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ON SOME DIFFERENTIAL PROPERTIES OF FUNCTIONS IN LIZORKIN-TRIEBEL-MORREY SPACES WITH DOMINANT MIXED DERIVATIVES

ALIK M. NAJAFOV[1,2] and AZIZGUL M. GASIMOVA[2]

[1]Azerbaijan University of Architecture and Construction
Baku, Azerbaijan
[2]Institute of Mathematics and Mechanics of NAS of Azerbaijan
Baku, Azerbaijan
E-mail: aliknajafov@gmail.com

[3]Sumgait State University
Sumgait, Azerbaijan
E-mail: aliknajafov@gmail.com
ezizgul.qasimova@mail.ru

Abstract

In this paper some differential properties of functions in Lizorkin-Triebel-Morrey spaces with dominant mixed derivatives introduced and studied via embedding theory.

1. Introduction

In connection with the study of different types of differential equations, for example equations in which the dominating mixed derivative, makes it possible to study function spaces with the dominant mixed derivative.

In this paper we introduce

$$S^{l}_{p,\theta,\varphi,\beta}F(G_{\varphi}).$$

Lizorkin-Triebel-Morrey-type spaces with dominant mixed derivatives and study the differential and difference-differential properties of functions with the help of the method of integral representations in this space.

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Let $G \subset R^n; 1 < p, \theta < \infty; \beta = (\beta_1, \ldots, \beta_n), \beta_j \in [0, 1], j \in e_n = \{1, 2, \ldots, n\}$ and $\varphi(t) = (\varphi_1(t_1), \ldots, \varphi_n(t_n)), \varphi_j(t_j) > 0 (t_j > 0)$ be continuously-differentiable functions, $\lim_{t_j \to +0} \varphi_j(t_j) = 0, \lim_{t_j \to +\infty} \varphi_j(t_j) = K \leq \infty, j \in e_n$. We denote by A the set of such vector-functions.

Note that the spaces with parameters constructed and studied in Morrey's papers [8, 9]. After these results in the papers of V. P. Il'in [4], A. S. Ross [17], Yu. V. Netrusov [16], A. Mazzucato [7], V. Kokilashvili, A. Meskhi, H. Rafeiro [6], V. S. Guliyev [2, 3], Y. Sawano [19], E. Nakai [15], Xu Jingshi [20], A. M. Najafov [10-14], L. Kadimova and R. Kerbalayeva [5] etc. this theory were developed and generalized.

For any $x \in R^n$ we assume

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) = G \cap \left\{ y : | y_j - x_j | < \frac{1}{2} \varphi_j(t_j), \ j \in e_n \right\}$$

and $m_j > 0$ is entire, $l_j > 0, k_j \geq 0$ are entire and $m_j > l_j > k_j \geq 0; \ j \in e_n$.

Definition 1.1. Denote by $S^l_{p, \theta, \varphi, \beta} F(G_\varphi)$ the Banach space of locally summable functions on G with finite norm

$$\| f \|_{S^l_{p, \theta, \varphi, \beta} F(G_\varphi)} = \sum_{e \subseteq e_n} \left\| \left\{ \int_{0^e}^{t_0^e} \left[\frac{\delta^{m^e}(\varphi_i(t)) D^{k^e} f(\cdot)}{\prod_{j \in e} (\varphi_j(t_j))^{(l_j - k_j)}} \right]^\theta \prod_{j \in e} \frac{d\varphi_j(t_j)}{\varphi_j(t_j)} \right\}^{\frac{1}{\theta}} \right\|_{p, \varphi, \beta}, \quad (1.1)$$

here

$$\| f \|_{p, \varphi, \beta; G} = \| f \|_{L_{p, \varphi, \beta;}(G)} = \sup_{\substack{x \in G \\ t_j > 0}; \ j \in e_n} \left(\prod_{j \in e_n} | \varphi_j([t_j]_1) |^{-\beta_j} \| f \|_{p, G_{\varphi(t)}(x)} \right), \quad (1.2)$$

$$\delta^{m^e}(\varphi(t)) f(x) = \left(\prod_{j \in e} \delta_j^{m_j}(t_j) \right) f(x), \delta_j^{m_j}(t_j) f(x) = \int_{-1}^{1} | \Delta_j^{m_j}(t_j u, G_\varphi) f(x) | du,$$

$[t_j]_1 = \min\{1, t_j\}, \ j \in e_n$ and $t_0 = (t_{01}, \ldots, t_{0n})$ is a fixed positive vector, and

$$\int_{a^e}^{b^e} f(x)dx^e = \left(\prod_{j\in e}\int_{a_j}^{b_j} dx_j\right) f(x),$$

i.e., integration is carried out only with respect to the variables x_j whose indices belong to *e*.

