we have the following corollary.

Corollary 2.3. *Let* $(X, \preceq)$ *be partially ordered set such that there exists a complete generalized bicomplex valued metric space on* X *with degenerated*

$d_{\mathbb{BC}}(x, y) + d_{\mathbb{BC}}(u, v) + d_{\mathbb{BC}}(w, z)$ *for all* $x, y, u, v, w, z \in X$ *and* $S, T : X \to X$ *be a mapping satisfying the condition: for every comparable* $x, y \in X,$

$$d_{\mathbb{BC}}(Sx, Ty) \preceq_{i_2} \frac{\alpha[1 + d_{\mathbb{BC}}(x, Sx)]d_{\mathbb{BC}}(y, Ty)}{1 + d_{\mathbb{BC}}(x, y)} + \beta d_{\mathbb{BC}}(x, y) \tag{2.3.1}$$

where $\alpha \in \mathbb{R}^+$ *and* $\beta \in [0, 1).$

Also, suppose that S is T is weakly isotone increasing on X. If S or T is continuous or for any nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X, we have $x_n \preceq z$ then S and T have a unique common fixed point in X. Moreover, the set of common fixed points of S and T is totally ordered if and only if S and T have a unique common fixed point in X.

By choosing $p = 0, q = 0, s = 0, \mu = 1, \lambda = 1$ and $r = 1$ we have the following corollary.

Corollary 2.4. *Let* $(X, \preceq)$ *be partially ordered set such that there exists a complete generalized bicomplex valued metric space on* X *with degenerated* $d_{\mathbb{BC}}(x, y) + d_{\mathbb{BC}}(u, v) + d_{\mathbb{BC}}(w, z)$ *for all* $x, y, u, v, w, z \in X$ *and* $S, T : X \to X$ *be a mapping satisfying the condition: for every comparable* $x, y \in X$,

$$d_{\mathbb{BC}}(Sx, Ty) \preceq_{i_2} \frac{\alpha[d_{\mathbb{BC}}(x, Sx)]d_{\mathbb{BC}}(y, Ty)}{d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}(y, Sx) + d_{\mathbb{BC}}(x, y)}$$

$$+ \beta d_{\mathbb{BC}}(x, y) + \delta[d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(y, Ty)] \tag{2.4.1}$$

where $\alpha \in \mathbb{R}^+$ *and* $\beta, \delta \in [0, 1)$ *with* $\beta + 2\delta < 1$ *and* $d_{\mathbb{BC}}(Sx, Ty) = 0$ *when* $d_{\mathbb{BC}}(y, Sx) + d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}(x, y) = 0$.

Also, suppose that S is T is weakly isotone increasing on X. If S or T is continuous or for any nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X, we have $x_n \preceq z$ then S and T have a unique common fixed point in X. Moreover, the set of common fixed points of S and T is totally ordered if and only if S and T have a unique common fixed point in X.

By choosing $p = 0, q = 0, s = 0, \mu = 0, \lambda = 0$ and $r = 1$ we have the following corollary.

Corollary 2.5. *Let* $(X, \preceq)$ *be partially ordered set such that there exists a complete generalized bicomplex valued metric space on* X *with degenerated* $d_{\mathbb{BC}}(x, y) + d_{\mathbb{BC}}(u, v) + d_{\mathbb{BC}}(w, z)$ *for all* $x, y, u, v, w, z \in X$ *and* $S, T : X \to X$ *be a mapping satisfying the condition: for every comparable* $x, y \in X$,

$$d_{\mathbb{BC}}(Sx, Ty) \preceq_{i_2} \frac{\alpha[d_{\mathbb{BC}}(x, Sx)]d_{\mathbb{BC}}(y, Ty)}{d_{\mathbb{BC}}(x, y)}$$

$$+\beta d_{\mathbb{BC}}(x, y) + \delta[d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(y, Ty)] \tag{2.5.1}$$

where $\alpha \in \mathbb{R}^+$ *and* $\beta, \delta \in [0, 1)$ *with* $\beta + 2\delta < 1$ *and* $d_{\mathbb{BC}}(Sx, Ty) = 0$ *when* $d_{\mathbb{BC}}(x, y) = 0$.

Also, suppose that S is T is weakly isotone increasing on X. If S or T is continuous or for any nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X, we have $x_n \preceq z$ then S and T have a unique common fixed point in X. Moreover, the set of common fixed points of S and T is totally ordered if and only if S and T have a unique common fixed point in X.

By choosing $\alpha = 0$ and $\delta = 0$ in Corollary 2.2, we have the following corollary.

Corollary 2.6. *Let* $(X, \preceq)$ *be partially ordered set such that there exists a complete generalized bicomplex valued metric space on* X *and* $T : X \to X$ *be a mapping satisfying the condition: for every comparable* $x, y \in X$,

$$d_{\mathbb{BC}}(Tx, Ty) \preceq_{i_2} \beta d_{\mathbb{BC}}(x, y) \tag{2:6:1}$$

where $\beta \in [0, 1)$.

Also, suppose that T is weakly increasing on X. If T is continuous or for any nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X, we have $x_n \preceq z$ then T has a unique common fixed point in X. Moreover, the set of fixed points T is totally ordered if and only if T has a unique common fixed point in X.

The following is an example in support of Theorem 2.1.

