Advances and Applications in Mathematical Sciences
Volume 21, Issue 5, March 2022, Pages 2403-2419

A NOVEL NUMERICAL SCHEME BASED ON BERNOULLI WAVELETS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract

This article presents a novel numerical scheme based on the operational matrix of Bernoulli wavelets to transform fractional differential equations with variable coefficients into simultaneous algebraic equations. The uniform convergence analysis for Bernoulli wavelets expansion is investigated. The computational algorithm of the proposed method is also presented for the numerical solutions of fractional differential equations with variable coefficients. Applicability and efficiency of the proposed numerical scheme are illustrated by some numerical examples.

1. Introduction

Fractional Calculus is a valuable tool and an old mathematical topic from 17th century. For many researchers in various fields of science and technology, fractional differential equations have been the focus of interest in recent years. As a result, finding solutions to fractional differential equations is an essential part of scientific research.

Furthermore, analytic solutions to the majority of fractional differential equations are not available. Due to this fact, many numerical schemes have

2020 Mathematics Subject Classification: 26A33, 65T60, 34A08.
Keywords: Riemann-Liouville fractional integral, Caputo fractional derivative, Bernoulli wavelets, Block pulse functions, Fractional integration operational matrix.
Received May 25, 2021; Accepted July 31, 2021

been suggested to find approximate solutions of fractional differential equations, namely, Variational Iteration Method, Finite volume method, Finite element method, Adomian Decomposition Method, and Wavelet methods [1, 4, 8-12].

There have been several methods for solving fractional differential equations with variable coefficients. These types of problems have been studied by many authors using different methods [2, 5, 7].

In this work, we present a numerical method based on Bernoulli wavelets for solving fractional differential equations with variable coefficients. Our aim is to derive Bernoulli wavelets' operational matrices to convert the fractional differential equations with variable coefficients into a system of algebraic equations. It not only simplifies the problem, but also speeds up the computation. Numerical solutions of the proposed method are compared with the solutions of some existing numerical techniques and analytical solutions to manifest the applicability, the computational efficiency and the high precision of the proposed method.

The article is summarized as follows. In Section 2, some basic definitions and mathematical preliminaries of fractional calculus are given. Section 3 is devoted to the basic formulation of Bernoulli wavelets, function approximation, and the operational matrix of fractional order integration for Bernoulli wavelets. Computational algorithm of the scheme is given in Section 4. In Section 5, some numerical examples and absolute errors are presented. Finally, we conclude our work in Section 6.

2. Preliminaries

In this section, we briefly recall some essential definitions and preliminary mathematical facts of fractional calculus which are used further in this paper.

Definition 2.1. Fractional integral of order $\gamma \geq 0$ of $h(x) \in L^1(\mathbb{R}^+)$ in terms of Riemann-Liouville, is defined by

$$(I^{\gamma}h)(x) = \begin{cases} \frac{1}{\Gamma(\gamma)} \int_0^x \frac{h(\zeta)}{(x-\zeta)^{(1-\gamma)}} d\zeta, & \gamma > 0, \\ h(x), & \gamma = 0. \end{cases} \tag{1}$$

Fractional derivative of order $\gamma > 0$ of $h(x) \in L^1(\mathbb{R}^+)$ in terms of Riemann-Liouville, is normally defined by

$$(I^{-\gamma}h)(x) = \left(\frac{d}{dx}\right)^s (I^{s-\gamma}h)(x), \; s-1 < \gamma \leq s, \tag{2}$$

where s is a positive integer and $x > 0$.

Definition 2.2. The Caputo fractional derivative of order $\gamma \geq 0$ of $h(x) \in L^1(\mathbb{R}^+)$ is given by

$$(D^{\gamma}h)(x) = \frac{1}{\Gamma(s-\gamma)}\int_0^x (x-\zeta)^{s-\gamma-1}h^{(s)}(\zeta)d\zeta, \quad s-1 < \gamma \leq s, \tag{3}$$

where s is a positive integer and $x > 0$.

