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ON GENERALIZED BIDERIVATIONS OF TRIANGULAR ALGEBRAS

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Abstract

Let $\mathfrak{A} = \begin{pmatrix} A & M \\ & B \end{pmatrix}$ be a triangular algebra, where A and B are unital algebras over a commutative ring with unity and M is a unital (A, B) bi module which is faithful as a left A module and faithful as a right B module. The aim of this paper is to investigate the form of generalized biderivation on triangular algebras. In fact, we have shown that, under certain conditions involving derivation, generalized biderivation becomes an inner biderivation.

1. Introduction

Throughout this paper, A will denote the triangular algebra of the form $\mathfrak{A} = \begin{pmatrix} A & M \\ & B \end{pmatrix}$ where A and B are unital algebras over a commutative ring with identity and M is a unital (A, B) bi module which is faithful as a left A module $(am = 0$ implies $a = 0)$ and faithful as a right B module $(mb = 0$ implies $b = 0)$.

Benkovikc [1] obtained some interesting results on derivation of triangular algebras and introduce the notion of biderivation on triangular algebras as follows: Let A be an algebra over a commutative ring R and $Z(A)$

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be its centre. A linear map $d : A \to A$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$. The commutator of the elements x, y of A is denoted by $[x, y] = xy - yx$. If any derivation d is of the form $d(x) = [x, a]$ for any $a \in A$, then this derivation is called the inner derivation. Biderivation is a bilinear map $\phi : A * A \to A$ which is a derivation with respect to both the components that means $\phi(x, yz) = \phi(x, y)z + y\phi(x, z)$ and $\phi(xy, z) = \phi(x, z)y + x\phi(y, z)$ for all $x, y, z \in A$. A bilinear map $\phi : \mathfrak{A} * \mathfrak{A} \to \mathfrak{A}$ said to be generalized biderivation if there exist a biderivation $D : \mathfrak{A} * \mathfrak{A} \to \mathfrak{A}$ such that

$$\phi(xy, z) = \phi(x, y)z + xD(y, z) \tag{1.1}$$

and

$$\phi(x, yz) = \phi(x, y)z + yD(x, z) \tag{1.2}$$

for all $x, y, z \in \mathfrak{A}$. If the triangular algebra is non commutative then the map $\phi(x, y) = \lambda[x, y]$ for all $x, y \in \mathfrak{A}$ and $\lambda \in Z(A)$ is an example of inner biderivation.

Several useful results on biderivation have been proved by many authors. Bresar [3-6] deduced many interesting results on rings which are very useful in the study of commuting maps. Cheung [7, 8] investigated the commuting maps and Lie derivations on triangular algebras. Benkovikc [2] proved that under certain conditions biderivation on triangular algebras acts as an inner biderivation. He also defines extremal biderivation which occurs naturally on triangular algebras. Benkovikc proved that under certain conditions a biderivation of triangular algebras is a sum of the inner biderivation and an extremal biderivation. Benkovikc and Eremita [1] studied the commuting traces and commutativity preserving maps on triangular algebras and obtained some interesting results.

In this paper, we have investigated the form of generalized biderivation on triangular algebra and extended the results obtained by Benkovikc [2] on biderivation of triangular algebras to generalized biderivation on triangular algebras.

Definition 1.1. Let C denote the commutative ring with unity and two unital algebras over C are denoted by A and B. Consider M as a unital (A, B) bimodule which is faithful as a left A module and faithful as a right B module. An algebra of the form

$$\mathfrak{A} = Tri(A, M, B) = \begin{pmatrix} a & m \\ & b \end{pmatrix}$$

for all $a \in A$, $m \in M$ and $b \in B$, under usual matrix operation is said to be a triangular algebra.

Here we use two natural projections which are denoted by π_A and π_B defined as $\pi_A : \mathfrak{A} \to A$ such that

$$\pi_A\left(\begin{pmatrix} a & m \\ & b \end{pmatrix}\right) = a$$

and $\pi_B : \mathfrak{A} \to B$ such that

$$\pi_B\left(\begin{pmatrix} a & m \\ & b \end{pmatrix}\right) = b.$$

Definition 1.2. The centre of triangular algebra is defined as $Z = \begin{pmatrix} a & 0 \\ & b \end{pmatrix} : am = mb$ for all $m \in M$.