Example 2.7. Let $X = [0, \infty)$ with the partial order: $x \preceq y \Leftrightarrow x = y$ or $x, y \in [0, 1]$ with $y \leq x$. We define $d_{\mathbb{BC}} X \times X \to \mathbb{C}_2$ by

$d_{\mathbb{BC}}(x, y) = i_1 i_2 |x - y|$ then $(X, d_{\mathbb{BC}})$ is a complete bicomplex valued metric space with degenerated $d_{\mathbb{BC}}(x, u) + d_{\mathbb{BC}}(u, v) + d(v, z)$ for all $x, y, u, v \in X$.

Let $S, T: X \times X \to R$ be defined by

$$Sx = \begin{cases} \frac{x}{4} \text{ if } x \in [0, 1] \\ \frac{x}{2(1+x)} \text{ if } x \in (1, \infty) \end{cases} \text{ and } Tx = \begin{cases} \frac{x}{8} \text{ if } x \in [0, 1] \\ \frac{x}{4(1+x)} \text{ if } x \in (1, \infty) \end{cases}$$

We first check that S is T-weakly isotone increasing.

Case(i) When $x \in [0, 1]$, we have

$$Sx = \frac{x}{4} \geq \frac{x}{36} = TSx \geq \frac{x}{144} = STSx.$$

Case(ii) When $x \in (1, \infty)$, we have $Sx = \frac{x}{2(1+x)} \leq x$ and $Tx = \frac{x}{4(1+x)} \leq x$. Thus for all $x \in (1, \infty)$, we have $TSx = \frac{Sx}{4(1+Sx)} \leq Sx$ and $STSx = \frac{TSx}{2(1+TSx)} \leq TSx$ therefore $Sx \preceq TSx \preceq STSx$ for all $x \in X$. Hence S is T-weakly isotone increasing.

We now verify inequality (2.1.1) with $r = 1, s = 1, t = 1, p = 1, \alpha = \frac{1}{4}$, $\mu = \frac{1}{8}, \beta = \frac{1}{6}$ and $\delta = \frac{1}{3}$.

Let us consider two comparable elements $x, y \in X$ with $y \leq x$ then we have following cases:

Case(1) When $x \in [0, 1]$, so $y \in [0, 1]$, then $x \leq \frac{y}{2}$ or $\frac{y}{2} \leq x$.

(i) When $x \leq \frac{y}{2}$, we have

$$d_{\mathbb{BC}}(Sx, Ty) = d_{\mathbb{BC}}\left(\frac{x}{4}, \frac{y}{8}\right) = z\frac{1}{4}\left(\frac{y}{2} - x\right) \preceq_{i_2} \frac{z}{8}y,$$

where $z = i_1 i_2$.

Hence

$$d_{\mathbb{BC}}(Sx,\ Ty) \preceq_{i_2} z\frac{1}{3}\left(\frac{3x}{4}+\frac{7y}{8}\right) = \delta[d_{\mathbb{BC}}(x,\ Sx)+d_{\mathbb{BC}}(y,\ Ty)]$$

$$\preceq_{i_2} \frac{\alpha[p+d_{\mathbb{BC}}(x,\ Sx)d_{\mathbb{BC}}(y,\ Ty)^r d_{\mathbb{BC}}(x,\ Ty)^q d_{\mathbb{BC}}(y,\ Sx)^s}{\mu d_{\mathbb{BC}}(y,\ Sx)+\lambda d_{\mathbb{BC}}(x,\ Ty)+d_{\mathbb{BC}}(x,\ y)}$$

$$+\beta d_{\mathbb{BC}}(x,\ y)+\delta[d_{\mathbb{BC}}(x,\ Sx)+d_{\mathbb{BC}}(y,\ Ty)]$$

(ii) When $x \geq \frac{y}{2}$, we have

$$d_{\mathbb{BC}}(Sx,\ Ty) = d_{\mathbb{BC}}\left(\frac{x}{4},\ \frac{y}{8}\right) = z\frac{1}{4}\left(x-\frac{y}{2}\right) \preceq_{i_2} \frac{z}{4}x,$$

where $z = i_1 i_2$.

Hence

$$d_{\mathbb{BC}}(Sx,\ Ty) \preceq_{i_2} z\frac{1}{3}\left(\frac{3x}{4}+\frac{7y}{8}\right) = \delta[d_{\mathbb{BC}}(x,\ Sx)+d_{\mathbb{BC}}(y,\ Ty)]$$

$$\preceq_{i_2} \frac{\alpha[p+d_{\mathbb{BC}}(x,\ Sx)d_{\mathbb{BC}}(y,\ Ty)^r d_{\mathbb{BC}}(x,\ Ty)^q d_{\mathbb{BC}}(y,\ Sx)}{\mu d_{\mathbb{BC}}(y,\ Sx)+\lambda d_{\mathbb{BC}}(x,\ Ty)+d_{\mathbb{BC}}(x,\ y)}$$

$$+\beta d_{\mathbb{BC}}(x,\ y)+\frac{1}{3}[d_{\mathbb{BC}}(x,\ Sx)+d_{\mathbb{BC}}(y,\ Ty)].$$