If $h(x) \in L^1(\mathbb{R}^+)$ and $\gamma \geq 0$ then it has the following two basic properties.

$$(D^{\gamma}I^{\gamma}h)(x) = h(x), \tag{4}$$

and

$$(I^{\gamma}D^{\gamma}h)(x) = h(x) - \sum_{l=0}^{s-1} h^{(l)}(0^+)\frac{h^l}{l!}, \; s-1 < \gamma \leq s, \tag{5}$$

where s is a positive integer, $x > 0$ and $h^{(l)}(0^+) := \lim_{x\to 0^+} D^l h(x)$, $l = 0, 1, 2, \ldots, s-1$.

3. Bernoulli Wavelets

We here discuss Bernoulli polynomials and some of their properties in order to construct Bernoulli wavelets.

3.1. Properties of Bernoulli polynomials and Bernoulli wavelets

Wavelets represent a family of functions constructed from dilations and translations of a single function $\Psi(x)$ called the mother wavelet. When the dilation parameter c and the translation parameter d change continuously, the following family of continuous wavelets is obtained.

$$\psi_{cd}(x) = |c|^{-\frac{1}{2}} \psi\left(\frac{x-d}{c}\right), \; c, d \in \mathbb{R}, \; c \neq 0. \tag{6}$$

If the discrete values are selected for the translation and dilation parameters, that is, $c = c_0^{-p}$, $d = qd_0c_0^{-p}$, $c_0 > 1$, $d_0 > 0$, $p, q \in \mathbb{Z}^+$, then we have the following family of discrete wavelets,

$$\psi_{pq}(x) = |c_0|^{\frac{p}{2}} \psi(c_0^p x - qd_0), \tag{7}$$

where the functions ψ_{pq} form a wavelet basis for $L^2(\mathbb{R})$. Specifically, when $c_0 = 2$ and $d_0 = 1$, the functions $\psi_{pq}(x)$ form an orthonormal basis for $L^2(\mathbb{R})$.

The Bernoulli wavelets are defined on [0, 1) as

$$\psi_{pq}(x) = \begin{cases} 2^{\frac{j-1}{2}} \widetilde{E}_n(2^{j-1}x - m + 1), & \frac{m-1}{2^{j-1}} \leq x < \frac{m}{2^{j-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

for $m = 1, 2, \ldots, 2^{j-1}$, $n = 0, 1, \ldots, N-1$ and $j, N \in \mathbb{N}$, where

$$\widetilde{E}_n(x) = \begin{cases} 1, & n = 0, \\ \dfrac{1}{\sqrt{(\dfrac{(-1)^{n-1}(n!)^2}{(2n)!}\beta_{2n}}} E_n(x), & n > 0, \end{cases}$$

the coefficient $\dfrac{1}{\sqrt{(\dfrac{(-1)^{n-1}(n!)^2}{(2n)!}\beta_{2n}}}$ is used for normality, the dilation parameter is $2^{-(j-1)}$ and the translation parameter is $(m-1)2^{-(j-1)}$. Here $E_n(x)$, $n = 0, 1, \ldots, N-1$, denote Bernoulli polynomials of order n which can be defined by the relation

$$E_n(x) = \sum_{r=0}^{n} \binom{n}{r} \beta_r x^{n-r}, \tag{9}$$

where β_r, $(r = 0, 1, 2, \ldots, n)$ are Bernoulli numbers. Bernoulli numbers can be defined by the following generating function

$$\frac{x}{e^x - 1} = \sum_{r=0}^{\infty} \beta_r \frac{x^r}{r!}. \tag{10}$$

The first few Bernoulli numbers are

$$\beta_0 = 1,\ \beta_1 = -\frac{1}{2},\ \beta_2 = \frac{1}{6},\ \beta_4 = -\frac{1}{30},\ \ldots, \tag{11}$$

with $\beta_{2r+1} = 0,\ r = 1, 2, 3, \ldots$

The first few Bernoulli polynomials are

$$E_0(x) = 1,\ E_1(x) = x - \frac{1}{2},\ E_2(x) = x^2 - x + \frac{1}{6},\ E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \ldots \tag{12}$$

The properties of Bernoulli polynomials and Bernoulli wavelets have been discussed by Sahu and Saha Ray [10] and Jiao et al. [13].