The spaces $S^l_{p,\theta,\varphi,\beta}F(G_\varphi)$ in the case $\beta_j = 0 (j = 1, 2, \ldots, n)$ concides with the space $S^l_{p,\theta}F(G)$ Lizorkin-Triebel with dominant mixed derivatives, defined and studied in [11] with finite norm

$$\| f \|_{S^l_{p,\theta}F(G_\varphi)} = \sum_{e\subseteq e_n} \left\| \left\{ \int_{0^e}^{t_0^e} \left[\frac{\delta^{m^e}(\varphi(t))D^{k^e} f(\cdot)}{\prod_{j\in e}(\varphi_j(t_j))^{l_j - k_j}} \right]^\theta \prod_{j\in e} \frac{d\varphi_j(t_j)}{\varphi_j(t_j)} \right\}^{\frac{1}{\theta}} \right\|_p . \tag{1.3}$$

For any $t_j > 0$ there exists a constant $C > 0$ such that $\prod_{j\in e}(\varphi_j(t_j[t_j]_1)) \le C,$ then the following embedding holds

$$L_{p,\varphi,\beta}(G) \hookrightarrow L_p(G) \quad S^l_{p,\theta,\varphi,\beta}F(G_\varphi) \hookrightarrow S^l_{p,\theta}F(G_\varphi),$$

i.e.,

$$\| f \|_{p,G} \le c \| f \|_{p,\varphi,\beta,G} \text{ and } \| f \|_{S^l_{p,\theta}F(G_\varphi)} \le c \| f \|_{S^l_{p,\theta,\varphi,\beta}F(G_\varphi)}. \tag{1.4}$$

In the case $1 < \theta \le r \le s \le \sigma < \infty$ and $\theta \le p \le \sigma$, we have $S^l_{p,\theta,\varphi,\beta}B(G_\varphi) \hookrightarrow S^l_{p,r,\varphi,\beta}F(G_\varphi) \hookrightarrow S^l_{p,s,\varphi,\beta}F(G_\varphi) \hookrightarrow S^l_{p,\sigma,\varphi,\beta}B(G_\varphi)$. The space $S^l_{p,\theta,\varphi,\beta}B(G_\varphi)$ studied in [13].

Theorem 1.2. *Let* $1 < p < \infty, 1 < \theta < \infty,$ *and* $f \in S^l_{p,\theta}F(G_\varphi)$. *Then one can construct the sequence* $h_s = h_s(x) (s = 1, 2, \ldots)$ *of infinitely differentiable finite functions in* R^n *such that*

$$\lim_{s\to\infty} \| f - h_s \|_{S^l_{p,\theta}F(G)} = 0.$$

Let $\Psi_e(\cdot, y) \in C_0^\infty(R^n)$, and

$$S(\Psi_e) \subset I_{\varphi(t)} = \left\{ y : | y_j | < \frac{1}{2} \varphi_j(t_j),\ j \in e_n \right\}.$$

Assume that for any $0 < T_j \le 1$, $j \in e_n$

$$V = \bigcup_{\substack{0 < t_j \le T_j \\ j \in e_n}} \left\{ y : \left(\frac{y}{\varphi_j(t^e + T^{e'})} \right) \in S(\Psi_e) \right\},$$

where $t^e + T^{e'} = t_j (j \in e)$; $t^e + T^{e'} = T_j (j \in e_n \setminus e = e')$, $U \subset G$, $V \subset I_{\varphi(t)}$, and suppose that $U + V \subset G$.

