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EXPRESSION OF RATIOS OF PENTATOPE NUMBERS AS CONTINUED FRACTIONS

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Abstract

In this paper an attempt has been made to express higher dimensional polygonal numbers as continued fractions namely square based number and pentatope numbers. Here ratio of successive pentatope numbers and square based figurate numbers has been expressed as continued fractions. As the two dimensional polygonal numbers always lies as base this ratio of squares of polygonal numbers in various fashions have been expressed as continued fractions and some results proved based on this.

Notations:

1. $\langle p_0, p_1, p_2, p_3, p_n\rangle$–Continued fraction expansion
2. PT_n – Pentatope number
3. SF_n Square based figurate number

1. Introduction

As is well known any rational number can be written as a continued fraction. Based on this, in [1, 2, 3, 4, 5] the rational numbers of various kinds represented by polygonal numbers has been discussed. In [6], sums of

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Squares of polygonal numbers is quite appealing and hence in this paper a trial has been made to express ratios of squares of polygonal numbers and its nature is analysed. As a next step to polygonal numbers, figurate numbers of higher order namely pentatope numbers and square based figurate numbers are considered here. As triangles form a base for any polygonal number, the squares of various polygonal numbers with respect to squares of triangular numbers have been studied [7, 8, 9]. Consecutive pentatope numbers as ratios and pentatope numbers with respect to square based figurate numbers have also been analyzed. There are so many inter relationship of continued fractions with other mathematical concepts has been mentioned in 'Continued fraction and signal processing' [10].

1.2 Continued Fraction. An expression of the form

$$\frac{a}{b} = p_0 + \cfrac{q_0}{p_1 \cfrac{q_1}{p_2 \cfrac{q_2}{p_3 + \cfrac{q_3}{\ddots}}}}$$

where p_i, q_i are real or complex numbers is called a continued fraction.

1.3 Theorem. *The continued fraction* $\frac{PT_n}{PT_{n+1}}$ *of is given by*

$$\frac{PT_n}{PT_{n+1}} = \begin{cases} \langle 0, 1, k \rangle, & \text{if } n = 4k \\ \langle 0, 1, k, 4 \rangle, & \text{if } n = 4k+1 \\ \langle 0, 1, k, 2 \rangle, & \text{if } n = 4k+2 \\ \langle 0, 1, k, 1, 3 \rangle, & \text{if } n = 4k+3 \end{cases}$$

Proof.

$$\frac{PT_n}{PT_{n+1}} = \frac{n(n+1)(n+2)(n+3)/24}{(n+1)(n+2)(n+3)(n+4)/24} = \frac{n}{n+4}$$

Case (i) Take $n = 4k$

$$\frac{PT_n}{PT_{n+1}} = \frac{4k}{4k+4} = 0 + \cfrac{1}{\cfrac{4k+4}{4k}}$$

$$= 0 \frac{1}{1 + \frac{4}{4k}}$$

$$= 0 + \frac{1}{1 + \frac{1}{k}}$$

$$= \langle 0,\ 1,\ k \rangle$$

Case (ii) Take $n = 4k + 1$

$$\frac{PT_n}{PT_{n+1}} = \frac{4k + 1}{4k + 5}$$

$$= 0 + \frac{1}{\frac{4k + 5}{4k + 1}}$$

$$= 0 + \frac{1}{1 + \frac{4}{4k + 1}}$$

$$= 0 + \frac{1}{1 + \frac{1}{\frac{4k + 1}{4}}}$$

$$= 0 + \frac{1}{k + \frac{1}{k + \frac{1}{4}}}$$

$$= \langle 0,\ 1,\ k,\ 4 \rangle$$

Case (iii) Take $n = 4k + 2$

$$\frac{PT_n}{PT_{n+1}} = \frac{4k + 2}{4k + 6}$$

$$= 0 + \frac{1}{\frac{4k + 6}{4k + 2}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{4}{4k+2}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\cfrac{4k+2}{4}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{k + \cfrac{2}{4}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{k + \cfrac{1}{2}}}$$

