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ON STABILITY ANALYSIS AND EXISTENCE OF ATANGANA-BALEANU FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING P-LAPLACIAN OPERATOR

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Abstract

In this article, we study on the existence and uniqueness of solutions for a Atangana-Baleanu fractional differential equations with dependence on the lipschitz first derivative conditions with singularity and involving p-laplacian operator in the Banach's space. We develop a Guo-Krasnoselskii theorem in the frame of Atangana-Baleanu fractional integral. An example is given to illustrate the main results and investigate the stability in the sense of Ulam.

1. Introduction

Fractional calculus has been decrepit as elongated as ordinary calculus,

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the expedition of research in this field has only enormously heightened. Now a days, Fractional differential equation have proved to be the valuable tools in mathematical modeling. Mathematical modeling have captivate the thinking of many researchers in assorted discipline. Particularly fractional order model have been interest of many researchers in various fields such as Medical and engineering fields aerodynamics, rheology, Cosmology, fusion low light, analysis in the nursing bed design evaluation, economic growth model [17, 20, 25, 29, 30]. Few of the recent studies on ABC-derivatives such as, Jarad et al. investigated a ODE's in the form of AB derivative [18]. Ravichandran et al. [13] discussed in details the AB-fractional integro-differential equations. Atangana and Koca find the chaos in a simple nonlinear system with AB-fractional derivatives [9]. Many researchers give attention to the study of existence and uniqueness of positive solution for the fractional equation with p-Laplacian operator. In [37], analyzed the solution related to the existence of positive solutions for the fractional differential equation with the integral boundary conditions and p-Laplacian operator.

Recently, Pandiyammal and Karthik Raja [34] have studied the following ABC-fractional differential equation for the existence of a solution:

$$\begin{cases} {}^{ABC}{}_0D^{\alpha}u(t) = f(t,\, u(t),\, u'(t,\, u(t))) \\ u(0) = u_0 \end{cases} \tag{1}$$

Where ${}^{ABC}{}_0D^{\alpha}$ is the Atangana Baleanu caputo fractional differential operator and $\alpha \in (1,\, 2)$, ${}^{ABC}{}_0D^{\alpha}u(t),\, f(t,\, u(t),\, u'(t,\, u(t))) \in C[0,\, 1]$.

To develop this work, we follow [37] to get the existence solution and HU-stability of the following nonlinear ABC-fractional differential equation with p-Laplacian operator:

Where $k \in (0,\, 2]$ and consider c and $f(t,\, u(t),\, \mathfrak{D}u(t)) \in C[0,\, 1]$ are continuous functions. Then (1) becomes,

$$\begin{cases} {}^{ABC}{}_0D^{k}u(t) = -f(t,\, u(t),\, \mathfrak{D}u(t)) \\ \Phi_p[\,{}^{ABC}{}_0D^{k}(t)]\,|_{t=0} = 0,\; u(1) = 0 \end{cases} \tag{2}$$

The D^{α} and D^{k} are ABC-fractional order differential operators, and

$\Phi_p(r) = |r|^{p-2}r$ is non-linear operator such that $1/p + 1/q = 1$ and $\Phi_p^{-1} = \Phi_q$. Aside the positive solution $u(t)$ of the hinted fractional differential equation (2), where $u(t) > 0$ for $t \in (0, 1]$ (2). Our suggested p-laplacian ABC-fractional differential equation with the operator, is more general than (2).

Towards on the problem (2), we apply a fixed point theorem of the alternative, for contractions on generalized complete metric space to study a generalized Ulam-Hyers stability for (2). We formulate the problem to an alternate fractional integral form of the problem, based on the classical results and ${}^{AB}I_0^{\alpha}$, ${}^{AB}I_0^{k}$ and a Green function. Incessantly we examine the Green function for the application its complexion of positively variation. Finally, an example is given to illustrate our main results.

Definition 1.1. Fractional ABC derivative in Caputo sense of the function $u \in H^1(a, b)$, $a < b$ and α in $[0, 1]$. The Caputo Atanganabaleanu fractional derivative of u of order α is defined by

$$({}^{ABC}{}_{\alpha}D^{\alpha}u)(t) = \frac{B(\alpha)}{(1-\alpha)}\int_0^t u'(s)E_{\alpha}\left[-\alpha\frac{(t-s)^{\alpha}}{(1-\alpha)}\right]ds. \tag{3}$$

Where is the Mittag-Leffler function defined by $E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)}$ and $B(\alpha) > 0$ is a normalizing function satisfying $B(0) = B(1) = 1$. The Riemann Atangana-Baleanu fractional derivative of u of order α is defined by

$$({}^{ABR}{}_{\alpha}D^{\alpha}u)(t) = \frac{B(\alpha)}{(1-\alpha)}\frac{d}{dt}\int_0^t u(s)E_{\alpha}\left[-\alpha\frac{(t-s)^{\alpha}}{(1-\alpha)}\right]ds. \tag{4}$$

Definition 1.2. The fractional AB-integral of the function $u \in H^1(a, b)$, $b > a$, $0 < \alpha < 1$ is given by

$$({}^{AB}{}_{a}I^{\alpha}u)(t) = \frac{1-\alpha}{B(\alpha)}u(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds \tag{5}$$