Lemma 1.1. *Let* $1 \le p \le q \le r \le \infty$, $0 < \eta_j, t_j \le T_j \le 1 (j \in e_n)$; $v = (v_1, \dots, v_n)$, $v_j \ge 0$ *be entire* $(j \in e)$, $f_e \in L_{p, \varphi, \beta}(G)$ *and*

$$\mu_j = l_j - v_j - (1 - \beta_j p)\left(\frac{1}{p} - \frac{1}{q} \right),$$

$$A_\eta^e(x) = \prod_{j \in e'} (\varphi_j(T_j))^{-1-v_j} \int_{0^e}^{\eta^e} \int_{R^n} \psi(x, y; t^e + T^{e'}) f_e(x + y, t)$$

$$\times \prod_{j \in e} (\varphi_j(t_j))^{-2-v_j} \prod_{j \in e} \varphi_j'(t_j) dt^e dy, \tag{1.5}$$

$$A_{\eta, T}^e(x) = \prod_{j \in e'} (\varphi_j(T_j))^{-1-v_j} \int_{\eta^e}^{T^e} \int_{R^n} \psi(x, y; t^e + T^{e'}) f_e(x + y, t)$$

$$\times \prod_{j \in e} (\varphi_j(t_j))^{-2-v_j} \prod_{j \in e} \varphi_j'(t_j) dt_j, \tag{1.6}$$

where

$$| f_e(x, t) | \le C \int_{-1}^{1^e} | \delta^{m^e}(\varphi_j(t)) f(x + \vartheta \varphi_i(t)) | d\vartheta^e.$$

Then for any $\bar{x} \in U$ *the following inequalities are valid*

$$\sup_{\bar{x}\in U}\| A_{\eta}^{e} \|_{q,U_{\psi(\xi)}(\bar{x})} \le C_1 \| \prod_{j\in e_n} (\varphi_j(t_j))^{-l_j} \delta^{m^e}(\varphi(t)) \|_{p,\varphi,\beta,G}$$

$$\times \prod_{j\in e'} (\varphi_j(T_j))^{-\upsilon_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{j\in e_n} (\psi_j[\xi_j]_1)^{\beta_j \frac{p}{q}} \prod_{j\in e'} (\varphi_j(\eta_j))^{-\mu_j} (\mu_j > 0), \quad (1.7)$$

$$\sup_{\bar{x}U} \| A_{\eta,T}^{e} \|_{q,U_{\psi(\xi)}(\bar{x})} \le C_2 \| \prod_{j\in e_n} (\varphi_j(t_j))^{-l_j} \delta^{m^e}(\varphi(t)) \|_{p,\varphi,\beta;G}$$

$$\times \prod_{j\in e'} (\varphi_j(T_j))^{-\upsilon_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{j\in e_n} (\psi_j[\xi_j]_1)^{\beta_j \frac{p}{q}}$$

$$\times \begin{cases} \prod_{j\in e} (\varphi_j(T_j))^{-\mu_j} & \text{for } \mu_j > 0 \\ \prod_{j\in e} \ln \dfrac{\varphi_j(T_j)}{\varphi_j(\eta_j)} & \text{for } \mu_j = 0 \\ \prod_{j\in e} (\varphi_j(\eta_j))^{-\mu_j} & \text{for } \mu_j < 0, \end{cases} \quad (1.8)$$

here $U_{\psi(\xi)}(\bar{x}) = \{x : | x_j - \bar{x}_j | < \frac{1}{2} \psi_j(\xi_j),\ j \in e_n\}$ *and* $\psi \in A$, C_1, C_2 *are the constants independent of* f, ξ, η *and* T.

Proof. Applying sequentially Minkowskii generalized inequality for any $\bar{x} \in U$

$$\| A_{\eta}^{e} \|_{q,U_{\psi(\xi)}(\bar{x})} \le \prod_{j\in e} (\varphi_j(T_j))^{-1-\upsilon_j} \int_{0^e}^{\eta^e} \| A(\cdot, t^e + T^{e'}) \|_{q,\, U_{\psi(\xi)}(\bar{x})}$$

$$\times \prod_{j\in e} (\varphi_j(t_j))^{-2-\upsilon_j} \prod_{j\in e} \varphi'_j(t_j) dt_j, \quad (1.9)$$

where

$$A(\cdot, t^e + T^{e'}) = \int_{R^n} \psi(x,\ y;\ t^e + T^{e'}) f_e(x + y,\ t) dy. \quad (1.10)$$

From the Hölder inequality $(q \le r)$ we have

$$\| A(\cdot, t^e + T^{e'}) \|_{q, U_{\psi(\xi)}(\bar{x})} \leq \| A(\cdot, t^e + T^{e'}) \|_{r, U_{\psi(\xi)}(\bar{x})} \prod_{j \in e_n} (\psi_j(\xi_j))^{\frac{1}{q} - \frac{1}{r}}. \quad (1.11)$$

Further, we will assume that there exists a function $\psi_1(x)$ such that $|\psi(x, y, z)| \leq C|\psi_1(x)|$ for all $(y, z) \in R^n \times R^n$.