$$= \langle 0, 1, k, 2 \rangle$$

Case (iv) Take $n = 4k + 3$

$$\frac{PT_n}{PT_{n+1}} = \frac{4k+3}{4k+7}$$

$$= 0 + \cfrac{1}{\cfrac{4k+7}{4k+3}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{4}{4k+3}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\cfrac{4k+3}{4}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{k + \cfrac{3}{4}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{k + \cfrac{1}{\frac{4}{3}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{k + \cfrac{1}{1 + \frac{1}{3}}}}$$

$$= \langle 0, 1, k, 1, 3 \rangle$$

1. 4 Theorem. *The continued fraction of* $\frac{PT_n}{SF_n}$ *is given by*

$$\frac{PT_n}{SF_n} = \begin{cases} \langle 0, 1, 1, k \rangle, & \text{if } n = 4k+1 \\ \langle 0, 1, 1, k, 4 \rangle, & \text{if } n = 4k+2 \\ \langle 0, 1, 1, k, 2 \rangle, & \text{if } n = 4k+3 \\ \langle 0, 1, 1, k, 1, 3 \rangle, & \text{if } n = 4k+4 \end{cases}$$

Proof.

$$\frac{PT_n}{SF_n} = \frac{n+3}{2n+2}$$

Case (i) Take $n = 4k+1$

$$\frac{PT_n}{SF_n} = \frac{4k+4}{8k+4}$$

$$= 0 + \cfrac{1}{\frac{8k+4}{4k+4}}$$

$$= 0 + \cfrac{1}{1 + \frac{4k}{4k+4}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\frac{4k+4}{4k}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{4k}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{k}}}$$

$$\frac{PT_n}{SF_n} = \langle 1, 1, k \rangle$$

Case (ii) Take $n = 4k + 2$

$$\frac{PT_n}{SF_n} = \frac{4k+5}{8k+6} = 0 \, \cfrac{1}{\cfrac{8k+6}{4k+5}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{4k+1}{4k+5}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\cfrac{4k+5}{4k+1}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\cfrac{4}{4k+1}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4k+1}{4}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{k + \cfrac{1}{4}}}}$$

$$= \langle 0, 1, 1, k, 4 \rangle$$

Case (iii) Take $n = 4k + 3$

$$\frac{PT_n}{SF_n} = \frac{4k+6}{8k+8}$$

$$= 0 + \cfrac{1}{\cfrac{8k+8}{4k+6}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{4k+2}{4k+6}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\cfrac{4k+6}{4k+2}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{\cfrac{4}{4k+2}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4k+2}{4}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{k + \cfrac{2}{4}}}}$$

$$= 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{k + \cfrac{1}{2}}}}$$

$$= \langle 0, 1, 1, k, 2 \rangle$$

Case (iv) Take $n = 4k + 4$

$$\frac{PT_n}{SF_n} = \frac{4k+7}{8k+10} = 0\,\frac{1}{\dfrac{8k+10}{4k+7}}$$

$$= 0 + \frac{1}{1 + \dfrac{4k+3}{4k+7}}$$

$$= 0 + \frac{1}{1 + \dfrac{1}{\dfrac{4k+7}{4k+3}}}$$

$$= 0 + \frac{1}{1 + \dfrac{1}{\dfrac{4}{4k+3}}}$$

$$= 0 + \frac{1}{1 + \dfrac{1}{k + \dfrac{1}{\dfrac{4}{3}}}}$$

$$= 0 + \frac{1}{1 + \dfrac{1}{1 + \dfrac{1}{k + \dfrac{1}{1 + \dfrac{1}{3}}}}}$$

$$= \langle 0, 1, 1, k, 1, 3 \rangle$$

2. Continued fraction versus squares of polygonal numbers. $T^2_{m,n}$ with respect to $T^2_{3,n}$ has been considered in two categories.

The first category is given in table 1. The continued fraction of ratio of squares of polygonal number $T^2_{m,n}$ with triangular numbers. $T^2_{m,n}$ is significant in the sense that the continued fraction expression consist of only two terms, the second one of which is $(m-4)^2 \cdot (m-4)$ constitute the whole number set namely $1, 2, 3, \ldots$ The illustrations are exhibited below.