Case(2)**.** When $x > 1$, we have $x = y$, then

$$d_{\mathbb{BC}}(Sx, Ty) = d\left(\frac{x}{2(1+x)}, \frac{x}{4(1+x)}\right) = z\frac{x}{4(1+x)} \preceq_{i_2} \frac{1}{3}z\frac{5x+8x^2}{4(1+x)}$$

$$= \frac{1}{3}[d_{\mathbb{BC}}(x,\ Sx)+d_{\mathbb{BC}}(y,\ Ty)]$$

$$\preceq_{i_2} \frac{\alpha[p+d_{\mathbb{BC}}(x,\ Sx)]d_{\mathbb{BC}}(y,\ Ty)^r d_{\mathbb{BC}}(x,\ Ty)^q d_{\mathbb{BC}}(x,\ Sx)^s}{\mu d_{\mathbb{BC}}(y,\ Sx)+\lambda d_{\mathbb{BC}}(x,\ Ty)+d_{\mathbb{BC}}(x,\ y)}+\beta d_{\mathbb{BC}}(x,\ y)$$

$$+\ \delta[d_{\mathbb{BC}}(x,\ Sx)+d_{\mathbb{BC}}(y,\ Ty)],$$

thus the inequality (2.1.1) is verified .

Hence, S and T satisfies all the conditions of Theorem 2.1 with '0' is the unique common fixed point of S and T.

Here we note that inequality (1.11.1) fails to hold for any a and b at $x = 2$ and $y = 2$, since

$$d_{\mathbb{BC}}(Sx, Ty) = d_{\mathbb{BC}}\left(\frac{1}{3}, \frac{1}{6}\right) = k\frac{1}{6} \not\precsim_{i_2} bk^2 \frac{55}{18}$$

$$= b\frac{d_{\mathbb{BC}}(x, Sx)d_{\mathbb{BC}}(y, Ty)}{1 + d_{\mathbb{BC}}(x, y)} + ad_{\mathbb{BC}}(x, y).$$

This shows that condition (2.1.1) is more general than (1.11.1)

Example 2.8. Let $X = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$, we define $d_{\mathbb{BC}} : X \times X \to \mathbb{C}_2$ by

$$d_{\mathbb{BC}}\left(1, \frac{1}{2}\right) = d_{\mathbb{BC}}\left(\frac{1}{2}, 1\right) = i_1 i_2,$$

$$d_{\mathbb{BC}}\left(1, \frac{1}{3}\right) = d_{\mathbb{BC}}\left(\frac{1}{3}, 1\right) = 2i_1 i_2,$$

$$d_{\mathbb{BC}}\left(1, \frac{1}{4}\right) = d_{\mathbb{BC}}\left(\frac{1}{4}, 1\right) = i_1 i_2$$

$$d_{\mathbb{BC}}\left(\frac{1}{2}, \frac{1}{3}\right) = d_{\mathbb{BC}}\left(\frac{1}{3}, \frac{1}{2}\right) = 5i_1 i_2$$

$$d_{\mathbb{BC}}\left(\frac{1}{4}, \frac{1}{2}\right) = d_{\mathbb{BC}}\left(\frac{1}{2}, \frac{1}{4}\right) = 4i_1 i_2 \; d_{\mathbb{BC}}\left(\frac{1}{3}, \frac{1}{4}\right) = d_{\mathbb{BC}}\left(\frac{1}{4}, \frac{1}{3}\right) = 3i_1 i_2 \text{ and}$$

$d_{\mathbb{BC}}(x, x) = 0$ for all $x \in X$.

It is clear that $(X, d_{\mathbb{BC}})$ is a generalized bicomplex valued metric space. Also, $(X, d_{\mathbb{BC}})$ is not a bicomplex valued metric space, since

$$5i_1 i_2 = d_{\mathbb{BC}}\left(\frac{1}{2}, \frac{1}{3}\right) \not\precsim d_{\mathbb{BC}}\left(\frac{1}{2}, 1\right) + d_{\mathbb{BC}}\left(1, \frac{1}{3}\right) = 3i_1 i_2.$$

We define $\preceq$ on X by $x \preceq y$ if $y \geq x$ We define $S, T : X \to X$ by

$$S1 = S\frac{1}{3} = S\frac{1}{4} = \frac{1}{4}, S\frac{1}{2} = \frac{1}{3} \text{ and } T1 = T\frac{1}{2} = \frac{1}{3}, T\frac{1}{3} = T\frac{1}{4} = \frac{1}{4}.$$

Then clearly, S and T are weakly isotone maps. Also, S and T satisfies all the conditions of Theorem 2.1 with $p = 1$, $\alpha = 7$, $q = r = 1$, $s = 2$, $\beta = \frac{5}{6}$, $\delta = \frac{1}{15}$ with $\frac{1}{4}$ is the unique common fixed point of S and T.

Here, we note that conditions (1.10.1) and (1.11.1) fails to hold for any β and δ at $x = 1$ and $y = 1$ since

$$d_{\mathbb{BC}}(Sx, Ty) = d_{\mathbb{BC}}\left(\frac{1}{4}, \frac{1}{3}\right) = 3i_1 i_2 \not\preceq_{i_2} 3\delta i_1 i_2 = \beta d_{\mathbb{BC}}(1, 1) + \delta\left[d_{\mathbb{BC}}\left(1, \frac{1}{4}\right)\right.$$

$$\left. + d_{\mathbb{BC}}(1, \frac{1}{3})\right] = \delta[d_{\mathbb{BC}}(y, Ty)]$$

since $\delta < 1$ and

$$d_{\mathbb{BC}}(Sx, Ty) = d_{\mathbb{BC}}\left(\frac{1}{4}, \frac{1}{3}\right) = 3i_1 i_2 \not\preceq_{i_2} 0 = \beta d_{\mathbb{BC}}(1, 1)$$

since $\beta < 1$.