Lemma 1.3. *The ABC fractional derivative and ABC fractional integral of the function u satisfy the Newton-Leibniz formula* [31]

$$({}^{AB}{}_0I^{\alpha}({}^{ABC}{}_0D^{\alpha}))(t) = u(t) - u(0)E_{\alpha}(\lambda t^{\alpha}) - \frac{\alpha}{1-\alpha}u(0)E_{\alpha,\alpha-1}(\lambda t^{\alpha}))$$

$$= u(t) - (0). \tag{6}$$

Definition 1.4. The Riemann-Liouville fractional integral of a function f of order $\alpha > 0$, $f : (0, +\infty) \to \mathbb{R}$ is given by [24]

$$I^{\alpha}f(t) = \frac{1}{r(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds. \tag{7}$$

Where for $\mathrm{Re}(\alpha) > 0$, we have

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-s}s^{\alpha-1}ds. \tag{8}$$

Definition 1.5. The fractional order derivative in Caputo sense for a continuous function $f : (0 + \infty) \to \mathbb{R}$ is given by [24]

$$D^k f(t) = \frac{1}{\Gamma(n-k)}\int_0^t (t-s)^{n-k-1} f^n(s)ds. \tag{9}$$

For $n = [k] + 1$, where $[k]$ is integer part of such that the integral is well defined on $(0, \infty)$ range.

Lemma 1.6. *For a fractional order* $\alpha \in (n-1, n]$, $f \in C^{n-1}$ *the following equation is satisfied*

$$I^{\alpha}D^{\alpha}f(t) = f(t) + q_0 + q_1 t + q_2 t^2 + \ldots + q_n t^{n-1} \tag{10}$$

for $q_k \in \mathbb{R}$ *the for* $k = 1, 2, \ldots, n-1$. *Let us consider the well known Guo-Krasnoselskii theorem for the existence of a positive solution.*

Theorem 1.7. *Consider a Banach space* Y *and let* $P \in Y$ *be a cone. Suppose that* B_1, B_2 *are two bounded subsets of* Y *such that* $0 \in B_1$, $\overline{B_1} \subset B_2$, *and the operator* $F : P \cap (\overline{B}_2 / B_1) \to P$ *be continuous such that* [21, 39].

$(A_1) \| Fz \| \leq \| z \|$ *if* $z \in P \cap \partial B_1$ *and* $\| Fz \| \geq \| z \|$ *if* $z \in P \cap \partial B_1$ *or*

$(A_2) \| Fz \| \geq \| z \|$ *if* $z \in P \cap \partial B_1$ *and* $(A_1) \| Fz) \leq \| z \|$ *if* $z \in P \cap \partial B_1$ *or*

Then F has a fixed point in $P \cap (\overline{B}_2 / B_1)$.

Lemma 1.8. *Let* Φ_p *be the nonlinear* Φ_p*-operator. Then a For* $1 < p \leq 2, \alpha_1 \alpha_2 > 0$ *and* $| \alpha_1 |, | \alpha_1 | \geq \xi > 0$, *then*

$$| \Phi_p(\alpha_1) - \Phi_p(\alpha_2) | \leq (p-1) \xi^{p-2} | \alpha_1 - \alpha_2 | \tag{11}$$

If $p > 2$ *and* $| \alpha_1 |, | \alpha_2 | \leq \xi^*$, *then*

$$| \Phi_p(\alpha_1) - \Phi_p(\alpha_2) | \leq (p-1) \xi_1^{p-2} | \alpha_1 - \alpha_2 |. \tag{12}$$

Proposition 1.9 [13, 15]. $f'(u) \in D$ *satisfy the Lipschitz condition. i.e., There exist a constant* $k > 0$ *such that*

$$\| f'(u) - f'(v) \| \leq k(\| u - v \|), \, u, v \in D. \tag{13}$$

2. Green Function and Its Properties

Theorem 2.1. *For* $\alpha, k \in (1, 2]$ *and* $f(t, u(t), \mathfrak{D}u(t)) \in C[0, 1]$ *such that* $f(t, u(0), \mathfrak{D}u(0)) = 0$, $u(t)$ *is a solution of* (2) *if and only if*

$$u(t) = \int_0^1 \mathcal{G}^k(t, s) \Phi_p({}^{AB}I_0^{\alpha}[f(t, u), \mathfrak{D}u(t))]) dt. \tag{14}$$

Where

$$\mathcal{G}^k(t, s) = \begin{cases} \dfrac{k}{B(k)} \dfrac{(1-s)^{k-1}}{r(k)} - \dfrac{k}{B(k)} \dfrac{(t-s)^{k-1}}{r(k)} & s \leq t \\ \dfrac{k}{B(k)} \dfrac{(1-s)^{k-1}}{r(k)} & s \geq t. \end{cases} \tag{15}$$

Proof. Now we affix the AB-fractional integral operator ${}^{AB}I_0^{\alpha}$ on the equation (2) and by using the lemma (1.6) then the problem (2) becomes as below

$$\Phi_p[{}^{ABC}D^K u(t)] = -{}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))] + c_0. \tag{16}$$