Moreover,

$$\int_0^1 E_m(x)E_n(x)dx = (-1)^{m-1} \frac{n!\,m!}{(n+m)!} \beta_{m+n},\ n,\quad m \geq 1, \tag{13}$$

and

$$\int_0^1 |\, E_n(x)\,|dx \leq 16 \frac{n!}{(2\pi)^{n+1}},\ n \geq 0. \tag{14}$$

Let $\psi(x) = [\psi_1(x),\ \psi_2(x),\ \ldots,\ \psi_k(x)]^T$, where $\psi_i(x) = \psi_{mn}(x), i = N(m-1) + n + 1, k = 2^{j-1}N, m = 1, 2, \ldots, 2^{j-1}, n = 0, 1, \ldots, N-1$ and $j,\ N \in \mathbb{N}$. Then Bernoulli wavelets have the following orthonormality properties.

$$\langle \psi_r(x),\ \psi_s(x)dx \rangle = \int_0^1 \psi_r(x)\psi_s(x)dx = \{1,\ r = s,\ 0,\ r \neq s, \tag{15}$$

and

$$\int_0^1 \Psi(x)\Psi^T(x)dx = E, \tag{16}$$

where $\langle .,. \rangle$ denotes the inner product and E indicates identity matrix.

3.2. Function approximation

A function $h(x) \in L^2[0, 1)$ can be expressed in terms of Bernoulli wavelets as

$$h(x) = \sum_{m=0}^{\infty} \sum_{n \in Z} a_{mn} \psi_{mn}(x), \tag{17}$$

where the coefficients a_{mn} are given by

$$a_{mn} = \langle h(x), \psi_{mn}(x) \rangle = \int_0^1 h(x) \psi_{mn}(x) dx.$$

By truncating the infinite series in Equation (17), $h(x)$ is approximated as

$$\widetilde{h(x)} \approx \sum_{m=1}^{2^{j-1}} \sum_{n=0}^{N-1} a_{mn} \psi_{mn}(x). \tag{18}$$

For simplicity, Equation (18) is written as

$$\widetilde{h(x)} = \sum_{i=1}^{k} a_i \psi_i(x) = A^T \Psi(x), \tag{19}$$

where $a_i = a_{mn}, \psi_i = \psi_{mn}, k = 2^{j-1} N, A = [a_1, a_2, \ldots, a_k]^T$, (20)

and

$$\Psi(x) = [\psi_1(x), \psi_2(x), \ldots, \psi_k(x)]^T. \tag{21}$$

The index i is determined by the relation $i = N(m-1) + n + 1$.

We define the Bernoulli wavelet coefficient matrix $\phi_{k \times k}, k = 2^{j-1} N$, at the collocation points $x_r = \frac{2r-1}{2k}, r = 1, 2, \ldots, k$ as

$$\phi_{k \times k} = \left[\Psi\left(\frac{1}{2k}\right), \Psi\left(\frac{3}{2k}\right), \ldots, \Psi\left(\frac{2k-1}{2k}\right) \right]. \tag{22}$$

Specifically, the Bernoulli wavelet coefficient matrix for $j = 2$ and $N = 3$ becomes

$$\phi_{6\times 6} = \begin{pmatrix} 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 \\ -1.6330 & 0 & 1.6330 & 0 & 0 & 0 \\ 0.5270 & -1.5811 & 0.5270 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & -1.6330 & 0 & 1.6330 \\ 0 & 0 & 0 & 0.5270 & -15811 & 0.5270 \end{pmatrix}. \quad (23)$$

Here, we have

$$\widetilde{h}_k = [\widetilde{h}(x_1), \widetilde{h}(x_2), \ldots, \widetilde{h}(x_k)] = A^T \phi_{k\times k}.$$

Since the Bernoulli wavelet coefficient matrix $\phi_{k\times k}$ is invertible, it is possible to obtain the Bernoulli wavelet coefficient vector A^T by $\widetilde{h}_k \phi_{k\times k}^{-1}$.