Cheung [7] gives the definition of unique algebra isomorphism which is defined as: If $\pi_A(Z(\mathfrak{A})) \subseteq Z(A)$ and $\pi_B(Z(\mathfrak{A})) \subseteq Z(B)$ then there exist a unique algebra isomorphism $\tau : \pi_A(Z(\mathfrak{A})) \to \pi_B(Z(\mathfrak{A}))$ such that

$$am = m\tau(a)$$

for all $m \in M$. In this paper, we use the notation 1_A and 1_B for the identities of A and B and 1 as the identity of triangular algebra. Throughout this paper we will use two identities e and f which are defined as $e = \begin{pmatrix} 1_A & 0 \\ & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ & 1_B \end{pmatrix} = 1 - e$. Here e and f are orthogonal idempotents of $\mathfrak{A}$. So $\mathfrak{A}$ may also be represented as $\mathfrak{A} = 1\mathfrak{A}1 = (e + f)\mathfrak{A}(e + f) = e\mathfrak{A}e + e\mathfrak{A}f + f\mathfrak{A}f$, where $e\mathfrak{A}e$ is isomorphic to A, $f\mathfrak{A}f$ is isomorphic to B and $e\mathfrak{A}f$ is isomorphic to M. From this, we can conclude that

$a = eae \in A = e\mathfrak{A}e$, $b = fbf \in B = f\mathfrak{A}f$ and $m = emf \in M = e\mathfrak{A}f$. Hence every element of $\mathfrak{A}$ can be written as

$$x = eae + emf + fbf = a + m + b,$$

where $a \in A$, $b \in B$, $m \in M$.

2. Generalized Biderivation on Triangular Algebra

Theorem 2.1. *Let* $\mathfrak{A} = tri(A, M, B)$ *be a triangular algebra. If the following conditions hold*:

(i) $\pi_A(Z(\mathfrak{A})) = Z(A)$ *and* $\pi_B(Z(\mathfrak{A})) = Z(B)$,

(ii) *at least one of the algebra A and B is non commutative*,

(iii) *if* $\alpha a = 0$, $\alpha \in Z(\mathfrak{A})$, $0 \neq a \in \mathfrak{A}$ *then* $\alpha = 0$,

(iv) *each derivation of* A *is inner*, *then every generalized biderivation* $\phi : \mathfrak{A} * \mathfrak{A} \to \mathfrak{A}$ *that satisfies* $\phi(e, e) = 0 = \phi(f, f)$ *is an inner biderivation.*

To prove the main result we use the following lemmas:

Lemma 2.1. ([2], Corollary 3.4) *If every derivation of the triangular algebra* $Tri(A, M, B)$ *is inner, then every bimodule homomorphism* $f : M \to M$ *is of the standard form i.e., there exist* $a_0 \in A$, $b_0 \in B$ *such that*

$$f(m) = a_0 m + m b_0.$$

Lemma 2.2. ([2], Lemma 4.2) *Let* $D : \mathfrak{A} * \mathfrak{A} \to \mathfrak{A}$ *be a biderivation. Then*

(i) $D(x, 1) = 0 = D(1, x)$ *for all* $x \in \mathfrak{A}$,

(ii) $D(x, 0) = 0 = D(0, x)$ *for all* $x \in \mathfrak{A}$,

(iii) $D(e, e) = -D(e, f) = -D(f, e) = D(f, f)$ *and*

(iv) $D(x, y)[u, v] = [x, y]D(u, v)$, *for all* $x, y, u, v \in \mathfrak{A}$.

In the proof of Theorem 4.11 [2], *author find some results that will be used in the proof of our main theorem which we state as in the following remark*:

Remark 2.1. ([2], Theorem 4.11): Let $\mathfrak{A} = Tri(A, M, B)$ be a triangular algebra. $D : \mathfrak{A} * \mathfrak{A} \to \mathfrak{A}$ be a biderivation of $\mathfrak{A}$ that satisfies $D(e, e) = 0$, then $D(a, b) = 0 = D(b, a)$, for all $a \in A, b \in B$ and $D(m, n) = 0$ for all $m, n \in M$.