By using the condition $\Phi_p[{}^{ABC}D^K u(t)]_{t=0} = 0$, then we have $c_0 = 0$. Thus we get

$$\Phi_p[{}^{ABC}D^K u(t)] = -{}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))]. \tag{17}$$

Then we have from the equation (17)

$${}^{ABC}D^K u(t) = -\Phi_p({}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))]). \tag{18}$$

${}^{AB}I^k$ and by apply the lemma (1.3)

$$u(t) = -{}^{AB}I^k(\Phi_q({}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))])) \tag{19}$$

And by using the condition from (2) $u(1) = 0$,

$$c_1 = {}^{AB}I_k(\Phi_q({}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))]))\,|_{t=1} \tag{20}$$

By using the equations (19) and (20) we get

$$u(t) = {}^{AB}I_k(\Phi_q({}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))]))\,|_{t=1}$$

$$- {}^{AB}I_k(\Phi_q({}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))]))$$

$$= \left[\frac{1-k}{B(k)} + \frac{k}{B(k)} I_{t=1}^k\right]\Phi_q({}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))])$$

$$= \left[\frac{k}{B(k)} I_{t=1}^k - \frac{k}{B(k)} I_t^k\right]\Phi_q({}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))]) \tag{21}$$

$$u(t) = \int_0^1 \mathcal{G}^k(t, s)\Phi_q({}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))])ds. \tag{22}$$

Lemma 2.2. *The function* $\mathcal{G}^k(t, s)$ *defined by the equation* (15), *which satisfy*

(C1) $0 < \mathcal{G}^k(t, s)$ *for all* $s, t \leq (0, 1)$;

(C2) *The function* $\mathcal{G}^k(t, s)$ *is a decreasing multivalued function and*

$\mathcal{G}^k(0, s) = \max_{t \in [0, 1]} \mathcal{G}^k(t, s)$ *and*

(C_3) *Based on the assumption of* $0 \leq t^{k-1} \leq 0.5$, *for* $\mathcal{G}^k(t, s) = t^{\alpha - 1} \max_{t \in [0, 1]} \mathcal{G}^k(t, s)$ *for* $s, t \in (0, 1)$.

Proof. First we prove (C_1), we assume two cases.

Case 1. For $s \leq t$.

$$\mathcal{G}^k(t, s) = \frac{k}{B(k)} \frac{(1-s)^{k-1}}{\Gamma(k)} - \frac{k}{B(k)} \frac{(t-s)^{k-1}}{\Gamma(k)}$$

$$= \frac{k}{B(k)} \left[\frac{(1-s)^{k-1}}{\Gamma(k)} - t^{k-1} \frac{\left(1 - \frac{s}{t}\right)^{k-1}}{\Gamma(k)} \right] \tag{23}$$

$$\geq \frac{k}{B(k)} \left[\frac{(1-s)^{k-1}}{\Gamma(k)} - t^{k-1} \frac{(t-s)^{k-1}}{\Gamma(k)} \right] \geq 0.$$

Case 2. For $t \leq s$ we have

$$\mathcal{G}^k(t, s) = \frac{k}{B(k)} \frac{(1-s)^{k-1}}{\Gamma(k)} > 0 \tag{24}$$

From the equation (23), and (24), it is shown that $\mathcal{G}^k(t, s) > 0$ for all $0 < s, t < 1$.

To examine the proof of (C_2)

Case 1. For $s \leq t$.

$$\frac{\partial}{\partial t} \mathcal{G}^k(t, s) = \frac{\partial}{\partial t} \left[\frac{k}{B(k)} \frac{(1-s)^{k-1}}{\Gamma(k)} - \frac{k}{B(k)} \frac{(t-s)^{k-1}}{\Gamma(k)} \right]$$

$$= -\frac{k}{B(k)} \frac{(t-s)^{k-1}}{\Gamma(k)} < 0. \tag{25}$$

Case 2. For, $t \leq s$ we evaluate

$$\frac{\partial}{\partial t}\mathcal{G}^k(t, s) = \frac{\partial}{\partial t}\frac{k}{B(k)}\frac{(1-s)^{k-1}}{\Gamma(k)} = 0 \tag{26}$$

From equation (25) and (26) we have $\frac{\partial}{\partial t}\mathcal{G}^k(t, s) \leq 0$ for $s, t \in (0, 1)$, which implies that the Green function $\mathcal{G}^k(t, s)$ decreasing with respect to t.

For $t \leq s$ we get

$$\max_{t\in[0, 1]}\mathcal{G}^k(t, s) = \lim_{t\to 0}\left[\frac{k}{B(k)}\frac{(1-s)^{k-1}}{\Gamma(k)} - \frac{k}{B(k)}\frac{(t-s)^{k1}}{\Gamma(k)}\right]$$

$$= \frac{k}{B(k)}\frac{(1-s)^{k-1}}{\Gamma(k)} = \mathcal{G}^k(0, s) \tag{27}$$

In the same for $s \geq t$, we have

$$\max_{t\in[0, 1]}\mathcal{G}^k(t, s) = \lim_{t\to 0}\mathcal{G}^k(t, s) = \frac{k}{B(k)}\frac{(1-s)^{k-1}}{\Gamma(k)} = \mathcal{G}^k(0, s) \tag{28}$$