This shows that condition (2.1.1) is more general than (1.10.1) and (1.11.1).

Theorem 2.9. *Let* $(X, \preceq)$ *be partially ordered set such that there exists a complete generalized bicomplex valued metric space on* X *with degenerated* $d_{\mathbb{BC}}(x, y) + d_{\mathbb{BC}}(u, v) + d_{\mathbb{BC}}(w, z)$ *for all* $x, y, u, v, w, z \in X$ *and* $S, T : X \to X$ *be a mapping satisfying the condition: for every comparable* $x, y \in X$,

$$d_{\mathbb{BC}}(Sx, Ty) \preceq_{i_2} \alpha[d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(y, Ty)]$$

$$+ \frac{\beta d_{\mathbb{BC}}(x, Sx) d_{\mathbb{BC}}^2(x, Ty) + d_{\mathbb{BC}}(y, Ty) d_{\mathbb{BC}}^2(y, Sx)}{d_{\mathbb{BC}}^2(x, Ty) + d_{\mathbb{BC}}^2(y, Sx)}$$

$$+ \frac{\gamma d_{\mathbb{BC}}(x, Sx) d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}^2(x, y) + d_{\mathbb{BC}}(y, Sx) d_{\mathbb{BC}}(x, y)}{d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}(x, y)} \tag{2.9.1}$$

where $\alpha, \beta, \gamma \geq 0$ *with* $2\alpha + \beta + \gamma < 1$ *and*

$d_{\mathbb{BC}}(Sx, Ty) = 0$ *when* $d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}(x, y) = 0$ *or*

$d^2_{\mathbb{BC}}(x, Ty) + d^2_{\mathbb{BC}}(y, Sx) = 0.$

Also, suppose that S is T is weakly isotone increasing on X. If S or T is continuous or for any nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X, we have $x_n \preceq z$ then S and T have a unique common fixed point in X. Moreover, the set of common fixed points of S and T is totally ordered if and only if S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Without loss of generality, suppose that $Sx_0 \neq x_0$. Let us define a sequence $\{x_n\}$ in X as follows:

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}, \text{ for } n = 0, 1, 2, 3, \ldots \tag{2.9.2}$$

Now, in view of our assumption, we have

$$x_1 = Sx_0 \preceq TSx_0 = Tx_1 = x_2 \preceq STSx_0 = STx_1 \preceq x_3.$$

By repeating this process, we get

$$x_1 \preceq x_2 \preceq x_3 \preceq x_4 \preceq \ldots \preceq x_n \preceq x_{n+1} \ldots \tag{2.9.3}$$

Assume that $d(x_{2n}, x_{2n+1}) > 0$ for all $n \in \mathbb{N}$.

Otherwise, if $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N}$ then we have $x_{2n} = Sx_{2n}$. We now show that x_{2n} is a common fixed point of S and T. Suppose that $d(x_{2n+1}, x_{2n+2}) \geq 0$, since x_{2n+1}, x_{2n+2} are comparable, we have $d(Sx_{2n}, Tx_{2n+1}) \preceq_{i_2} \alpha[d_{\mathbb{BC}}(x_{2n}, Sx_{2n}) + d_{\mathbb{BC}}(x_{2n+1}, Tx_{2n+1})]$

$$+ \beta \frac{d_{\mathbb{BC}}(x_{2n}, Sx_{2n})d^2_{\mathbb{BC}}(x_{2n+1}, Tx_{2n+1}) + d_{\mathbb{BC}}(x_{2n+1}, Tx_{2n+1})d^2_{\mathbb{BC}}(x_{2n+1}, Sx_{2n})}{d^2_{\mathbb{BC}}(x_{2n}, Tx_{2n+1}) + d^2_{\mathbb{BC}}(x_{2n+1}, Sx_{2n})}$$

$$+ \gamma \frac{d_{\mathbb{BC}}(x_{2n}, Sx_{2n})d_{\mathbb{BC}}(x_{2n}, Tx_{2n+1}) + d^2_{\mathbb{BC}}(x_{2n+1}, x_{2n}) + d_{\mathbb{BC}}(Sx_{2n}, x_{2n})d_{\mathbb{BC}}(x_{2n}, x_{2n+1})}{d_{\mathbb{BC}}(x_{2n}, Sx_{2n}) + d_{\mathbb{BC}}(x_{2n}, Tx_{2n+1}) + d_{\mathbb{BC}}(x_{2n}, x_{2n+1})}$$

$(1 - \alpha)d_{\mathbb{BC}}(x_{2n+1}, x_{2n+2}) \preceq_{i_2} 0$ this implies $x_{2n+1} = x_{2n+2}$.

Hence x_{2n} is a common fixed point of S and T.

Thus, we suppose that $x_n \neq x_{n+1}$ for all $n \in N$.

We now show that

$$d_{\mathbb{BC}}(x_{2n},\, x_{2n+1}) \preceq_{i_2} \left(\frac{\alpha+\beta+\gamma}{1-\alpha}\right) d_{\mathbb{BC}}(x_{2n},\, x_{2n-1})$$

for all $n \in \mathbb{N}$.