Let χ be a characteristic function of the set $S(\psi)$. Again applying the Holder inequality for representing function in the form (1.10) for $1 \leq p \leq r \leq \infty$, $s \leq r$ as $\left(\frac{1}{s} = 1 - \frac{1}{p} + \frac{1}{r}\right)$, we get

$$\| A(\cdot, t^e + T^{e'}) \|_{r, U_{\psi(\xi)}(\bar{x})} \leq \sup_{x \in U_{\psi(\xi)}(\bar{x})} \left(\int_{R^n} | f_e(x + y, t) |^p \chi\left(\frac{y}{\varphi(t^e + T^{e'})} \right) dy \right)^{\frac{1}{p} - \frac{1}{r}}$$

$$\times \sup_{y \in V} \left(\int_{U_{\varphi(\xi)}(\bar{x})} | f_e(x + y, t) |^p dx \right)^{\frac{1}{r}} \left(\int_{R^n} \left| \Psi_i\left(\frac{y}{\psi(t^e + T^{e'})} \right) \right|^s dy \right)^{\frac{1}{s}}. \quad (1.12)$$

For any $x \in U$ we have

$$\int_{R^n} | f_e(x + y, t) |^p \chi\left(\frac{y}{\varphi(t^e + T^{e'})} \right) dy$$

$$\leq \int_{(U+V)_{\varphi(t)}(\bar{x})} | f_e(y, t) |^p dy \leq \int_{G_{\varphi(t)}(\bar{x})} | f_e(y, t) |^p dy$$

$$\leq \prod_{j \in e'} (\varphi_j(T_j))^{\beta_j p} \prod_{j \in e} (\varphi_j(t_j))^{\beta_j p} \prod_{j \in e} (\varphi_j(T_j))^{l_j p}$$

$$\| \prod_{j \in e} (\varphi_j(t_j))^{l_j p} \delta^{m^e} (\varphi_i(t)) f \|_{p, \varphi, \beta}. \quad (1.13)$$

For $y \in V(U_\psi + V \subset G_\varphi)$

$$\int_{U_{\psi(\xi)}(\bar{x})} | f_e(x + y, t) |^p dx \leq \int_{(U+V)_{\psi_\xi}(\bar{x})} | f_e(y, t) |^p dx$$

$$\leq \prod_{j \in e} (\varphi_j(t_j))^{p l_j} \| \prod_{j \in e} (\varphi_j(t_j))^{-l_j} \delta^{m^e} (\varphi(t)) f \|_{p, \varphi, \beta} \prod_{j \in e_n} (\psi_j([\xi_j]_1))^{\beta_j p}. \quad (1.14)$$

and

$$\int_{R^n} \left| \Psi_1\left(\frac{y}{\varphi(t^e + T^{e'})}\right)\right|^s dy = \| \Psi_1 \|_s^s \prod_{j \in e} (\varphi_j(t_j)). \tag{1.15}$$

From inequalities (1.11)-(1.15) it follows that

$$\| A(\cdot, t^e + T^{e'}) \|_{r, U_{\psi(\xi)}(\bar{x})} \leq \| \Psi_1 \|_s, \| \prod_{j \in e} (\varphi_j(t_j))^{-l_j} \delta^{m^e}(\varphi(t)) f \|_{p, \varphi, \beta}$$

$$\times \prod_{j \in e'} (\varphi_j(T_j))^{1-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{j \in e} (\varphi_j(t_j))^{1-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)}$$

$$\times \prod_{j \in e'} (\psi_j([\xi_j]_1))^{\left(\frac{1}{q}-\frac{1}{r}\right)} \prod_{j \in e} (\psi_j([\xi_j]_1))^{\frac{\beta_j p}{q}}. \tag{1.16}$$

Substituting inequalities in (1.9) for $(r = q)$, we obtain (1.7). Inequality (1.8) is proved in the same way.

Corollary 1.3. *From inequality* (1.7) *for* $\beta_{1j} = \dfrac{\beta_j p}{q}$, $j \in e_n$ *it follows that*

$$\| A_\eta^e \|_{q, \psi, \beta_1; U} \leq \| \prod_{j \in e} (\varphi_j(t_j))^{-l_j} \delta^{m^e}(\varphi(t)) f \|_{p, \varphi, \beta}, \tag{1.17}$$

where C is a constant independent of f.