3.3. Operational matrix of fractional order integration

In this section, we explore the basic idea of finding the operational matrix of fractional order integration for the Bernoulli wavelets.

A k-set of Block pulse functions (BPFs) over the interval [0, 1) is defined as

$$b_r(x) = \begin{cases} 1, & (r-1)/k \le x < r/k, \\ 0, & \text{otherwise}, \end{cases} \quad (24)$$

where $r = 1, 2, 3, \ldots k$.

It is known that any square integrable function $h(x)$ defined on the interval [0, 1) can be extended in terms of BPFs, and by using orthogonality of BPFs as

$$h(x) \simeq \sum_{r=1}^{k} h_r b_r(x) = h^T B_k(x), \quad (25)$$

where $h = [h_1, h_2, \ldots, h_k]^T$, h_r for $r = 1, 2, \ldots, k$ are given by

$$h_r = \frac{1}{k}\int_{(r-1)/k}^{r/k} h(x) b_r(x) dx, \text{ and } B_k(x) = [b_1(x), b_2(x), \ldots, b_k(x)]^T.$$

There is a connection between the block pulse functions and Bernoulli wavelets, which is,

$$\Psi(x) = \phi_{k\times k} B_k(x). \tag{26}$$

The block pulse operational matrix $H^{\beta}, \beta \geq 0$ of fractional integration of order $\beta \geq 0$ is defined by Kilicman [6] as,

$$(I^{\beta} B_k)(x) \approx H^{\beta} B_k(x), \tag{27}$$

where

$$H^{\beta} = \frac{1}{k^{\beta}} \frac{1}{\Gamma(\beta+2)} \begin{pmatrix} 1 & \zeta_1 & \zeta_2 & \zeta_3 & \cdots & \zeta_{k-1} \\ 0 & 1 & \zeta_1 & \zeta_2 & \cdots & \zeta_{k-2} \\ 0 & 0 & 1 & \zeta_1 & \cdots & \zeta_{k-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & \zeta_1 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix},$$

with $\zeta_j = (j+1)^{\beta+1} - 2j^{\beta+1} + (j-1)^{\beta+1}$.

The fractional integration of order $\beta \geq 0$ of the vector $\Psi(x)$ defined in Equation (21) can be approximated as

$$(I^{\beta}\Psi)(x) \approx p^{\beta}_{k\times k}\Psi(x), \tag{28}$$

where $p^{\beta}_{k\times k}$ is called Bernoulli wavelet operational matrix of order $\beta \geq 0$.

Using Equations (26) and (27), we attain

$$(I^{\beta}\Psi)(x) \approx (I^{\beta}\phi_{k\times k} B_k)(x) = \phi_{k\times k}(I^{\beta} B_k)(x) \approx \phi_{k\times k} H^{\beta} B_k(x). \tag{29}$$

Thus combining Equations (28) and (29), we attain

$$P^{\beta}_{k\times k}\Psi(x) \approx (I^{\beta}\Psi)(x) \approx \phi_{k\times k} H^{\beta} B_k(x) = \phi_{k\times k} H^{\beta} \phi^{-1}_{k\times k}\Psi(x), \text{ and so} \tag{30}$$

$$P^{\beta}_{k\times k} \approx \phi_{k\times k} H^{\beta} \phi^{-1}_{k\times k}. \tag{31}$$