For (C_3), we have two cases,

Case 1. Let $t \geq s$ then based on the assumption $0 \leq t^{k-1} \leq \frac{1}{2}$ we have

$$\mathcal{G}^k(t, s) = \frac{k}{B(k)}\frac{(1-s)^{k-1}}{\Gamma(k)} - t^{k-1}\frac{k}{B(k)}\frac{\left(1-\frac{s}{t}\right)^{k-1}}{\Gamma(k)}$$

$$\geq t^{k-1}\frac{k}{B(k)}\frac{(1-s)^{k-1}}{\Gamma(k)} = t^{k-1}\mathcal{G}^k(0, s). \tag{29}$$

Case 2. For $s \geq t$, then

$$\mathcal{G}^k(t, s) = \frac{k}{B(k)}\frac{(1-s)^{k-1}}{\Gamma(k)} - t^{k-1}$$

$$\max_{t\in[0, 1]}\mathcal{G}^k(t, s) = t^{k-1}\mathcal{G}^k(0, s). \tag{30}$$

From the equations (29) and we proved the assumption (C_3) □

3. Existence of Solutions

Let us consider the Banach space $X = C[0, 1]$ with the norm $\| u \| = \max_{t \in [0, 1]} \{| u(t) | : u \in X\}$ and let H be a nonnegative cone in the space X where $H = \{u \in X : u(t) \geq t^{\zeta} \| u \|, t \in [0, 1]\}$.

Let $B(r) = \{u \in H : \| u \| < B(r)\}, \partial B(r) = \{u \in H : \| u \| = r\}$ by using the theorem (2.1), the solution of equation (2) is given by

$$u(t) = \int_0^1 \mathcal{G}^k(t, s) \Phi_q ({}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))]) ds \tag{31}$$

Consider the function $F : H \setminus 0 \to X$ by

$$Fu(t) = \int_0^1 \mathcal{G}^k(t, s) \Phi_q ({}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))]) ds \tag{32}$$

Thus $u(t)$ is equivalent to a fixed point of F, implies that

$$u(t) = Fu(t). \tag{33}$$

Here we assume the following assumptions

(R_1) $N : (0, 1) \to [0, +\infty)$ is discontinuous on $(0, 1)$ and nonvanishing and $\| N \| = \max_{t \in [0, 1]} | N(t) | < +\infty$;

(R_2) Let $u \in C[0, 1]$ and $f \in (J \times PC^1 \times J, J)$ is a piecewise continuous function and there exists a positive constants $\mathfrak{M}_1$, $\mathfrak{M}_2$ and $\mathfrak{W}$ such that

$$| f(t, u_1, v_1) - f(t, u_2, v_2) | \leq \mathfrak{M}_1(| u_1 - u_2 | + | v_1 - v_2 |) \tag{34}$$

for each u_1, u_2, v_1, v_2 in Y, $\mathfrak{M}_2 = \max_{t \in R} \| f(t, 0, 0) \|$ and $\mathfrak{M} = \max \{\mathfrak{M}_1, \mathfrak{M}_2\}$. Let $Y = C[R, X]$ be the set of continuous functions on R with in the Banach space X values

(R_3) Let $u' \in C[a, b]$ satisfy the Lipschitz condition. i.e., There exists a positive constants $\mathfrak{N}_1$, $\mathfrak{N}_2$ and N such that

$$\| \mathfrak{D}(t, u) - \mathfrak{D}(t, v) \| \leq \mathfrak{N}_1(\| u - v \|),$$

for all u, v in $Y \cdot \mathfrak{N}_2 = \max_{t \in D} \| \mathfrak{D}(t, 0) \|$ and $\mathfrak{N} = \max\{\mathfrak{N}_1, \mathfrak{N}_2\}$.

(R_4) For each $\lambda > 0$, Let $B_\lambda \in \{u \in Y : \| u \| \leq \lambda\} \subset Y$ then B_λ is clearly bounded, closed and convex subset in $C([0, 1], R)$.

Lemma 3.1. *If* (R_1) *and* (R_3) *are satisfied, then the estimate*

$\| \mathfrak{D}u(t) \| \leq t(\mathfrak{N}_1 \| u \| + \mathfrak{N}_2)$, $\| \mathfrak{D}u(t) - \mathfrak{D}v(t) \| \leq \mathfrak{N}t \| u - v \|$ *are satisfied for any* $t \in R$ *and* $u, v \in Y$. *Take is* $(\mathfrak{M} + \mathfrak{N}t)$.

Theorem 3.2. *If the conditions* (R_1)-(R_4) *are satisfied and* $0 \leq t^{k-1} \leq \frac{1}{2}$ *then the function F is completely continuous operator*.