By considering the condition (2.9.3), we have

$$d(Sx_{2n},\, Tx_{2n+1}) \preceq_{i_2} \alpha[d_{\mathbb{BC}}(x_{2n},\, Sx_{2n}) + d_{\mathbb{BC}}(x_{2n+1},\, Tx_{2n+1})]$$

$$+\, \beta \frac{d_{\mathbb{BC}}(x_{2n},\, Sx_{2n}) d_{\mathbb{BC}}^2(x_{2n+1},\, Tx_{2n+1}) + d_{\mathbb{BC}}(x_{2n+1},\, Tx_{2n+1}) d_{\mathbb{BC}}^2(x_{2n+1},\, Sx_{2n})}{d_{\mathbb{BC}}^2(x_{2n},\, Tx_{2n+1}) + d_{\mathbb{BC}}^2(x_{2n+1},\, Sx_{2n})}$$

$$+\, \gamma \frac{d_{\mathbb{BC}}(x_{2n}, Sx_{2n}) d_{\mathbb{BC}}(x_{2n}, Tx_{2n+1}) + d_{\mathbb{BC}}^2(x_{2n+1}, x_{2n}) + d_{\mathbb{BC}}(Sx_{2n}, x_{2n}) d_{\mathbb{BC}}(x_{2n}, x_{2n+1})}{d_{\mathbb{BC}}(x_{2n}, Sx_{2n}) + d_{\mathbb{BC}}(x_{2n}, Tx_{2n+1}) + d_{\mathbb{BC}}(x_{2n}, x_{2n+1})}$$

which implies

$$d_{\mathbb{BC}}(x_{2n+1},\, x_{2n+2}) \preceq_{i_2} \left(\frac{\alpha+\beta+\gamma}{1-\alpha}\right) d_{\mathbb{BC}}(x_{2n},\, x_{2n+1}). \tag{2.9.4}$$

Similarly,

$$d_{\mathbb{BC}}(x_{2n},\, x_{2n+1}) \preceq_{i_2} \left(\frac{\alpha+\beta+\gamma}{1-\alpha}\right) d_{\mathbb{BC}}(x_{2n},\, x_{2n-1}) \tag{2:9:5}$$

Thus from (2.9.4) and (2.9.5), we get

$$d_{\mathbb{BC}}(x_{2n},\, x_{2n+1}) \preceq_{i_2} \left(\frac{\alpha+\beta+\gamma}{1-\alpha}\right) d_{\mathbb{BC}}(x_{2n},\, x_{2n-1}) \tag{2:9:6}$$

for all $n \in \mathbb{N}$.

Let $\wp = \dfrac{\alpha+\beta+\gamma}{1-\alpha}$. (2.9.7)

Thus, from (2.9.6), it follows that

$$d_{\mathbb{BC}}(x_n,\, x_{n+1}) \preceq_{i_2} \wp\, d_{\mathbb{BC}}(x_n,\, x_{n-1}) \tag{2.9.8}$$

Hence for all $n \geq 0$, we have

$$d_{\mathbb{BC}}(x_n,\, x_{n+1}) \preceq_{i_2} \wp^n d_{\mathbb{BC}}(x_n,\, x_{n-1}). \tag{2.9.9}$$

Following on the same lines of proof of Cauchy sequence

Therefore $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete there exists $v \in X$ such that $\lim_{n\to\infty} x_n = v$.

Suppose that S is continuous then

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Sx_{2n} = S(\lim_{n\to\infty} x_{2n}) = Sv.$$

Hence $v = Sv$.

We now show that $Tv = v$.

$$\| d_{\mathbb{BC}}(Tv, Sv) \| \leq \| \alpha[d_{\mathbb{BC}}(v, Sv) + d_{\mathbb{BC}}(v, Tv)] \|$$

$$+ \beta \frac{\| d_{\mathbb{BC}}(v, Sv) d_{\mathbb{BC}}^2(v, Tv) + d_{\mathbb{BC}}(v, Tv) d_{\mathbb{BC}}^2(v, Sv) \|}{\| d_{\mathbb{BC}}^2 + (v, Tv) + d_{\mathbb{BC}}^2(v, Sv) \|}$$

$$+ \gamma \frac{\| \gamma d_{\mathbb{BC}}(v, Sv) d_{\mathbb{BC}}(v, Tv) + d_{\mathbb{BC}}^2(v, v) + d_{\mathbb{BC}}(v, Sv) d_{\mathbb{BC}}(v, v) \|}{\| d_{\mathbb{BC}}(v, Sv) + d_{\mathbb{BC}}(v, Tv) + d_{\mathbb{BC}}(v, y) \|}$$

which implies $\| (1-\alpha) d_{\mathbb{BC}}(Tv, v) \| \leq 0$.

Therefore $v = Tv$. Hence $Tv = Sv = v$. Thus S and T have a common fixed point.

Next, suppose that neither S nor T is continuous, then we have $x_n \preceq v$ for all $n \in N$.