For example, the Bernoulli wavelet operational matrix of the fractional order integration for $j = 2,\ N = 3$ and $\beta = 0.5$ yields

$$P_{6\times 6}^{0.5} = \begin{pmatrix} 0.5282 & 0.1819 & -0.0298 & 0.4438 & -0.0871 & 0.0256 \\ -0.1452 & 0.2243 & 0.1329 & 0.0799 & -0.0449 & 0.0198 \\ -0.0598 & -0.0964 & 0.1688 & -0.0417 & -1.8589e-04 & 0.0029 \\ 0 & 0 & 0 & 0.5282 & 0.1819 & -0.0298 \\ 0 & 0 & 0 & -0.1452 & 0.2243 & 0.1329 \\ 0 & 0 & 0 & -0.0598 & -0.0964 & 0.1688 \end{pmatrix}. \tag{32}$$

Since the operational matrix $P_{6\times 6}^{0.5}$ contains several zeros, the proposed technique reduces the computation greatly.

3.4. Convergence analysis

In the following theorem, we establish the convergence of the Bernoulli wavelets expansion [10].

Theorem 3.1. *If* $h(x) \in L^2[0, 1)$ *is a continuous function and* $|h(x)| \leq \eta$, $\eta \in \mathbb{R}$, *then the Bernoulli wavelets expansion of* $h(x)$ *defined in Equation* (17) *converges uniformly and also*

$$|a_{mn}| < \eta \frac{F}{2^{\frac{j-1}{2}}} \frac{16n!}{(2\pi)^{n+1}}, \tag{33}$$

where

$$F = \frac{1}{\sqrt{\left(\frac{(-1)^{n-1}(n!)^2}{(2n)!}\right)\beta_{2n}}}.$$

Proof. Any function $h(x) \in L^2[0, 1)$ can be approximated in terms of Bernoulli wavelets as

$$h(x) \simeq \sum_{m=1}^{2^{j-1}} \sum_{n=0}^{N-1} a_{mn}\psi_{mn}(x), \tag{34}$$

Here

$$a_{mn} = \int_0^1 h(x)\psi_{mn}(x)dx$$

$$= \sum_{m} \int_{I_{j,m}} h(x)\psi_{mn}(x)dx, \text{ where } I_{j,m} = \left[\frac{m-1}{2^{j-1}}, \frac{m}{2^{j-1}}\right), m = 1, 2, \ldots, 2^{j-1}, \tag{35}$$

$$= 2^{\frac{j-1}{2}} F \sum_{m} \int_{I_{j,m}} h(x) E_n(2^{j-1}x - m + 1)dx, \text{ where } F = \frac{1}{\sqrt{\left(\frac{(-1)^{n-1}(n!)^2}{(2n)!}\beta_{2n}\right.}}. \tag{36}$$

Using $2^{j-1}x - m + 1 = t$, we have

$$a_{mn} = \frac{F}{2^{j-1}} \sum_{m} \int_{I_{j,m}} h\left(\frac{t+m-1}{2^{j-1}}\right) E_n(x)dt, \tag{37}$$

and so

$$|a_{mn}| = \left| \frac{F}{2^{\frac{j-1}{2}}} \sum_{m} \int_{I_{j,m}} h\left(\frac{t+m-1}{2^{j-1}}\right) E_n(t)dt \right| \leq \frac{F\eta}{2^{\frac{j-1}{2}}} \int_0^1 |E_n(t)|dt$$

$$< \frac{F\eta}{2^{\frac{j-1}{2}}} 16 \frac{n!}{(2\pi)^{n+1}}. \tag{38}$$

Thus the series $\sum_{m=1}^{2^{j-1}} \sum_{n=0}^{N-1} a_{mn}$ is absolutely convergent, and so the series $\sum_{m=1}^{2^{j-1}} \sum_{n=0}^{N-1} a_{mn}\psi_{mn}(x)$ is uniformly convergent. □

4. Algorithm for the Proposed Numerical Scheme

Step 1. Assign the values for j and N for step size $k = 2^{j-1}N$ in Equation (20).

Step 2. Compute Bernoulli wavelet coefficient matrix $\phi_{k\times k}$ at the collocation points $x_r = \frac{2r-1}{2k}$, $r = 1, 2, \ldots, k$ from Equation (22).

Step 3. Compute the block pulse operational matrix H^{β} from Equation (27).