Proof. For every $u \in \overline{B(r_2)}/B(r_1)$ from the lemma (2.2) and the equation (32) we get,

$$Fu(t) = \int_0^1 \mathcal{G}^k(t; s)\Phi_q({}^{AB}I_0^\alpha[f(t, u(t), \mathfrak{D}u(t))])ds$$

$$\leq \int_0^1 \mathcal{G}^k(0, s)\Phi_q({}^{AB}I_0^\alpha[f(t, u(t), \mathfrak{D}u(t))])ds \tag{35}$$

and

$$Fu(t) = \int_0^1 \mathcal{G}^k(t, s)\Phi_q({}^{AB}I_0^\alpha[f(t, u(t), \mathfrak{D}u(t))])ds$$

$$\leq t^{k-1}\int_0^1 \mathcal{G}^k(0, s)\Phi_q({}^{AB}I_0^\alpha[f(t, u(t), \mathfrak{D}u(t))])ds \tag{36}$$

by using the equation (35) and (36) we get on with

$$Fu(t) \geq t^{k-1} \| Fu(t) \|, \; t \in [0, 1] \tag{37}$$

Which implies that $F : \overline{B(r_2)}/B(r_1) \to H$. Next we prove that F is continuous, we prove that $\| F(u_n) - F(u) \| \to 0$ as $n \to \infty$ as follows:

$$\| Fu_n(t) - Fu(t) \| = \left| \int_0^1 \mathcal{G}^k(t, s)\Phi_q({}^{AB}I_0^\alpha[f(t, u_n(t), \mathfrak{D}u_n(t))])ds \right.$$

$$- \int_0^1 \mathcal{G}^k(t, s)\Phi_q({}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))])ds \Bigg|$$

$$\leq \int_0^1 \mathcal{G}^k(t, s)\Phi_q({}^{AB}I_0^{\alpha}[f(t, u_n(t), \mathfrak{D}u_n(t))])$$

$$- ({}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))]) \,|\, ds \tag{38}$$

by using the equation (38) and the continuity of f we get $| Fu_n(t) - Fu(t) | \to 0$ as $n \to +\infty$, this shows that F is continuous. Here for the uniformly continuous of F by equation (22) and the assumption (R_2), we get

$$| Fu(t) | = \left| \int_0^1 \mathcal{G}^k(t, s)\Phi_q({}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))])ds \right|$$

$$= \Bigg| \int_0^1 \mathcal{G}^k(t, s)\Phi_q\Bigg[\frac{1-\alpha}{B(\alpha)} f(t, u(t), \mathfrak{D}u(t))$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}({}_aI^{\alpha} f(t, u(t), \mathfrak{D}u(t))\Bigg]ds \Bigg|$$

$$\leq \int_0^1 | \mathcal{G}^k(0, s) | \Phi_q\Bigg[\frac{1-\alpha}{B(\alpha)} \| f(t, u(t), \mathfrak{D}u(t)) \|$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \| {}_aI^{\alpha} f(t, u(t), \mathfrak{D}u(t)) \| \Bigg]ds$$

$$= \int_0^1 | \mathcal{G}^k(t, s) | \Phi_q\Bigg[\frac{1-\alpha}{B(\alpha)} (\mathfrak{M}\| u \| + \mathfrak{N}t\| u \|)$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} (\mathfrak{M}\| u \| + \mathfrak{N}t\| u \|)({}_aI^{\alpha})\Bigg]ds$$

$$= \int_0^1 | \mathcal{G}^k(0, s) | \Phi_q(\mathfrak{M} + \mathfrak{N}t)\Bigg[\frac{1-\alpha}{B(\alpha)}\| u \| + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \| u \| ({}_aI^{\alpha})\Bigg]ds$$

$$= \int_0^1 | \mathcal{G}^k(0, s) | \Phi_q \rho\lambda\Bigg[\frac{1-\alpha}{B(\alpha)} + \frac{(1-0)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\Bigg]ds < \infty \tag{39}$$

From the equation (38) the function F is uniformly bounded. By using the assumption (R_3), Theorem (2.1) and (3.2), the operator F is equicontinuity, for any $t_1, t_2 \in [0, 1]$, we have

$$| Fu(t_1) - Fu(t_2) | = \Bigg| \int_0^1 \mathcal{G}^k(t_1, s)\Phi_q({}^{AB}I_0^{\alpha}[f(t, u(t_1), \mathfrak{D}u(t_1))]ds$$

$$- \int_0^1 \mathcal{G}^k(t_2, s)\Phi_q({}^{AB}I_0^{\alpha}[f(t, u(t_2), \mathfrak{D}u(t_2))]ds \Bigg|$$

$$\leq \int_0^1 | \mathcal{G}^k(t_1, s) | \Phi_q\Bigg[\frac{1-\alpha}{B(\alpha)} \| f(t, u(t_1), \mathfrak{D}u(t_1)) \|$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \| {}_aI^{\alpha} f(t, u(t_1), \mathfrak{D}u(t_1)) \| \Bigg] ds$$

$$- \int_0^1 | \mathcal{G}^k(t_2, s) | \Phi_q\Bigg[\frac{1-\alpha}{B(\alpha)} \| f(t, u(t_2), \mathfrak{D}u(t_2)) \|$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \| {}_aI^{\alpha} f(t, u(t_2), \mathfrak{D}u(t_2)) \| \Bigg] ds$$

$$\leq \int_0^1 | \mathcal{G}^k(t_1, s) | \Phi_q\Bigg[\frac{1-\alpha}{B(\alpha)} (\mathfrak{M} + \mathfrak{N}t) \| u \|$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} (\mathfrak{M} + \mathfrak{N}t) \| u \| ({}_aI^{\alpha}) \Bigg] ds$$

$$- \int_0^1 | \mathcal{G}^k(t_2, s) | \Phi_q\Bigg[\frac{1-\alpha}{B(\alpha)} (\mathfrak{M} + \mathfrak{N}t) \| u \|$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} (\mathfrak{M} + \mathfrak{N}t) \| u \| ({}_aI^{\alpha}) \Bigg] ds$$

$$\leq \left[\frac{(t_1 - a)^k}{B(k)\Gamma(k)} - \frac{(t_2 - a)^k}{B(k)\Gamma(k)} \right] \Phi_q \rho\lambda \left[\frac{1-\alpha}{B(\alpha)} - \frac{(1-0)^{\alpha}}{B(\alpha)\Gamma(\alpha)} \right] \tag{40}$$