Table 1.

$\frac{T_{3,n}^2}{T_{m,n}^2}$	Value of $m-4$	Continued fraction
$\frac{T_{3,n}^2}{T_{5,n}^2}$	1	$\langle 0, 1^2 \rangle$
$\frac{T_{3,n}^2}{T_{6,n}^2}$	2	$\langle 0, 2^2 \rangle$
$\frac{T_{3,n}^2}{T_{7,n}^2}$	3	$\langle 0, 3^2 \rangle$
$\frac{T_{3,n}^2}{T_{8,n}^2}$	4	$\langle 0, 4^2 \rangle$
$\frac{T_{3,n}^2}{T_{9,n}^2}$	5	$\langle 0, 5^2 \rangle$

In the second category equally spaced values of has been taken. It can be noted that at the $(4n-13)^{th}$ place the continued fraction of $\frac{T_{3,n}^2}{T_{m,n}^2}$ is of the form $\langle 0, (m, 3)(m-2), 4 \rangle$. Some more illustrations for this given below.

Table 2.

$\frac{T_{3,n}^2}{T_{m,n}^2}$	Value of $m-4$	Continued fraction
$\frac{T_{3,n}^2}{T_{5,n}^2}$	7	$\langle 0, 6, 4 \rangle$
$\frac{T_{3,n}^2}{T_{6,n}^2}$	11	$\langle 0, 12, 4 \rangle$

$\frac{T_{3,n}^2}{T_{7,n}^2}$	15	$\langle 0, 20, 4\rangle$
$\frac{T_{3,n}^2}{T_{8,n}^2}$	19	$\langle 0, 30, 4\rangle$
$\frac{T_{3,n}^2}{T_{9,n}^2}$	23	$\langle 0, 42, 4\rangle$
$\frac{T_{3,n}^2}{T_{10,n}^2}$	27	$\langle 0, 56, 4\rangle$

3. Conclusion

It is inferred that the figurate numbers of higher dimension may be studied in any combination. Also the squares of figurate numbers may be extended to higher dimension. It is interesting to note that as the dimension of the figurate numbers increases the number of categories of ratios also increases. The ratios of squares may be analyzed in further study.

References

[1] A. Gnanam and S. Krithika, Ratios of Polygonal numbers as Continued Fractions, International Journal of Engineering Science, Advanced Computing and Bio Technology 8(3) (2017), 143-155.

[2] P. Balamurugan and A. Gnanam, Pattern classification of continued fractions with triangular number as base, International Journal of Pure and Applied Mathematics 119(15) (2018), 3413-3418.

[3] P. Balamurugan and A. Gnanam, Pattern classification of continued fractions with centered polygonal number as base, International Journal of Management, IT and Engineering 9(1) (2019), 156-162.

[4] P. Balamurugan, A. Gnanam and B. Anitha, Expression of ratios of polygonal numbers as continued fractions, Advances and Applications in Mathematical Sciences 18(10) (2019), 997-1006.

[5] P. Balamurugan, A. Gnanam and R. Senthilkumar, Expression of ratio of continued fractions of consecutive orders and rank, International Journal of Advanced Science and Technology 29(7s) (2020), 1550-1556.

[6] A. Gnanam and B. Anitha, Sums of Squares of polygonal numbers, Advances in pure mathematics 6(4) (2016), 297-301.

[7] Ivan Niven, Herbert S. Zukerman and Hugh L. Montgomery, An introduction to theory of Numbers, 5th edition, John Wiley and Sons, 1991.

[8] M. B. Nathanson, Sums of polygonal numbers, In analytic Number Theory and Diophantine Problem: Proceedings of a Conference at Oklahoma State University, 1984 (Ed. A. Adolphsonetal), M. ABoston, Birkhauser (1987), 305-316.

[9] S. G. Telang, Number Theory, TataMcgrawHill publishing Company, New York, 1996.

[10] Tomas Sauer, Continued Fractions and Signal Processing, Springer Undergraduate Texts in Mathematics and Technology, ISSN 1867-5514 (electronic), 2021.