Step 4. Construct Bernoulli wavelet operational matrix $P^{\beta}_{k\times k}$ of order $\beta \geq 0$ using Equation (31).

Step 5. Dispersing the coefficients of the given fractional differential equations at the collocation points, construct diagonal matrices.

Step 6. Express all Caputo fractional derivatives in the given fractional differential equations in terms of Bernoulli wavelets.

Step 7. Solve the system of algebraic equations using MATLAB2015a to compute the unknown vector.

Step 8. Compute the solution using the unknown vector and the Bernoulli wavelet operational matrix.

5. Numerical Experiments

To show the applicability and the effectiveness of the proposed numerical scheme, we consider here some fractional differential equations with variable coefficients.

Example 5.1. Consider the following fractional order linear differential equation with variable coefficients

$$r[D^2 h(x)] + s(x)[D^{\gamma_2} h(x)] + t(x)[Dh(x)] + u(x)[D^{\gamma_1} h(x)] + v(x)h(x) = w(x), \quad (39)$$

with $0 \leq x < 1,\ 0 < \gamma 1 \leq 1,\ 1 < \gamma_2 \leq 2,\ h(0) = 2$ and $h'(0) = 0$,

where $r \in \mathbb{R},\ s(x),\ t(x),\ u(x),\ v(x),\ h(x),\ w(x) \in L^2[0, 1)$,

$$w(x) = -r - \frac{s(x)}{\Gamma(3-\gamma 2)} x^{2-\gamma_2} - t(x)x - \frac{u(x)}{\Gamma(3-\gamma_2)} x^{2-\gamma_2} + u(x)(2 - \frac{1}{2}x^2).$$

Suppose

$$D^2 h(x) \simeq A^T \Psi(x) \text{ where } A = [a_1, a_2, \ldots, a_k]^T, \text{ and}$$

$$w(x) \simeq W^T \Psi(x) \text{ where } w = [w_1, w_2, \ldots, w_k]. \quad (40)$$

$$\text{Then } D^{\gamma_2} h(x) = A^T P^{2-\gamma_2}_{k\times k} \Psi(x), \quad (41)$$

$$D^{\gamma_1} h(x) = A^T P_{k\times k}^{2-\gamma_1} \Psi(x), \tag{42}$$

$$Dh(x) = A^T P_{k\times k} \Psi(x), \tag{43}$$

$$\text{and } h(x) = A^T P_{k\times k}^{2} \Psi(x) + 2. \tag{44}$$

Using Equations (40)-(44) in (39), we attain

$$[rA^T]\Psi(x) + [A^T P_{k\times k}^{2-\gamma_2}]\Psi(x)s(x) + [A^T P_{k\times k}]\Psi(x)t(x) + [A^T P_{k\times k}^{2-\gamma_1}]\Psi(x)u(x)$$
$$+ [A^T P_{k\times k}^{2}]\Psi(x)v(x) + 2v(x) = W^T \Psi(x). \tag{45}$$

Dispersing the coefficients $s(x)$, $t(x)$, $u(x)$, $v(x)$ at the collocation points, construct the following matrices.

$$S = \begin{pmatrix} s(x_1) & 0 & \dots & 0 \\ 0 & s(x_1) & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & s(x_k) \end{pmatrix}, T = \begin{pmatrix} t(x_1) & 0 & \dots & 0 \\ 0 & t(x_2) & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & t(x_k) \end{pmatrix},$$

$$U = \begin{pmatrix} u(x_1) & 0 & \dots & 0 \\ 0 & u(x_1) & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u(x_k) \end{pmatrix}, V = \begin{pmatrix} v(x_1) & 0 & \dots & 0 \\ 0 & v(x_2) & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & v(x_k) \end{pmatrix}.$$

Table 1. Maximum absolute errors for various choices of j and N.

k	48 ($j = 3$, $N = 3$)	96 ($j = 4$, $N = 3$)	192 ($j = 5$, $N = 3$)	384 ($j = 6$, $N = 3$)	768 ($j = 7$, $N = 3$)
The Proposed method	1.5100e-05	3.8168e-06	9.6282e-07	2.4249e-07	6.0990e-08

Table 2. Adams type Predictor-Corrector method [3].

