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MATHEMATICAL MODELS OF SOLVING PARTIAL DIFFERENTIAL EQUATIONS IN VARIATIONAL ITERATION METHOD

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Abstract

In this paper, we submitted a good tool to solve linear and nonlinear partial differential equations which is called variational iteration method. This method makes hard problems so easy to solve, in our paper we gave some various examples for linear and nonlinear partial differential equations by using this method. The results show that the present method is very effective and simple and provide the analytic solutions.

1. Introduction

Recently, many mathematicians seek new techniques to find exact or approximate solutions for nonlinear partial differential equations which describe different fields of science, physical phenomena, engineering, mechanics, and so on. Some modern methods have been appeared like variational Iteration method which is analytic technique for solving linear and nonlinear problems. The variational iteration method has been shown to solve effectively, easily, and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions. He applied his method to autonomous ordinary differential systems [4] and nonlinear

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equations with convolution product nonlinearity [5], Ganji and Sadighi to nonlinear heat transfer [6], Lu to two-point boundary value problems [7]. Wazwaz gave a completely comparison between the two methods [8], revealing the variational iteration method has many merits over the Adomian method; it can completely overcome the difficulty arising in the calculation of the Adomian polynomial. Though the variational iteration method leads to fast convergent solutions, unnecessary calculation arises in the solution procedure. In this paper, we have applied the variational iteration method (VIM) [8] to solve a nonlinear partial differential equation [9], three-dimensional linear parabolic equation [10] and the one dimensional parabolic-like equation with variable coefficients [11] and with given initial conditions. The main advantage of the method is the fact that it provides its user with an analytical solution.

2. Analysis of Variational Iteration Method

To illustrate the basic idea of He's variational iteration method [12-14], we consider the following nonlinear functional equation:

$$Lx(\tau) + Nx(\tau) = g(\tau), \tag{1}$$

where $Lx(\tau)$ is a linear operator, $Nx(\tau)$ a nonlinear operator and $g(\tau)$ an inhomogeneous term. Inokuti et al. [2] suggested a method of general Lagrange multiplier. Then, we can construct a correct functional as follows:

$$x_{n+1}(\tau) = x_n(\tau) + \int_0^{\tau} \lambda(s)\left(Lx_n(\tau) + N\tilde{x}_n(\tau) - g(s)\right)ds, \tag{2}$$

where $\lambda(s)$ is a Lagrange multiplier that can be identified optimally via the variational theory [12-14]. The subscript n denotes the n^{th} approximation, and $x_n(\tau)$ is considered to be restricted variation, that is, $\delta\tilde{x}_n(\tau) = 0$. In this method the Lagrange multiplier $\lambda(s)$ is first determined optimally. The successive approximation $x_{n+1}(\tau)$, $n \geq 0$, of the solution $x(\tau)$ can be readily obtained by using this determined Lagrange multiplier with any selective function $x_0(\tau)$. Consequently, the solution is given by $x(\tau) = \lim_{n\to\infty} x_n(\tau)$.

3. Examples

Example 1. Consider the following in homogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin x \tag{3}$$

with initial condition

$$u(0, x) = 2\cos x + \sin x \tag{4}$$

To solve Equation (3) by variational iteration method, we will have

$$u_{n+1} = u_n + \int_0^\tau \lambda(s)(u_n''(x, s) + \sin x - u_n'(t, s))ds \tag{5}$$

we obtain for Equation (5) the following stationary conditions:

$$\lambda'(s) = 0 \tag{6}$$

$$1 + \lambda(s)|_{s=t} = 0. \tag{7}$$

Equation (6) is called Lagrange Euler equation, and Equation (7) natural boundary condition.