The equation (40) tends to zero because $t_1 \to t_2$. Thus the function operator $F : \overline{B(r_2)} \setminus B(r_1) \to H$ is an equicontinuous operator. From the Arzela Ascoli theorem the function operator is compact. This completes the proof the function operator F is compact in $\overline{B(r_2)} \setminus B(r_1)$.

Therefore $F : \overline{B(r_2)} \setminus B(r_1) \to H$ is completely continuous. □

Now we define the height $f(t, u(t), \mathfrak{D}u(t))$ for $r > 0$ and

$$\begin{cases} \Phi_{\max}(t, r) = \max_{t \in (0, 1)}\{f(t, u(t), \mathfrak{D}u(t)) : t^{k-1}r \le u \le r\} \\ \Phi_{\min}(t, r) = \min_{t \in (0, 1)}\{f(t, u(t), \mathfrak{D}u(t)) : t^{k-1}r \le u \le r\}. \end{cases} \tag{41}$$

Theorem 3.3. *Let* (R_1)-(R_4) *clinch true and there exist* $\zeta_1, \zeta_2 \in \mathbb{R}^+$ *such that* $(\mu_1)\, \zeta_1 \le \int_0^1 \mathcal{G}^k(0, s)\Phi_q({}^{AB}I_0^{\alpha})\, \Phi_{\min}(\gamma, \zeta_1)ds < +\infty$ *and*

$$\int_0^1 \mathcal{G}^k(0, s)\Phi_q({}^{AB}I_0^{\alpha})\, \Phi_{\min}(\gamma, \zeta_1)ds < +\infty$$

or

$$(\mu_2) \le \int_0^1 \mathcal{G}^k(0, s)\Phi_q({}^{AB}l_0^{\alpha})\, \Phi_{\min}(\gamma, \zeta_1)ds < \zeta_1$$

and

$$\zeta_2 \le \int_0^1 \mathcal{G}^k(0, s)\Phi_q({}^{AB}l_0^{\alpha})\, \Phi_{\min}(\gamma, \zeta_2)ds < +\infty$$

is satisfied. Then, the ABC-fractional differential equation with operator Φ_p (41) *has a positive solution* $u \in H$ *and* $\zeta_1 \le \| u \| \le \zeta_2$.

Proof. By using the generality consider the case (μ_1). If $u \in \partial B(\zeta_1)$ then we have $\| u \| = \zeta_1$ and $t^{k-1}\zeta_1 \le u \le \zeta_1, t \in [0, 1]$ by using the equation (41) for $t \in (0, 1), \Phi_{\min}(t, u) \le f(t, u(t), \mathfrak{D}u(t))$ we get

$$\| Fu(t) \| = \max_{t \in [0, 1]} \int_0^1 | \mathcal{G}^k(t, s) | \Phi_q\Bigg[\frac{1-\alpha}{B(\alpha)} \| f(t, u(t), \mathfrak{D}u(t)) \|$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \| {}_a l^{\alpha} f(t, u(t), \mathfrak{D}u(t)) \| \Bigg] ds$$

$$\ge t^{k-1} \int_0^1 | \mathcal{G}^k(0, s) | \Phi_q\Bigg[\frac{1-\alpha}{B(\alpha)} \| f(t, u(t), \mathfrak{D}u(t)) \|$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \| f(t, u(t), \mathfrak{D}u(t)) \| ({}_a I^{\alpha}) \Big] ds$$

$$\geq \int_0^1 | \mathcal{G}^k(0, s) | \Phi_q(t, \zeta_1) \rho \Big[\frac{1-\alpha}{B(\alpha)} \Phi_{\min}(t, \zeta_1)$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \Phi_{\min}(t, \zeta_1) \int_0^{\tau} (\tau - \eta)^{\alpha - 1} d\eta \Big] ds$$

$$\geq \zeta_1 = \| u \| \tag{42}$$

Let $u \in \partial B(\zeta_2)$ then $\| u \| = \zeta_2$ and $t^{k-1}\zeta_2 \leq u \leq \zeta_2$ for $0 \leq t \leq 1$ by using the equation (41) for $t \in (0, 1), \Phi_{\max}(t, u) \leq f(t, u(t), \mathfrak{D}u(t))$ which implies

$$\| Fu(t) \| = \max_{t \in [0, 1]} \int_0^1 | \mathcal{G}^k(t, s) | \Phi_q \Big[\frac{1-\alpha}{B(\alpha)} \| f(t, u(t), \mathfrak{D}u(t)) \|$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \| {}_a I^{\alpha} f(t, u(t), \mathfrak{D}u(t)) \| \Big] ds$$