2. Main Results

We prove two theorems on the properties of functions in the space $S^l_{p,\theta,\varphi,\beta}F(G_\varphi)$.

Theorem 2.1. *Let the domain* $G \subset R^n$ *satisfy the condition of exible* φ-*horn* [14], $1 < p < q \leq \infty$ *and let* $\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)$, $\upsilon_j \geq 0$ *be entire* $j \in e_n$, $1 < \theta < \infty$; $\mu_j > 0 (j \in e_n)$, *and let* $f \in S^l_{p,\theta,\varphi,\beta}F(G_\varphi)$.

Then the following embedding holds

$$D^{v} : S^{l}_{p,\theta_1,\varphi,\beta}F(G_{\varphi}) \hookrightarrow L_{q,\psi,\beta^1}(G)$$

i.e., for $f \in S^{l}_{p,\theta,\varphi,\beta}F(G_{\varphi})$ *generalized derivatives* $D^{v}f$ *exist and the following inequalities are true*

$$\| D^{v} f \|_{q,G}$$

$$\leq C_1 \sum_{e\subseteq} \prod_{j\in e_n} (\varphi_j(T_j))^{s_{e,j}} \left\| \left\{ \int_{0^e}^{t_0^e} \left[\frac{\delta^{m^e}(\varphi(t)) D^{k^e} f(\cdot)}{\prod_{j\in e} (\varphi_j(t_j))^{l_j - k_j}} \right]^{\theta} \frac{d\varphi_j(t_j)}{\varphi_j(t_j)} \right\}^{\frac{1}{\theta}} \right\|_{p,\varphi,\beta}, \quad (2.1)$$

$$\| D^{v} f \|_{q,\psi,\beta^1;G} \leq C_2 \| f \|_{S^{l}_{p,\theta,\varphi,\beta}F(G)}, \ p \leq q < \infty. \quad (2.2)$$

In particular, if

$$\mu_{j,0} = l_j - v_j - (1 - \beta_j p)\frac{1}{p} > 0,$$

then $D^{v} f(x)$ *is continuous in the domain G, and*

$$\sup_{x\in G} | D^{v} f(x) |$$

$$\leq C_1 \sum_{e\subseteq} \prod_{j\in e_n} (\varphi_j(T_j))^{s^0_{e,j}} \left\| \left\{ \int_{0^e}^{t_0^e} \left[\frac{\delta^{m^e}(\varphi(t)) D^{k^e} f(\cdot)}{\prod_{j\in e} (\varphi_j(t_j))^{l_j - k_j}} \right]^{\theta} \frac{d\varphi_j(t_j)}{\varphi_j(t_j)} \right\}^{\frac{1}{\theta}} \right\|_{p,\varphi,\beta}, \quad (2.3)$$

where

$$s_{e,j} = \begin{cases} \mu_j \ \text{for } j \in e, \\ -v_j - (1 - \beta_j p)\left(\frac{1}{p} - \frac{1}{q}\right), \ j \in e' \end{cases}$$

$0 < T_j \leq \min\{1, t_{oj}\} (j \in e_n)$, C_1, C_2 *are the constants independents of* f, C_1 *independent of T.*

Proof. Under the conditions of our theorem, generalized derivatives $D^v f$ exist. Indeed, if $\mu_j > 0, \{j \in e_n\}$, then for $f \in S^l_{p,\theta,\varphi,\beta} F(G_\varphi) \to S^l_{p,\theta,\varphi,\beta} F(G_\varphi)$ there exist generalized derivatives $D^v f \in L_p(G)$ and for almost each point $x \in G$ the integral representation [18]

$$D^v f(x) = \sum_{e \subseteq} \prod_{j \in e'} (\varphi_j(T_j))^{-1-v_j}$$

$$\times \int_{0^e}^{T^e} \int_{R^n} M_e^{(v)}\left(\frac{y}{\varphi(t^e + T^{e'})}, \frac{\rho(\varphi(t^e + T^{e'}, x))}{3\varphi(t^e + T^{e'})} \right) f_e(x + t, t)$$

$$\times \prod_{j \in e_n} (\varphi_j(t_j))^{-2-v_j} \prod_{j \in e} \varphi'_j(t_j) dt^e dy \qquad (2.4)$$

with the kernels is valid and $0 \le T_j \le \min\{1, t_{j,0}\}$, $j \in e_n$, M_e, $(\cdot, y) \in C_0^\infty(R^n)$.