We now claim that v is a fixed point of S.

$$d_{\mathbb{BC}}(v, Sv) \preceq_{i_2} d_{\mathbb{BC}}(v, x_{2n+1}) + d_{\mathbb{BC}}(x_{2n+1}, x_{2n+2}) + d_{\mathbb{BC}}(x_{2n+2}, Sv)$$

$$\preceq_{i_2} d_{\mathbb{BC}}(v, x_{2n+1}) + d_{\mathbb{BC}}(x_{2n+1}, x_{2n+2}) + \alpha[d_{\mathbb{BC}}(v, Sv) + d_{\mathbb{BC}}(x_{2n+1}, Tx_{2n+1})]$$

$$+ \beta \frac{d_{\mathbb{BC}}(v, Sv) d_{\mathbb{BC}}^2(v, Tx_{2n+1}) + d_{\mathbb{BC}}(x_{2n+1}, Tx_{2n+1}) d_{\mathbb{BC}}^2(x_{2n+1}, Sv)}{d_{\mathbb{BC}}^2(v, Tx_{2n+1}) + d_{\mathbb{BC}}^2(x_{2n+1}, Sv)}$$

$$+ \gamma \frac{d_{\mathbb{BC}}(v, Sv) d_{\mathbb{BC}}(v, Tx_{2n+1}) + d_{\mathbb{BC}}^2(v, x_{2n+1}) + d_{\mathbb{BC}}(x_{2n+1}, Sv) d_{\mathbb{BC}}(v, x_{2n+1})}{d_{\mathbb{BC}}(v, Sv) + d_{\mathbb{BC}}(v, Tx_{2n+1}) + d_{\mathbb{BC}}(x_{2n+1}, v)}$$

$$\| d_{\mathbb{BC}}(v, Sv) \| \preceq_{i_2} \| d_{\mathbb{BC}}(v, x_{2n+1}) \| + \| d_{\mathbb{BC}}(x_{2n+1}, x_{2n+2}) \| + \| d_{\mathbb{BC}}(x_{2n+2}, Sv) \|$$

$$\preceq_{i_2} \| d_{\mathbb{BC}}(v, x_{2n+1}) \| + \| d_{\mathbb{BC}}(x_{2n+1}, x_{2n+2}) \|$$

$$+\alpha \| [d_{\mathbb{BC}}(v, Sv) \| + \| d_{\mathbb{BC}}(x_{2n+1}, Tx_{2n+1})] \|$$

$$+ \beta\sqrt{2} \frac{\| d_{\mathbb{BC}}(v, Sv) \| \| d^2_{\mathbb{BC}}(v, Tx_{2n+1}) \| + \| d_{\mathbb{BC}}(x_{2n+1}, Tx_{2n+1}) \| \| d^2_{\mathbb{BC}}(x_{2n+1}, Sv) \|}{\| d^2_{\mathbb{BC}}(v, Tx_{2n+1}) + d^2_{\mathbb{BC}}(x_{2n+1}, Sv) \|}$$

$$+ \gamma\sqrt{2} \frac{\| d_{\mathbb{BC}}(v, Sv) \| \| d_{\mathbb{BC}}(v, Tx_{2n+1}) \| + \| d^2_{\mathbb{BC}}(v, x_{2n+1}) \| + \| d_{\mathbb{BC}}(x_{2n+1}, Sv) \| \| d_{\mathbb{BC}}(v, x_{2n+1}) \|}{\| d_{\mathbb{BC}}(v, Sv) + d_{\mathbb{BC}}(v, Tx_{2n+1}) + d_{\mathbb{BC}}(x_{2n+1}, v) \|}$$

which on taking limits $n \to \infty.$ we get $(1 - \alpha) \| d_{\mathbb{BC}}(v, Sv) \| = 0,$ this implies v is a fixed point of S.

Similar to the above argument, we can show that $Tv = v,$ consequently, it follows that v is a common fixed point of S and T,

Next, we suppose that the set of all common fixed points of S and T is totally ordered.

If possible suppose that z, is another common fixed point of S and T. From condition (2.9.1), we have

$$d_{\mathbb{BC}}(v, z) = d_{\mathbb{BC}}(Sv, Tz) \preceq_{i_2} \alpha[d_{\mathbb{BC}}(v, Sv) + d_{\mathbb{BC}}(z, Tz)].$$

$$+ \frac{\beta d_{\mathbb{BC}}(v, Sv) d^2_{\mathbb{BC}}(z, Tz) + d_{\mathbb{BC}}(z, Tz) d^2_{\mathbb{BC}}(z, Sv)}{d^2_{\mathbb{BC}}(v, Tz) + d^2_{\mathbb{BC}}(z, Sz)}$$

$$+ \gamma \frac{d_{\mathbb{BC}}(v, Sv) d_{\mathbb{BC}}(v, Tz) + d^2_{\mathbb{BC}}(v, z) + d_{\mathbb{BC}}(z, Sv) d_{\mathbb{BC}}(v, z)}{d_{\mathbb{BC}}(v, Sv) + d_{\mathbb{BC}}(v, Tv) + d_{\mathbb{BC}}(v, z)}$$

thus $(1 - \gamma) d_{\mathbb{BC}}(v, z) \preceq_{i_2} 0,$ hence $v = z,$ since $\gamma < 1.$

Hence S and T have a unique common fixed point in X. Conversely, suppose that S and T have only one common fixed points, then the set of common fixed point of S and T being singleton, is totally ordered.

By choosing $S = T$ in Theorem 2.9, we have the following corollary.