Lemma 2.3. *Let* $\phi : \mathfrak{A} * \mathfrak{A} \to \mathfrak{A}$ *be a generalized biderivation. Then*

$$\phi(x, y)[u, v] = [x, y]D(u, v) \text{ for all } x, y, u, v \in \mathfrak{A}.$$

Proof. Since ϕ is a derivation in the first argument, then for all $x, y, u, v \in \mathfrak{A}$,

$$\phi(xu, yv) = \phi(x, yv)u + xD(u, yv),$$

As ϕ is a derivation on the second argument, we have

$$\begin{aligned}\phi(xu, yv) &= \phi(x, yv)u + xyD(u, v) + xD(u, y)v \\ &= \phi(x, y)vu + yD(x, v)u + xyD(u, v) + xD(u, y)v. \qquad (2.1)\end{aligned}$$

Also,

$$\phi(xu, yv) = \phi(x, y)vu + xD(u, y)v + yD(x, v)u + yxD(u, v). \qquad (2.2)$$

Now subtracting equation (2.1) from (2.2), we get

$$\phi(x, y)[u, v] = [x, y]D(u, v), \qquad (2.3)$$

for all $x, y, u, v \in \mathfrak{A}$. □

Lemma 2.4. *Let* $\phi : \mathfrak{A} * \mathfrak{A} \to \mathfrak{A}$ *be a generalized biderivation then* $\forall x \in \mathfrak{A}$

(i) $\phi(x, 1) = \phi(1, 1)x = \phi(1, x)$,

(ii) $\phi(x, 0) = 0 = \phi(0, x)$.

Proof. (i) Since ϕ is a generalized biderivation then there exist a biderivation D such that

$$\phi(x, 1) = \phi(1.x, 1)x = \phi(1, 1)x + 1D(x, 1)$$

Using Lemma (2.2), we have $D(x, 1) = 0$.

This implies that

$$\phi(x, 1) = \phi(1, 1)x$$

Similarly, we can prove that

$$\phi(1, x) = \phi(1, 1)x.$$

(ii) Again,

$$\phi(x, 0) = \phi(1.0, 0)x = \phi(x, 0)0 + 0D(x, 0) = 0$$

Similarly,

$$\phi(0, x) = 0. \qquad \square$$

Lemma 2.5. *Let* $\mathfrak{A} = Tri(A, M, B)$ *be a triangular algebra and* $\phi : \mathfrak{A} * \mathfrak{A} \to \mathfrak{A}$ *be a generalized biderivation. If* $x, y \in \mathfrak{A}$ *such that* $[x, y] = 0$ *then* $\phi(x, y) = e\phi(x, y)f + f\phi(x, y)f$.

Proof. Since we have $\phi(x, y) \in \mathfrak{A}$, then

$$\phi(x, y) = e\phi(x, y)e + e\phi(x, y)f + f\phi(x, y)f$$

Using Lemma (2.3), we have

$$\phi(x, y)[e, emf] = [x, y]D(e, emf) = 0$$

Since $m = em = emf$

$$0 = \phi(x, y)[e, emf] = \phi(x, y)em.$$

Hence

$$e\phi(x, y)eM = 0$$

But M is a faithful left A module, we see that

$$e\phi(x, y)e = 0,$$

therefore,

$$\phi(x, y) = e\phi(x, y)f + f\phi(x, y)f. \qquad \square$$

Proof of Theorem 2.1. Let $x, y \in \mathfrak{A}$, we can write

$x = a + m + b$ and $y = a' + m' + b'$, where a, $a, a' \in A$, $m, m' \in M$, $b, b' \in B$.

Since ϕ is a bilinear map, we can write

$$\begin{aligned}\phi(x, y) &= \phi(a + m + b, a' + m' + b')\\ &= \phi(a, a') + \phi(a, m') + \phi(a, b') + \phi(m, a') + \phi(m, m')\\ &+ \phi(m, b') + \phi(b, a') + \phi(b, m') + \phi(b, b') \qquad (2.4)\end{aligned}$$

for all $x, y \in \mathfrak{A}$. Now we will describe all the parts of the generalized biderivation. We will first find out the value of $\phi(a, b)$ and $\phi(b, a)$ for all $a \in A$, $b \in B$.