From the Minkowski inequality we have

$$\| D^v f \|_{q,G} \le \sum_{e \subseteq e_n} \| A_T^e \|_{q,G}. \qquad (2.5)$$

By means of inequality (1.7) for $U = G$, $\Psi = M_e$, $\eta_j = T_j (j \in e)$, $\xi \to \infty$ and $p \le \theta$, we get inequality (2.1). By means of inequality (1.8) for $\eta_j = T_j$, $(j \in e_n)$, $p \le \theta$ and from (1.17) we have (2.2).

Now let $\mu_{j,0} = \mu_j(q = \infty) > (j \in e_n)$, then from the identity (2.4), $\beta \le 0$, from (2.5) we get

$$\| D^v f - f^{(v)}_{\varphi(T)} \|_{\infty, G}$$

$$\le \sum_{\emptyset \ne e \subseteq e_n} \prod_{j \in e_n} (\varphi_j(T_j))^{\mu_{j,0}} \left\| \left\{ \int_{0^e}^{t_0^e} \left[\frac{\delta^{m^e}(\varphi(t)) D^{k^e} f(\cdot)}{\prod_{j \in e} (\varphi_j(t_j))^{l_j - k_j}} \right]^\theta \frac{d\varphi_j(t_j)}{\varphi_j(t_j)} \right\}^{\frac{1}{\theta}} \right\|_{p,\varphi,\beta}.$$

As $T_j \to 0$, $j \in e_n$, then $\| D^v f - f^{(v)}_{\varphi(T)} \|_{\infty, G}$. Since $f_{\varphi(T)}(x)$ is continuous on G the convergence on $L_\infty(G)$ coincides with the uniform convergence. Then the limit function $D^v f$ is continuous on G. Theorem 2.1 is proved.

Let γ be an n-dimensional vector.

Theorem 2.2. *Let the conditions of Theorem* 2.1 *be fulfilled. Then for* $\mu_j > 0 \, (f \in e_n)$ *the generalized derivatives* $D^v f$ *satisfies on G the generalized Holder condition*, *i.e. the following inequality is valid*:

$$\| \Delta(\gamma, G) D^v f \|_{q, G} \leq C \| f \|_{S^l_{p, \varphi, \beta} F(G_\varphi)} \prod_{j \in e_n} (\varphi_j(| \gamma_j |))^{\sigma_j}, \tag{2.6}$$

where $\sigma_j (j \in e_n)$ *is any number satisfying the inequality*:

$$\begin{cases} 0 \leq \sigma_j \leq 1, \, if \; \mu_j > 1 \, for \; j \in e, \\ 0 \leq \sigma_j \leq 1, \, if \; \mu_j = 1 \, for \; j \in e, \, and \; 0 \leq \sigma_j \leq 1 \; for \; j \in e' \\ 0 \leq \sigma_j \leq \mu_j, \, if \; \mu_j > 1 \, for \; j \in e, \end{cases} \tag{2.7}$$

If $\mu_{j, 0} > 0 \; (j \in e_n)$, *then*

$$\sup_{x \in G} | \Delta(\gamma, G) D^v f(x) | \leq C \| f \|_{S^l_{p, \varphi, \beta} F(G)} \prod_{j \in e_n} (\varphi_j(| \gamma_j |))^{\sigma^0_j}. \tag{2.8}$$

where σ^0_j *satisfy the same conditions as* σ_j, *but replaced* μ_j *in* $\mu_{j, 0}$.