The general Lagrange multipliers, therefore, can be identified as $\lambda(s) = -1$: we start with the initial guess $u_0(x, t) = 2\cos x + \sin x$ in the above iteration formula and obtain the following approximate solutions:

$$u_0(x, t) = 2\cos x + \sin x. \tag{8}$$

When $n = 0$, in Equation (5)

$$u_1 = u_0 + \int_0^t \lambda(s)(u_0''(x, s) + \sin x - u_0'(t, s))ds \tag{9}$$

$$u_1 = 2\cos x + \sin x + \int_0^t (-1)\left(\frac{d^2 u_0}{dx^2} + \sin x - \frac{du_0}{dt}\right)ds \tag{10}$$

$$u_1 = (2\cos x + \sin x)(x + 1) + t\sin x \tag{11}$$

$$u_2 = u_1 + \int_0^t \lambda(s)(u_1''(x, s) + \sin x - u_1'(t, s))ds \tag{12}$$

$$u_2 = (2\cos x + \sin x)(t+1) + t\sin x + \int_0^t (-1)\left(\frac{d^2 u_1}{dx^2} + \sin x - \frac{du_1}{dt}\right)ds \tag{13}$$

$$u_2 = (2\cos x + \sin x)(t+1) + t\sin x + t(\cos x + \sin x)(t+2). \tag{14}$$

And so on, proceedings as before the rest of components were obtained, and then the functions are

$$u(x, t) = (2\cos x + \sin x)(t+1) + t\sin x + t(\cos x + \sin x)(t+2). \tag{15}$$

Example 2. Consider the following nonlinear PDEs

$$u_t = \frac{1}{3} u u_x - 3x \tag{16}$$

with initial condition

$$u(0, x) = 2. \tag{17}$$

To solve Equation (16) by variational iteration method, we will have

$$u_{n+1} = u_n + \int_0^\tau \lambda(s)\left(u_n'(t, s) - \frac{1}{3} u_n u_n'(x, s) + 3x\right)ds \tag{18}$$

we obtain for Equation (18) the following stationary conditions

$$\lambda'(s) = 0 \tag{19}$$

$$1 + \lambda(s)|_{s=t} = 0. \tag{20}$$

Equation (19) is called Lagrange Euler equation, and Equation (20) natural boundary condition.

The general Lagrange multipliers, therefore, can be identified as $\lambda(s) = -1$: we start with the initial guess $u_0(x, t) = 2$ in the above iteration formula and obtain the following approximate solutions:

$$u_0(x, t) = 2. \tag{21}$$

When $n = 0$, in Equation (18)

$$u_1 = u_0 + \int_0^t \lambda(s)\left(\frac{du_0}{dt} - \frac{u_0}{3}\frac{du_0}{dx} + 3x\right)ds \tag{22}$$

$$u_1 = 2 + \int_0^t (-1)\left(\frac{du_0}{dt} - \frac{u_0}{3}\frac{du_0}{dx} + x\right)ds \tag{23}$$

$$u_1 = 2 - 3xt \tag{24}$$

$$u_2 = u_1 + \int_0^t \lambda(s)\left(\frac{du_1}{dt} - \frac{u_1}{3}\frac{du_1}{dx} + 3x\right)ds \tag{25}$$

$$u_2 = 2 + \int_0^t (-1)\left(\frac{du_1}{dt} - \frac{u_1}{3}\frac{du_1}{dx} + x\right)ds \tag{26}$$

$$u_1 = 2 + t^2 - 3xt(1 + t^2). \tag{27}$$

And so on, proceedings as before the rest of components were obtained, and then the functions are

$$u(x, t) = 2 + t^2 - 3xt(1 + t^2). \tag{28}$$

4. Conclusion

In this paper, we used the variational iteration method for solving some partial differential equations. The most useful iteration formulations are listed in a convenient form for later reference and systematic use. Furthermore, VIM was successful implemented in approximating the solutions of nonlinear systems of PDEs. The VIM reduced the size of calculations without a need to transform the nonlinear terms. It is obvious that the method gives rapidly convergent successive approximations through determining the Lagrange multiplier. He's variational iteration method gives several successive approximations by using the iteration of the correction functional. The VIM uses the initial values for selecting the zeroth approximation. The results obtained confirm the accuracy and efficiency of the method.

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