Step size	Maximum absolute errors
0.1	0,023658990000
0.01	0.000986218500
0.001	0.000043988230

Discreting Equation (45), we can achieve

$$rA^T\phi_{k\times k} + A^T P^{2-\gamma_2}\phi_{k\times k}.S + A^T P\phi_{k\times k}.T + A^T P^{2-\gamma_1}\phi_{k\times k}.U$$
$$+ [A^T P^2\phi_{k\times k} + Y].V = W^T\phi_{k\times k}, \qquad (46)$$

where $Y = [2, 2, \ldots, 2]_{1\times k}$. At the collocation points $x_i = (2i-1)/2k$, $i = 1, 2, \ldots, k$, we transform Equation (46) into a system of algebraic equations. Solving this system of algebraic equations using MATLAB2015a, we can easily obtain A^T.

Suppose

$r = 1$, $s(x) = x^{1/2}$, $t(x) = x^{1/3}$, $u(x) = x^{1/4}$, $v(x) = x^{1/5}$, $\gamma_1 = 0.333$, $\gamma_2 = 1.234$.

Then the exact solution of Equation (39) for $\gamma_1 = 0.333$ and $\gamma_2 = 1.234$ is $h(x) = 2 - \frac{1}{2}x^2$.

In Tables 1 and 2, the maximum absolute error obtained using Adams type Predictor-Corrector method is 4.40e-05 in 1000th step, while the maximum absolute error using the proposed method is 1.51e-05 in 48th step. We also see clearly from Table 1 that the numerical solutions are in perfect agreement with the exact solutions for larger values of k. Numerical results of this problem demonstrate that the proposed method converges rapidly and is more efficient than the Adams type predictor-corrector method [3]. Also from Figure 1, we see clearly that the numerical solutions are in perfect agreement with the exact solutions.

Example 5.2. Consider the following fractional differential equation

$$D^{1/3}h(x) + x^{1/3}h(x) = w(x),\ x \in [0, 4), \qquad (47)$$

with the initial state $h(0) = 0$ and $w(x) = \frac{3}{2\Gamma(2/3)}x^{(2/3)} + x^{(4/3)}$. The exact solution of Equation (47) is $h(x) = x$.

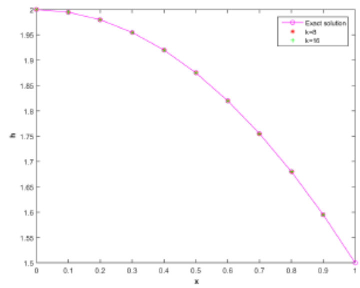

Figure 1. Comparison of Numerical solutions of Example 5.1 for $k = 8\,(j = 3,\ N = 2)$ and $k = 16\,(j = 4,\ N = 2)$ with the exact solutions.

Let $t = x/4$. Then $x = 4t,\ t \in [0, 1)$. Thus

$$D^{1/3}h(4t) + (4t)^{1/3}h(4t) = v(t), \tag{48}$$

where

$$v(t) = w(4t) = \frac{3.2^{1/3}}{\Gamma(2/3)}t^{2/3} + (4t)^{4/3},\ t \in [0, 1).$$

Approximating

$$D^{1/3}h(4t) \text{ as } A^T\Psi(t) \text{ where } A = [a_1, a_2, \ldots, a_k]^T, \tag{49}$$

we have

$$h(4t) = A^T P^{1/3}\Psi(t). \tag{50}$$

Similarly, $v(t)$ can be approximated by the Bernoulli wavelet functions as

$$v(t) = V^T\Psi(t), \text{ where } V = [v_1, v_2, \ldots, v_k]^T. \tag{51}$$

Using Equations (49), (50) and (51) in Equation (48), we have

$$A^T\Psi(t) + (4t)^{1/3}A^T P^{1/3}\Psi(t) = V^T\Psi(t). \tag{52}$$

Table 3. Absolute errors for various choices of j and for $N = 2$.