Corollary 2.10. *Let* $(X, \preceq)$ *be partially ordered set such that there exists a complete generalized bicomplex valued metric space on* X *with degenerated* $d_{\mathbb{BC}}(x, y) + d_{\mathbb{BC}}(u, v) + d_{\mathbb{BC}}(w, z)$ *for all* $x, y, u, v, w, z \in X$ *and* $T : X \to X$

be a mapping satisfying the condition: *for every comparable* $x, y \in X$,

$$d_{\mathbb{BC}}(Tx, Ty) \preceq_{i_2} \alpha[d_{\mathbb{BC}}(x, Tx) + d_{\mathbb{BC}}(y, Ty)]$$

$$+ \beta \frac{\alpha d_{\mathbb{BC}}(x, Tx) d_{\mathbb{BC}}^2(x, Ty) + d_{\mathbb{BC}}(y, Ty) d_{\mathbb{BC}}^2(y, Tx)}{d_{\mathbb{BC}}^2(x, Ty) + d_{\mathbb{BC}}^2(y, Tx)}$$

$$+ \gamma \frac{d_{\mathbb{BC}}(x, Tx) d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}^2(x, y) + d_{\mathbb{BC}}(y, Tx) d_{\mathbb{BC}}(x, y)}{d_{\mathbb{BC}}(x, Tx) + d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}(x, y)} \tag{2.10.1}$$

where with $2\alpha + \beta + \gamma < 1$ *and*

$d_{\mathbb{BC}}(Tx, Ty) = 0$ *when* $d_{\mathbb{BC}}(x, Tx) + d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}(x, y) = 0$ *or*

$d_{\mathbb{BC}}^2(x, Ty) + d_{\mathbb{BC}}^2(y, Tx) = 0.$

Also, suppose that T is weakly increasing on X. If T is continuous or for any nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X, we have $x_n \preceq z$ T has a unique common fixed point in X. Moreover, the set of fixed points of T is totally ordered if and only if T has a unique fixed point.

Example 2.11. Let $X = [-1, -2] \cup [0, 1] \cup (1, +\infty)$.

$d_{\mathbb{BC}}(0, -1) = d_{\mathbb{BC}}(-1, 0) = i_2 + i_1 i_2,$

$d_{\mathbb{BC}}(0, -2) = d_{\mathbb{BC}}(-2, 0) = i_2 + 5 i_1 i_2,$

and $d_{\mathbb{BC}}(-1, -2) = d_{\mathbb{BC}}(-2, -1) = i_2 + 5 i_1 i_2$

and $d_{\mathbb{BC}}(x, y) = (3 + 6i_1 + i_2 + 2i_1 i_2)|x - y|$, for all $x, y \in X$.

Then clearly, $(X, d_{\mathbb{BC}})$ is a generalized bicomplex valued metric space such that $d_{\mathbb{BC}}(x, y)$ is degenerated for all $x, y \in X$. Here we note that

$$5i_1 i_2 + i_2 = d_{\mathbb{BC}}(-1, -2) \succeq_{i_2} d_{\mathbb{BC}}(-1, 0) + d_{\mathbb{BC}}(0, -2) = i_2 + 3i_1 i_2.$$

Hence $(X, d_{\mathbb{BC}})$ is not a bicomplex valued metric space.

We define partial order on X by $x \preceq y \Leftrightarrow x = y$ or $x, y \in [0, 1]$ with $y \leq x$.

Let $S, T : X \times X \to R$ by

$$Sx = \begin{cases} 0 \ if \ x \in [-1, -2] \\ \frac{x}{6} \ if \ x \in [0, 1] \\ \frac{x}{4\sqrt{1+x^3}} \ if \ x \in (1, \infty) \end{cases} \quad \text{and } Tx = \begin{cases} 0 \ if \ x \in [-1, -2] \\ \frac{x}{36} \ if \ x \in [0, 1] \\ \frac{x}{8\sqrt{1+x^3}} \ if \ x \in (1, \infty) \end{cases}$$

We first check that S is T-weakly isotone increasing.

Case (i)**.** When $x \in [-1, -2]$, then we have

$$Sx = 0 \geq 0 = TSx \geq 0 = STSx.$$

Case (ii)**.** When $x \in [0, 1]$ we have

$$Sx = \frac{x}{6} \geq \frac{x}{216} = TSx \geq \frac{x}{1296} = STSx.$$

Case (iii)**.** When $x \in (1, \infty)$, we have

$$Sx = \frac{x}{4\sqrt{1+x^3}} \leq x \text{ and } Tx = \frac{x}{\sqrt{8(1+x^3)}} \leq x.$$

Thus for all $x \in (1, \infty)$, we have

$$TSx = \frac{x}{4(\sqrt{1+Sx^3})} \leq Sx \text{ and } STSx = \frac{TSx}{8(\sqrt{1+TSx^3})} \leq TSx \text{ therefore}$$

$Sx \preceq TSx \preceq STSx$ for all $x \in X$.

Hence S is T-weakly isotone increasing.

Let us consider two comparable elements $x, y \in X$ with $y \leq x$ then we have following cases.

Case (1)**.** When $x \in [0, 1]$, so $y \in [0, 1]$, then $x \leq \frac{y}{6}$ or $\frac{y}{6} \leq x$.

(i) When $x \leq \frac{y}{6}$, we have

$$d_{\mathbb{BC}}(Sx, Ty) = d_{\mathbb{BC}}\left(\frac{x}{6}, \frac{y}{36}\right) = z\frac{1}{6}\left(\frac{y}{6} - x\right) \preceq_{i_2} \frac{z}{36}y,$$

where $z = (3 + 6i_1 + i_2 + 2i_1 i_2)$.