By Lemma (2.5), we can write

$$\begin{aligned}\phi(a, b) &= e\phi(a, b)f + f\phi(a, b)f\\ &= e\phi(ae, fb)f + f\phi(ae, b)f\\ &= e\phi a(m, b)ef + eaD(e, b)f + f\phi(a, b)ef + faD(e, b)f. \qquad (2.5)\end{aligned}$$

Using Remark (2.1), we have

$$D(a, b) = 0, \text{ for all } a \in A, b \in B.$$

This implies that

$$\phi(a, b) = 0.$$

Also,

$$\begin{aligned}\phi(b, a) &= e\phi(b, a)f + f\phi(b, a)f\\ &= e\phi(b, ae)f + f\phi(b, ae)f\\ &= e\phi(b, a)ef + eaD(b, e)f + f\phi(b, a)ef + faD(b, e)f. \qquad (2.6)\end{aligned}$$

Again, using Remark (2.1), we have

$$\phi(b, a) = 0, \text{ for all } a \in A, b \in B.$$

Our next step is to find out the value of $\phi(a, m)$ and $\phi(m, b)$, for all $a \in A$, $b \in B$, $m \in M$. Let us define a map $h : M \to M$ as $h(m) = \phi(e, m)$

for all $m \in M$. Then h is a bimodule homomorphism. Now, for all $a \in A, b \in B$ and $m \in M$, we have

$$h(am) = \phi(e, am)$$

$$= \phi(e, a)m + aD(e, m) = 0 + aD(e, m) \quad (2.7)$$

As,

$$\phi(e, m) = \phi(e, em)$$

$$= \phi(e, e)m + eD(e, m)$$

$$= eD(e, m) \quad (2.8)$$

Thus, we have,

$$h(am) = a\phi(e, m) = ah(m) \quad (2.9)$$

and

$$h(mb) = \phi(e, mb)$$

$$= \phi(e, m)b + mD(e, b)$$

$$= \phi(e, m)b$$

$$= h(m)b \quad (2.10)$$

So by Lemma (2.1,) h is of the standard form and we can write,

$$h(m) = a_0 m + mb_0,\ a_0 \in Z(A),\ b_0 \in Z(B)$$

Now we have,

$$\phi(e, m) = h(m) = (a_0 + \tau^{-1}(b_0))m = \alpha m$$

where $\alpha = a_0 + \tau^{-1}(b_0) \in \pi_A(Z(A))$.

Similarly, the map $g : M \to M$ defined by $g(m) = \phi(m, e)$ for all $m \in M$, is a bimodule homomorphism. So there exist $\beta \in \pi_A(Z(A))$ such that

$$\phi(m, e) = \beta m \text{ for all } m \in M.$$

We have to prove that

$$\phi(e, m) = \alpha m = -\phi(m, e)$$

For this, it is sufficient to prove that,

$$\alpha + \beta = 0.$$

According to the condition (ii), we may assume that, A is non commutative algebra.

Choose $a, a' \in A$ such that

$$[a, a'] \neq 0.$$

Since $\phi(e, m) = \alpha m$ and $\phi(m, e) = \beta m$, using Lemma (2.3), we have

$$\begin{aligned} \phi(a, a')[e, m] &= [a, a'] D(e, m) \\ &= [a, a'] \phi(e, m) \\ &= \alpha [a, a'] m \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \phi(a, a')[m, e] &= [a, a'] D(e, m) \\ &= [a, a'] \phi(e, m) \\ &= \beta [a, a'] m \end{aligned} \tag{2.12}$$

for all $m \in M$.

Adding equations (2.11) and (2.12), we get

$$(\alpha + \beta)[a, a']M = 0$$

The faithfulness of the left A module M implies that

$$(\alpha + \beta)[a, a'] = 0$$

Since $[a, a'] \neq 0$, we conclude using the condition (iii) that

$$\alpha + \beta = 0$$

So we have,

$$\phi(e, m) = \alpha m = -\phi(m, e)$$

Now, if B is non commutative, then

$$\begin{aligned} e[e, m]D(b, b') &= eD(e, m)[b, b'] \\ &= \phi(e, m)[b, b'] \\ &= \alpha m[b, b'] \end{aligned} \tag{2.13}$$

Also,

$$\begin{aligned} e[e, m]D(b, b') &= eD(e, m)[b, b'] \\ &= \phi(e, m)[b, b'] \\ &= \beta m[b, b'] \end{aligned} \tag{2.14}$$