x	$k = 8$ $(j = 3,\ N = 2)$		$k = 16$ $(j = 4,\ N = 2)$		$k = 32$ $(j = 5,\ N = 2)$	
	The proposed method	HWM	The proposed method	HWM	The proposed method	HWM
0.25	3.4042e-02	4.6972e-02	1.0979e-02	2.5554e-02	1.6860e-03	6.3723e-03
0.75	5.2261e-03	1.8818e-02	1.7086e-03	7.7490e-03	4.8561e-04	2.6655e-03
1.25	2.9291e-03	1.2333e-02	9.1120e-04	4.9465e-03	2.7618e-04	1.7841e-03
1.75	1.8933e-03	9.2464e-03	5.9425e-04	3.6780e-03	1.8461e-04	1.3549e-03
2.25	1.3507e-03	7.4107e-03	4.2765e-04	3.2651e-03	1.3471e-04	1.3549e-03
2.75	1.0238e-03	6.1850e-03	3.2668e-04	2.6691e-03	1.0387e-04	1.0953e-03
3.25	8.0889e-04	5.3060e-03	2.5985e-04	2.0961e-03	8.3183e-05	9.1997e-04
3.75	6.5879e-04	4.6437e-03	2.1286e-04	1.8332e-03	6.8503e-05	6.9676e-04

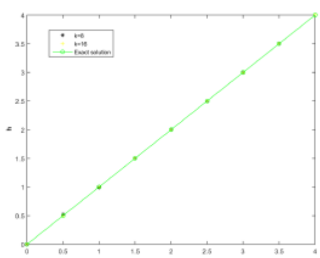

Figure 2. Comparison of Numerical solutions of Example 5.2 for $k = 8\,(j = 3,\ N = 2)$ and $k = 16\,(j = 4,\ N = 2)$ with the exact solutions.

Dispersing the coefficient $(4t)^{1/3}$ of Equation (52) at the collocation points, construct the following matrix.

$$R = \begin{pmatrix} (4t_1)^{(1/3)} & 0 & \cdots & 0 \\ 0 & (4t_2)^{(1/3)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (4t_k)^{(1/3)} \end{pmatrix}.$$

Discreting Equation (52), we get

$$A^T \phi_{k\times k} + A^T P^{1/3} \phi_{k\times k}.R = V^T \phi_{k\times k}. \tag{53}$$

At the collocation points $t_i = \frac{2i-1}{2^j N}$, $i = 1, 2, \ldots, 2^{j-1}N$, we transform Equation (53) into a system of algebraic equations. Solving this system of algebraic equations using MATLAB2015a, we can easily obtain the coefficients vector A^T. Then we get the numerical solutions $h(4t)$ of Equation (48). The numerical solutions $h(x)$ of Equation (47) are obtained by $h(x) = A^T P^{1/3} \psi(x/4)$.

The numerical results for $k = 8\,(j = 3,\ N = 2)$ and $k = 16\,(j = 4,\ N = 2)$ are shown in Figure 2. From Figure 2, we find easily that the numerical solutions are in good agreement with the exact solutions. The absolute errors for different values of k are shown in Table 3. We also see from Table 3 that as k increases, the errors become smaller and the proposed method is more accurate compared with the Haar wavelets method.

6. Conclusion

In this paper, an efficient numerical scheme based on Bernoulli wavelets for solving a class of fractional differential equations with variable coefficients was proposed. By the advantages of sparse and orthogonal nature, the proposed technique reduces the computation greatly to give numerical solutions with good coincidence. Tables 2 and 3 depict the advantages of the proposed method over other methods, namely Adams type Predictor-Corrector method and Haar wavelet method, in terms of less computational effort and time, accuracy and simplicity. Absolute errors and graphical representations in the two numerical examples demonstrate the high degree accuracy of the proposed numerical scheme.

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