Hence

$$d_{\mathbb{BC}}(Sx, Ty) \preceq_{i_2} z\frac{1}{5}\left(\frac{5x}{6} + \frac{35y}{36}\right) = \alpha[d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(y, Ty)]$$

$$\preceq_{i_2} \alpha[d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(y, Ty)] + \frac{\beta d_{\mathbb{BC}}(x, Sx) d_{\mathbb{BC}}^2(x, Ty) + d_{\mathbb{BC}}(y, Ty) d_{\mathbb{BC}}^2(y, Sx)}{d_{\mathbb{BC}}^2(x, Ty) + d_{\mathbb{BC}}^2(y, Sx)}$$

$$+ \frac{\gamma d_{\mathbb{BC}}(x, Sx) d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}^2(x, y) + d_{\mathbb{BC}}(y, Sx) d_{\mathbb{BC}}(x, y)}{d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}(x, y)}$$

(ii) When $x \geq \frac{y}{6}$, we have

$$d_{\mathbb{BC}}(Sx, Ty) = d_{\mathbb{BC}}\left(\frac{x}{6}, \frac{y}{36}\right) = z\frac{1}{6}\left(x - \frac{y}{6}\right) \preceq_{i_2} \frac{z}{6}x,$$

where $z = (3 + 6i_1 + i_2 + 2i_1 i_2)$.

Hence

$$d_{\mathbb{BC}}(Sx, Ty) \preceq_{i_2} z\frac{1}{5}\left(\frac{5x}{6} + \frac{35y}{36}\right) = \delta[d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(y, Ty)]$$

$$\preceq_{i_2} \alpha[d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(y, Ty)] + \frac{\beta d_{\mathbb{BC}}(x, Sx) d_{\mathbb{BC}}^2(x, Ty) + d_{\mathbb{BC}}(y, Ty) d_{\mathbb{BC}}^2(y, Sx)}{d_{\mathbb{BC}}^2(x, Ty) + d_{\mathbb{BC}}^2(y, Sx)}$$

$$+ \frac{\gamma d_{\mathbb{BC}}(x, Sx) d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}^2(x, y) + d_{\mathbb{BC}}(y, Sx) d_{\mathbb{BC}}(x, y)}{d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}(x, y)}$$

Case (2)**.** When $x > 1$, we have $x = y$, then

$$d_{\mathbb{BC}}(Sx, Ty) = d_{\mathbb{BC}}\left(\frac{x}{4\sqrt{1 + x^3}}, \frac{x}{8\sqrt{1 + x^3}}\right) = z\frac{x}{8\sqrt{1 + x^3}} \preceq_{i_2} \frac{1}{5}z\frac{16\sqrt{1 + x^3} - 3x}{8\sqrt{1 + x^2}}$$

$$= \frac{1}{5}[d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(y, Ty)]$$

$$\preceq_{i_2} \frac{1}{5}\left[d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(y, Ty)\right] + \frac{1}{6}\frac{d_{\mathbb{BC}}(x, Sx)d_{\mathbb{BC}}^2(x, Ty) + d_{\mathbb{BC}}(y, Ty)d_{\mathbb{BC}}^2(y, Sx)}{d_{\mathbb{BC}}^2(x, Ty) + d_{\mathbb{BC}}^2(y, Sx)}$$

$$+ \frac{1}{7}\frac{d_{\mathbb{BC}}(x, Sx)d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}^2(x, y) + d_{\mathbb{BC}}(y, Sx)d_{\mathbb{BC}}(x, y)}{d_{\mathbb{BC}}(x, Sx) + d_{\mathbb{BC}}(x, Ty) + d_{\mathbb{BC}}(x, y)}$$

hence the inequality (2.9.1) is verified. Also, 0 is the unique common fixed point of S and T. Thus all the conditions of Theorem 2.9 are verified.

References

[1] I. Altun, B. Damjanovic and D. Doric, Fixed point and common fixed point theorems on ordered cone metric spaces, Appl. Math. Lett. 23 (2010), 310-316.

[2] I. Beg, Sanjib Kumar Datta and Dipankar Pal, Fixed point in bicomplex valued metric spaces, Int. J. Nonlinear Anal. Appl. 12(2) (2021), 717-727.

[3] J. Choi, S. K. Datta, T. Biswas and M. N. Islam, Some fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces, Honam Mathematical Journal 39(1) (2017), 115-126.

[4] Fuli He, Z. Mostefaoui and M. Abdalla, Fixed point theorems for mizoguchi-takahashi type contraction in bicomplex-valued metric spaces and applications, Hindawi Journal of Function Spaces Volume 2020, Article ID 4070324, 7 pages.

[5] Hemant Kumar Nashine, Zoran Kadelburg and Stojan Radenovific, Common fixed point theorems for weakly isotone increasing mappings in ordered partial metric spaces, Mathematical and Computer Modelling 57 (2013), 2355-2365.

[6] I. H. Jebril, S. K. Datta, R. Sarkar and N. Biswas, Common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces, J. Interdisciplinary Math. 22(7) (2019), 1071-1082.

[7] M. E. Luna-Elizarrarfias, M. Shapiro, D. C. Struppa and A. Vajiac, Bicomplex Holomorphic Functions: The Algebra, Geometry and Analysis of Bicomplex Numbers. Frontiers in Mathematics, Springer, (2015).

[8] D. Rochon and M. Shapiro, On algebraic properties of bicomplex and hyperbolic numbers, An. Univ. Oradea Fasc. Mat. 11 (2004), 71-110.