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ON THE OPEN ss-COUNTABLY COMPACTNESS

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Abstract

In this article, I have introduced the new definition of topological concept namely, open ss-countably compact. I have proved that each of the compactness and the ss-countably compactness is stronger than of the open ss-countably compact, that is, compactness implies open ss-countably compact, also ss-countably compactness implies open ss-countably compactness, but the converse is not true. Also, I have shown that the continuous image of an open ss-countably compact is an open ss-countably compact. Finally, I have shown that it $X \times Y$ is an open ss-countably compact, then each of X and Y is an open ss-countably compact.

1. Introduction and Preliminaries

Let (X, τ) be a Hausdorff topological space. Let $\mathbb{N}$ denote the set of all natural numbers, let $\mathbb{R}$ be the set of all the real numbers, let $\mathbb{Z}$ be the set of all integers and $(x_n \mid n \in \mathbb{N})$ be a sequence of points in a set. Let A be a subset of X and $X \setminus A$ is a complement of the set A in X. If $K \subset \mathbb{N}$ then K_n will denote the set $\{k \in K, k \leq n\}$ and $| K_n |$ stands for the cardinality of K_n [3]. The natural density of K is defined by $d(K) = \lim_{n \to \infty} \frac{|K_n|}{n}$ if a limit exists.

Definition 1.1. A property of a topological space (X, τ) is said to be a topological property [4] if any a topological space (Y, δ) which is homeomorphic to (X, τ) has that property.

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Definition 1.2. A property of a topological space (X, τ) is said to be a hereditary property [4] if any subspace of (X, τ) has that property.

Definition 1.3. A sequence (x_n) in a topological space (X, τ) is said to be converge statistically [2] (or shortly s-converge) to $x \in X$ if for every neighborhood U of x, $d(\{n \in \mathbb{N} \mid x_n \in U\}) = 1.$ Here x will be called s-limit. In this case, we write $x = s - \lim_{n \to \infty} x_n$ or $(x_n) \xrightarrow{s} x$.

Definition 1.4. A point x is called a statistically sequential accumulation point of A (or is in the statistically sequential derived set) [1] if there is a sequence (x_n) of points in $A \setminus \{x\}$ such that $s - x = s - \lim_{n \to \infty} x_n$.

Definition 1.5. A subset A of X is called statistically sequentially countably compact (ss-countably compact) [1] if any infinite subset of A has at least one statistically sequential accumulation point of A.

2. Motivation and Main Results

Definition 2.1. A topological space (X, τ) is said to be an open ss-countably compact space if any an infinite proper open subset of X has at least one statistically sequential accumulation point. A subset A of X is said to be an open ss-countably compact if any an infinite proper open subset of A has at least one statistically sequential accumulation point in A.

Example 2.2.

1. Consider the real line $\mathbb{R}$ with usual topology is an open ss-countably compact space, since any infinite proper open subset of $\mathbb{R}$ has at least one statistically sequential accumulation point in $\mathbb{R}$.

2. Consider the real line $\mathbb{R}$ with cofinite topology. Any an open set A of $\mathbb{R}$ is of the form $\mathbb{R} \setminus A$ where A is a finite set. Since $\mathbb{R} \setminus A$ is an infinite set, then every element of $\mathbb{R}$ is a statistically sequential accumulation point of $\mathbb{R} \setminus A$.

3. A discrete topological space is not an open ss-countably compact space since the discrete topology does not have a statistically sequential accumulation point.

Theorem 2.3. *Let X be open ss-countably compact and Y be any space. If* $f : X \to Y$ *is continuous, then* $f(X)$ *is an open ss-countably compact.*

Proof. *Let* A be an infinite proper open subset of $f(X)$. Then $A = \{f(x) | x \in B\}$ where $B \subseteq X$ is infinite. Since X is open ss-countably compact, B has a statistically sequential accumulation point b. Let V_b be a neighborhood of $f(b)$. Since f is continuous, there exists some neighborhood U_b of b such that $f(U_b) \subseteq V_b$. Since b is a statistically sequential accumulation point of B, there exists some $y_n \in B$ such that $y_n \neq b$, $d(\{n \in \mathbb{N} | y_n \in U_b\}) = 1$. Thus, $f(y_n) \in f(U_b) \subseteq V_b$. Since $f(y_n) \in A \setminus f(b)$, $f(y_n) \overset{s}{\rightarrow} f(b)$. Since every neighborhood V_b of $f(b)$, $d(\{n \in \mathbb{N} | y_n \in V_b\}) = 1$, that is, $f(b)$ is a statistically sequential accumulation point of A. So, every infinite proper open subset A of $f(X)$ has a statistically sequential accumulation point. Therefore, $f(X)$ is an open ss-countably compact.

Corollary 2.4. *An open ss-countably compact is a topological property.*

The following Example 2.5 shows that open ss-countably compact is not a hereditary property.

Example 2.5. Consider the space $(\mathbb{R}, \tau_u)$ where τ_u is the usual topology. This space is an open ss-countably compact space, by Example 2.2(1). But $\mathbb{Z}$ as a subspace of $(\mathbb{R}, \tau_u)$ is not an open ss-countably compact space.

Theorem 2.6. *A compact space is an open ss-countably compact space.*

Proof. Let (X, τ) be a topological space. Suppose that (X, τ) is not an open ss-countably compact space, that is, there exists an infinite subset A of X, which has no statistically sequential accumulation point. Then for each $x \in A$, there exists an open set U containing x such that $d(\{n \in \mathbb{N} | x_n \in U\}) = 1$. Since x is not a statistically sequential accumulation point of A, $X \setminus A$ is a neighborhood of x containing no point of A. Thus, $X \setminus A$ is open and so A is closed. Now, $U_x \cup (X \setminus A)$ is an open cover of X. But this cover has no finite subcover for each U_x containing the element of A and also $A \cap (X \setminus A) = \phi$, which is a contradiction to (X, τ) is compact.

Theorem 2.7. *Every ss-countably compact space is an open ss-countably compact space*.

Proof. Let A be a proper open subset of the topological space (X, τ). Then the proof is proved, by the definition of open *ss*-countably compact.

Remark 2.8. The converse of the above Theorem 2.6 and Theorem 2.7 need not be true. By Example 2.2(1), $\mathbb{R}$ with usual topology is an open *ss*-countably compact but neither compact nor *ss*-countably compact.

Theorem 2.9. *If* $X \times Y$ *is an open ss-countably compact, then each of X and Y is an open ss-countably compact*.

Proof. Suppose that $X \times Y$ is an open *ss*-countably compact. Since the projection functions $P_1 : X \times Y \to X$ and $P_1 : X \times Y \to Y$ are continuous functions and onto functions, then each of X and Y is an open *ss*-countably compact by Theorem 2.3.

References

[1] H. Cakalh, A study on statistical convergence, Funct. Anal., Approx. Comput. 1(2) (2009), 19-24.

[2] G. Di. Maio and D. R. Kocinac, Statistical convergence in topology, Topology Appl. 156 (2008), 28-45.

[3] H. Fast, Surla convergence Statistique, Colloq. Math. 2 (1951), 241-244.

[4] L. Steen and J. A. Seebach Jr, Counterexamples in Topology, Renehart and Winston Inc., New York, 1970.

[5] V. Renukadevi and P. Vijayashanthi, On *I*-Frechet-Urysohn spaces and sequential *I*-convergence groups, Math. Moravica 23(1) (2019), 119-129.