Now, adding equation (2.13) and (2.14), we get

$$M(\alpha + \beta)[b, b'] = 0$$

By the faithfulness of right B module of M, we have

$$(\alpha + \beta)[b, b'] = 0$$

Again using condition (iii), we get

$$(\alpha + \beta) = 0$$

Similarly, we can also show that

$$\phi(f, m) = \alpha m = -\phi(m, f)$$

Let $a \in A$ and $m \in M$ are arbitrary elements, then we may write

$$\begin{aligned} \phi(a, m) &= \phi(ae, m) \\ &= \phi(a, m)e + aD(e, m) \end{aligned} \tag{2.15}$$

Since the inequality $D(a, f) = 0$ implies

$$\begin{aligned} \phi(a, m)e &= \phi(a, mf)e \\ &= \phi(a, m)fe + mD(a, f)e \end{aligned} \tag{2.16}$$

By equation (2.15)

$$\begin{aligned}\phi(a, m) &= D(e, m)\\ &= a\phi(e, m)\\ &= \alpha am \qquad (2.17)\end{aligned}$$

Similarly, we can prove that

$$\phi(m, b) = -\phi(b, m) = \alpha mb$$

Now, let us find out the value of $\phi(a, a')$ and $\phi(b, b')$.

Since using,

$$\begin{aligned}D(a, e) &= D(a, 1-f)\\ &= D(a, 1) - D(a, f)\\ &= 0 \qquad (2.18)\end{aligned}$$

We have,

$$\begin{aligned}\phi(a, e) &= \phi(ea, e)\\ &= \phi(e, e)a + eD(a, e)\\ &= 0 = \phi(e, a). \qquad (2.19)\end{aligned}$$

Now,

$$\begin{aligned}\phi(a, a') &= \phi(e(ae), a')\\ &= \phi(e, a')ae + eD(ae, a')\\ &= \phi(e, a')ae + eaD(e, a') + eD(a, a')e\\ &= eD(a, a')e \in e\mathfrak{A}e = A\end{aligned}$$

This implies that,

$$\phi(a, a') \in A.$$

By Lemma (2.3) and using $\phi(e, m) = eD(e, m)$, we have

$$\phi(a, a')[e, m] = [a, a']\phi(e, m)$$

$$= \alpha[a, a']m \tag{2.20}$$

This implies that

$$(\phi(a, a') - \alpha[a, a'])M = 0$$

and from the faithfulness of the left A module M, we obtain

$$\phi(a, a') = \alpha[a, a']$$

for all $a, a' \in A$.

Similarly, $\phi(b, b') \in f\mathfrak{A}f = B$ and

$$e[e, m]\phi(b, b') = eD(e, m)[b, b']$$

$$= \phi(e, m)[b, b']$$

$$= \alpha m[b, b'] \tag{2.21}$$

for all $m \in M$.

Hence $M(\phi(b, b') - \tau(\alpha)[b, b']) = 0$. The faithfulness of right B module M implies

$$\phi(b, b') = \tau(\alpha)[b, b']$$

Now, the only remaining part is to find the value of $\phi(m, n)$.

$$\phi(m, n) = \phi(em, n)$$

$$= \phi(e, n)m + eD(m, n) \tag{2.22}$$

Using Remark (2.1),

$$\phi(m, n) = \alpha nm + 0$$

$$= 0 \tag{2.23}$$

This implies that $\phi(m, n) = 0$.

At the end, let $\lambda = \alpha + \tau(\alpha) \in Z(A)$

Substituting all the values in equation (2.4)

$$\phi(x, y) = \alpha[a, a'] + \alpha a m' - \alpha a' m$$
$$+ \alpha m b' - \alpha m' b + \tau(\alpha)[b, b']$$
$$= \begin{pmatrix} \alpha & 0 \\ & \tau(\alpha) \end{pmatrix} \left[\begin{pmatrix} a & m \\ & b \end{pmatrix}, \begin{pmatrix} a' & m' \\ & b' \end{pmatrix} \right]$$
$$= \lambda[x, y] \tag{2.24}$$

for all $x, y \in \mathfrak{A}$.

Hence every generalized biderivation ϕ such that $\phi(e, e) = 0 = \phi(f, f)$, is an inner biderivation.

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