$$\geq t^{k-1} \int_0^1 | \mathcal{G}^k(0, s) | \Phi_q \Big[\frac{1-\alpha}{B(\alpha)} \| f(t, u(t), \mathfrak{D}u(t)) \|$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \| f(t, u(t), \mathfrak{D}u(t)) \| ({}_a I^{\alpha}) \Big] ds$$

$$\geq \int_0^1 | \mathcal{G}^k(0, s) | \Phi_q(t, \zeta_1) \rho \Big[\frac{1-\alpha}{B(\alpha)} \Phi_{\max}(t, \zeta_1)$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \Phi_{\max}(t, \zeta_1) \int_0^{\tau} (\tau - \eta)^{\alpha - 1} d\eta \Big] ds$$

$$\geq \zeta_2 = \| u \| \tag{43}$$

From the lemma (1.7) has a fixed point $\overline{B(1)} \setminus B(0)$. By lemma (2.2) and theorem (2.1) we have $a \leq \| u \| \leq b$ which yields that $u(t) \geq t^{k-1} \| u \| a t^{k-1} > 0$ for $t \in (0, 1)$. Hence u is a positive solution. □

4. Stability Analysis

The stability analysis of fractional equations for the stability group of homomorphism it is proposed by an open question of Ulam. If for a group of homomorphism $f : (C_1) \to (C_2)$ between the group (C_1) and a metric group (C_2) which is satisfying $d(f(xy)) = f(x)f(y) < \varepsilon$ for all $x, y \in (C_1)$ and then there exists a homomorphism $g : (C_1) \to (C_2)$ with $d(f(x), g(y)) < \varepsilon_1$ for $x \in (C_1)$ otherwise if we have an almost homomorphism then we get the small error.

In this section we investigate the nonlinear Φ_p operator for the problem (2) based on the Hyersulam stability for the ABC-fractional differential equation.

Definition 4.1. The equation (31) hyers Ulam stability for every $\delta > 0$, there exists a constant $C > 0$ such that the following be true if

$$\left| u(t) - \int_0^1 \mathcal{G}^k(t, s)\Phi_q({}^{AB}I_0^\alpha[f(t, u(t), \mathfrak{D}u(t))])ds \right| \leq \delta \tag{44}$$

there exists $h(t)$ satisfying that

$$h(t) - \int_0^1 \mathcal{G}^k(t, s)\Phi_q({}^{AB}I_0^\alpha[f(t, h(t), \mathfrak{D}h(t))])ds \tag{45}$$

such that

$$| u(t) - h(t) | \leq C\delta. \tag{46}$$

Theorem 4.2. *The singular ABC fractional differential equation with delay and* Φ_p *operator*, *the problrm* (2) *hyersulam stability provided that* (R_1)-(R_4) *are satisfied*.

Proof. By theorem (3.1) and definition (4.1) and let $u(t)$ be a solution of the fractional Differential equation with delay (31). Let $x(t)$ be a solution of the ABC-fractional DE with delay (3.1) and $y(t)$ be an approximate solution and satisfying (45). Then, we have

$$|u(t)-h(t)| = \left| \int_0^1 \mathcal{G}^k(t, s)\Phi_q({}^{AB}I_0^{\alpha}[f(t, u(t), \mathfrak{D}u(t))])ds \right| \leq \delta$$

$$- \int_0^1 \mathcal{G}^k(t, s)\Phi_q({}^{AB}I_0^{\alpha}[f(t, h(t), \mathfrak{D}h(t))])ds$$

$$\leq \int_0^1 |\mathcal{G}^k(t, s)|\Phi_q\left[\frac{1-\alpha}{B(\alpha)}\| f(t, u(t), \mathfrak{D}u(t))\|\right.$$

$$\left. + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^1 (t-\eta)^{\alpha-1}\| f(t, u(t), \mathfrak{D}u(t))\|\right] ds$$

$$- \int_0^1 |\mathcal{G}^k(t, s)|\Phi_q\left[\frac{1-\alpha}{B(\alpha)}\| f(t, h(t), \mathfrak{D}h(t))\|\right.$$

$$\left. + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^1 (t-\eta)^{\alpha-1}\| f(t, h(t), \mathfrak{D}h(t))\| d\eta\right] ds$$

$$\leq (p-1)\,\xi^{p-2}\left(\int_0^1 \mathcal{G}^k(t, s)\left[\frac{1-\alpha}{B(\alpha)}\| f(t, u(t), \mathfrak{D}u(t)) - f(t, h(t), \mathfrak{D}h(t))\|\right.\right.$$

$$\left. + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^1 (t-\eta)^{\alpha-1}\| f(t, u(t), \mathfrak{D}u(t)) - f(t, h(t), \mathfrak{D}h(t))\| d\eta\right] ds$$

$$\leq (p-1)\xi^{p-2}\left(\frac{(1-0)^k}{B(k)\Gamma(k)} + \frac{(1-0)^k}{B(k)\Gamma(k)}\right)\left[\frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(k)\Gamma(k)}\right](\mathfrak{M} + \mathfrak{N}t)\| u - v\|$$

$$\leq (p-1)\xi^{p-2}\left(\frac{(1-0)^k}{B(k)\Gamma(k)} + \frac{(1-0)^k}{B(k)\Gamma(k)}\right)\left[\frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(k)\Gamma(k)}\right]\rho\| u - v\| \qquad (47)$$

where

$$W = (p-1)\xi^{p-2}\left(\frac{(1-0)^k}{B(k)\Gamma(k)} + \frac{(1-0)^k}{B(k)\Gamma(k)}\right)\left[\frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(k)\Gamma(k)}\right]\rho.$$