Proof. According to Lemma 8.6 from [1] there exists a domain

$$G_u \subset G(u = \zeta_j r(x), \zeta_j > 0, \, j \in e_n \; r(x) = \rho(x, \partial G), \, x \in G).$$

Assume that $| \gamma_j | < u_j$, $j \in e_n$, then for any $x \in G_u$ the segment connecting the points x, $x + \gamma$ is contained in G. Consequently, for all the points of this segment, identity (2.4) with the same kernels are valid. After same transformations, from (2.4) we get

$$| \Delta(\gamma, G) D^{v} f(x) | \le C_1 \sum_{e \subseteq e_n} \prod_{j \in e'} (\varphi_j(T_j))^{-1-v_j}$$

$$\times \int_0^{|\gamma_1^e|} \dots \int_0^{|\gamma_n^e|} \prod_{j \in e} (\varphi_j(t_j))^{-2-v_j} \prod_{j \in e} \varphi_j'(t_j)$$

$$\times \int_{R^n} \left| M_e^{(v)} \left(\frac{y}{\varphi(t^e + T^{e'})}, \frac{\rho(\varphi(t^e + T^{e'}, x))}{3\varphi(t^e + T^{e'})} \right) \right|$$

$$\times | \Delta(\gamma, G) f_e(x + y) | dy dt^e + C^2 \sum_{e \subseteq e_n} \prod_{j \in e'} (\varphi_j(T_j))^{-2-v_j}$$

$$\times \prod_{j \in e_n} | \gamma_j | \int_{|\gamma_1^e|}^{T_1^e} \dots \int_{|\gamma_n^e|}^{T_n^e} \prod_{j \in e} (\varphi_j(t_j))^{-3-v_j} \prod_{j \in e} \varphi_j'(t_j)$$

$$\times \int_{R^n} \left| M_e^{(v+1)} \left(\frac{y}{\varphi(t^e + T^{e'})}, \frac{\rho(\varphi(t^e + T^{e'}, x))}{3\varphi(t^e + T^{e'})} \right) \right|$$

$$\times \int_0^1 \dots \int_0^1 | f_e(x + y + \vartheta, \omega_1, \gamma_1 + \dots + \omega_n \gamma_n) | dt^e d\omega dy dz$$

$$= C^1 \sum_{e \subseteq e_n} E^e_{|\gamma|}(x, \gamma) + C^2 \sum_{e \subseteq e_n} E^e_{|\gamma|, T}(x, \gamma), \tag{2.9}$$

where $0 < T_j \le \{1, t_{0,j}\}$, $j \in e_n$. We also assume that $| \gamma_j | < T_j (j \in e_n)$. Consequently, $| \gamma_j | < \min(u_j, T_j) (j \in e_n)$. If $x \in G \setminus G_u$, then by definition

$$\Delta(\gamma, G) D^{v} f(v) = 0.$$

From (2.9) we have

$$\| \Delta(\gamma, G) D^{v} f \|_{q, G} \le C^1 \sum_{e \subseteq e_n} \| E^e_{\gamma}(\cdot, \gamma) \|_{q, G_u}$$

$$+ C^2 \sum_{e \subseteq e_n} \| E^e_{\gamma, T}(\cdot, \gamma) \|_{q, G_u}. \tag{2.10}$$

By means of inequality (1.7), for $U = G$, $\eta_j = | \gamma_j | (j \in e)$ we have

$$\| E^e_{|\gamma|}(\cdot, \gamma) \|_{q, G_\omega} \leq C_1 \| \prod_{j \in e} (\varphi_j(t_j))^{-l_j} \delta^{m^e}(\varphi(t)) f \|_{p, \varphi, \beta} \prod_{j \in e} (\varphi_j(|\gamma_j|))^{\mu_j}, \quad (2.11)$$

and by means of inequality (1.8) for $U = G$, $\eta_j = |\gamma_j| (j \in e_n)$ we get

$$\| E_{|\gamma|, T}(\cdot, \gamma) \|_{q, G_\omega} \leq C_2 \| \prod_{j \in e} (\varphi_j(t_j))^{-l_j} \delta^{m^e}(\varphi(t)) f \|_{p, \varphi, \beta}$$

$$\prod_{j \in e'} (\varphi_j(|\gamma_j|)) \prod_{j \in e} (\varphi_j(|\gamma_j|))^{\mu_j - 1}. \quad (2.12)$$

From inequalities (2.10)-(2.12) we get the required inequality.

Now suppose that $|\gamma_j| \geq \min(u_j T_j)$, $(j \in e_n)$, then

$$\| \Delta(\gamma, G) D^v f \|_{q, G} \leq 2 \| D^v f \|_{q, G} \leq C(uT) \| D^v f \|_{q, G} \prod_{j \in e_n} (\varphi_j(|\gamma_j|))^{\sigma_j}.$$

Estimating for $\| D^v f \|_{q, G}$ by means of inequality (2.1)), we get estimation (2.9).

Theorem 2.2 is proved.

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