Thus equation (47) is Hyers-Ulam stable. Then the singular ABC-fractional differential equation with delay and the Φ_p operator in the equation (2) is Hyers-Ulam stable. □

5. Example

In this section we give an example which is based on the result in the section (3) and (4) is provided

Problem $t \in [0, 1]$, $p = 3$, $q = 1.5$, $b = 1$, $\alpha = k = \frac{3}{2}$, $a = 0.3$

$$f(t, u(t), \mathfrak{D}u(t)) = 2\left[\left(\frac{u(t)}{u'(t)}\right)^2 - (u(t))^2\right]$$

clearly the function $f \in C(0, 1) \times (0, +\infty) \times (0, +\infty) \to (0, +\infty)$. Here we assuming that the singular ABC fractional differential equation with operator:

$$\begin{cases} {}^{ABC}_{0}D^k u(t) + 2\left[\left(\frac{u(t)}{u'(t)}\right)^2 - (u(t))^2\right] = 0 \\ \Phi_p[{}^{ABC}_{0}D^k u(t)]|_{t=0} = 0 = \Phi_p[{}^{ABC}_{0}D^k (u(t))']|_{t=0}, \; u(1) = 0 = u'(0) \end{cases} \tag{48}$$

Now we consider

$$\Phi_{\max}(t, r) = \max_{t \in (0, 1)}\left\{2\left[\left(\frac{u(t)}{u'(t)}\right)^2 - (u(t))^2\right] : t^{\frac{1}{2}} r \le u \le r\right\}$$

$$\le 2\left[\left(\frac{r}{(t^{\frac{1}{2}} r)}\right)^2 - r^2\right] = 2\left[\left(2t^{\frac{1}{2}}\right)^2 - r^2\right] \tag{49}$$

$$\Phi_{\min}(t, r) = \min_{t \in (0, 1)}\left\{2\left[\left(\frac{u(t)}{u'(t)}\right)^2 - (u(t))^2\right] : t^{\frac{1}{2}} r \le u \le r\right\}$$

$$\le 2\left[\left(\frac{t^{\frac{1}{2}} r}{(tr)}\right)^2 - (t^{\frac{1}{2}} r)^2\right] = 2\left[\left(2t^{\frac{1}{2}}\right)^2 - (t^{\frac{1}{2}} r)^2\right] \tag{50}$$

This is the height function. Then for $t \in (0, 1)$, we have

$$\int_0^1 \mathcal{G}^k(0,\, s)\,\Phi_q(\,{}^{AB}I_0^{\alpha}(\Phi_{\max}(\eta,\, b)d\eta))ds$$

$$= \int_0^1 \mathcal{G}^k(0,\, s)\,\Phi_q\left[\frac{1-\alpha}{B(\alpha)}(2[(2t^{\frac{1}{2}})^2 - r^2])\right.$$

$$\left. + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{\tau}(\tau-\eta)^{\alpha-1}(2[(2t^{\frac{1}{2}})^2 - r^2])\,d\eta\right]ds$$

$$= \int_0^1 \mathcal{G}^k(0,\, s)\,\Phi_q\left[\frac{1-\alpha}{B(\alpha)}(2[(2t^{\frac{1}{2}})^2 - 1])\right.$$

$$\left. + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{s}(s-\eta)^{\alpha-1}(2[(2t^{\frac{1}{2}})^2 - 1])\,d\eta\right]ds$$

$$\leq 0.3043352 < 1 \tag{51}$$

and

$$\int_0^1 \mathcal{G}^k(0,\, s)\,\Phi_q\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\eta)^{\alpha-1}\psi_{\min}(\eta,\, a)d\eta\right)ds$$

$$= \int_0^1 \mathcal{G}^k(0,\, s)\,\Phi_q\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\eta)^{\alpha-1}\psi_{\min}\left(\eta,\, \frac{1}{100}\right)d\eta\right)ds$$

$$\geq \int_0^1 \mathcal{G}^k(0,\, s)\,\Phi_q\left[\frac{1-\alpha}{B(\alpha)}(2[(2s^{\frac{1}{2}})^2 - (s^{\frac{1}{2}}\,\frac{1}{100})^2])\right.$$

$$\left. + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^s (s-\eta)^{\alpha-1}(2[(2\eta^{\frac{1}{2}})^2 - (\eta^{\frac{1}{2}}\,\frac{1}{100})^2])d\eta\right]ds$$

$$= 0.5802446 > \frac{1}{100} \tag{52}$$

Based on the theorem (3.3) and the equation (48) (\ref{example}) has a solution and it satisfied $\frac{1}{100} < \| u \| < 1$.

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