Advances and Applications in Mathematical Sciences
Volume 20, Issue 7, May 2021, Pages 1239-1248

MATRIX GEOMETRIC METHOD FOR THE ANALYSIS OF *M/M*/1 MODEL UNDER REPAIR

PRADEEP K. JOSHI, SHEJAL GUPTA and K. N. RAJESHWARI

Department of Mathematics
IPS Academy, Indore, India
E-mail: pradeepkjoshi1@gmail.com

School of Mathematics
DAVV, Indore, India
E-mail: shejalgupta16@gmail.com

School of Mathematics
DAVV, Indore, India
E-mail: knr_k@yahoo.co.in

Abstract

In this paper, we consider an $M/M/1$ queue and analysed it under two conditions. In first condition the system is under repair due to breakdown and in second condition the system is in a regular busy period. Quasi birth and death process as well as matrix geometric method is used to find the distribution of number of customers in the system. Further, in the steady state, the average expected customers and average sojourn time is derived.

1. Introduction

Past few years back, queueing systems under repair due to breakdown is extensively studied due to its various applications in computer system, manufacturing units, business management etc. In this paper, the server initially serving the customers in a regular busy period but suddenly due to technical breakdown, the system goes for repair. During this period, the server serves the customer manually at a service rate less than the regular busy period. Chakravarthy et al. [8] studied the queueing system with server

2010 Mathematics Subject Classification: 60K20, 60K25, 90B22.
Keywords: *M/M*/1 Queue, Quasi Birth and Death Process, Matrix Geometric Method, Repair, Breakdown.
Received September 7, 2020; Accepted February 18, 2021

breakdowns, repairs, vacations and backup server. In their study, they have derived decomposition results for rate matrix and obtained the results in case of steady state. Rao et al. [6] analysed two-phase queueing system with impatient customers, server breakdowns and delayed repair and derived the probability generating function for the system size in various states. Madhu Jain [4] analysed machine repair problem using the concept of single working vacation under F-policy. Ayyapan [3] studied the $M^x/G_1, G_2/1$ queueing system with Bernoulli vacation, Breakdown and delayed repair.

Chun Xiu et al. [1] studied the $M/M/1$ queue with single working vacation in which the server is serving the customers at a slower rate during the start-up period using the concept of QBD process and matrix geometric method. In this paper, they have derived the distribution of queue size, mean of queue size as well as mean sojourn time in the steady state. Zhang [10] investigated the $M/M/1$ queue with multiple working vacations and N-policy. In their study, they derived the distribution of steady queue size using the quasi birth death process and matrix geometric method.

Shoukry et al. [7] used the matrix geometric method for the comparative study of the $M/M/1$ queueing models with and without breakdown ATM machines. The study shows that the various system performance measures like average queue length, average waiting time in case of system with breakdowns is greater than the systems without breakdown. Baba [9] studied the $M^x/M/1$ queue with multiple working vacation in which server serves at a lower rate rather than completely stopping the service. In this study, matrix analytic method is used to derive the PGF of stationary queue size distribution.

The rest of the paper is arranged in a following sequence: Section 2 consists of description of the model used as well as quasi birth death process. In this section, we obtain the expression for the rate matrix R. In section 3, we derive the expression for distribution of queue length in steady state. Section 4 deals with the average queue size and sojourn time in the steady state. We conclude the paper in section 5.

2. Description of the Model

Here we consider an $M/M/1$ queue with rate of arrival λ and rate of service μ_b. Initially the system is in a regular busy period and serving the customers with service rate μ_b. After some time, suddenly system breakdowns due to some technical problem. Hence, the system goes under repair during the breakdown. In this period, the server is providing the service manually at a rate of $\mu_r (\mu_r < \mu_b)$ exponentially with the parameter α. When the system get repaired then again server provides the service at a rate of μ_b.

Following assumptions are made for this model: -

(i) Service discipline is First Come First Served (FCFS) basis.

(ii) Repair time, inter-arrival time and service time are independent of each other.

Let, at time t, the number of customers in the system be denoted by $Q(t)$ and let state variables be

$$J(t) = \begin{cases} 0 & \text{the system is under repair due to technical breakdown at time } t \\ & 1, \text{the system is in a regular bust period at time } t. \end{cases}$$

Then $\{Q(t), J(t)\}$ is a Markov process with the state space

$$\Omega = \{(k, j) : k \geq 1, j = 0, 1\} \cup (0, 0).$$

Here state $(k, 0)$, $k \geq 1$ shows that the system is under repair due to breakdown and k customers are there in the waiting line; state $(k, 1)$, $k \geq 1$ shows that the system is in a regular busy period with k customers in the queue. Further state (0, 1) represents that the system is closed as $k = 0$.

The state transition matrix, according to lexicographical sequence can be written as

$$\tilde{Q} = \begin{bmatrix} A_{00} & A_{01} & & \\ B_{10} & A_1 & C & \\ & B & A_1 & C \\ & \ddots & \ddots & \ddots \\ & B & A_1 & C \\ & B & A & C \end{bmatrix}$$

Where

$$A_{00} = \begin{bmatrix} -(\lambda + \alpha) & \alpha \\ 0 & -\lambda \end{bmatrix}$$

$$A_{01} = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} B_{10} = (\mu_r, \mu_b)^T$$

$$B = \begin{bmatrix} \mu_r & 0 \\ 0 & \mu_b \end{bmatrix} \quad C = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -(\lambda + \mu_r) & 0 \\ 0 & -(\lambda + \mu_b) \end{bmatrix}$$

$$A = \begin{bmatrix} -(\lambda + \mu_r + \alpha) & \alpha \\ 0 & -(\lambda + \mu_b) \end{bmatrix}.$$

According to Neuts, this construction of $\tilde{Q}$ specifies that $\{Q(t), J(t)\}$ is a quasi-birth and death process.

Theorem 1. *The state transition rate matrix R satisfying the quadratic equation*

$$R^2B + RA + C = 0 \tag{1.1}$$

has a non-negative minimal solution given by

$$R = \begin{bmatrix} \delta & \frac{\alpha\delta}{\mu_b(1-\delta)} \\ 0 & \rho \end{bmatrix} \tag{1.2}$$

for $\rho = \frac{1}{\mu_b} < 1$ *where* $\delta = \frac{\lambda + \alpha + \mu_r - \sqrt{(\lambda + \alpha + \mu_r)^2 - 4\lambda\mu_r}}{2\mu_r}$ *and* $\rho = \frac{\lambda}{\mu_b}$.

Proof. Let R be of the following form

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}.$$

Substituting the value of A, B, C in equation (1.1) and solving, we get

$$r_{11} = \frac{\lambda + \alpha + \mu_r - \sqrt{(\lambda + \alpha + \mu_r)^2 - 4\lambda\mu_r}}{2\mu_r} r_{12} = \frac{r_{11}\alpha}{\mu_b(1 - r_{11})}$$

$$r_{21} = 0, \; r_{22} = \frac{\lambda}{\mu_b}.$$

Putting $r_{11} = \delta$ and $r_{22} = \rho$, we get the required non-negative minimal solution (1.2) of the quadratic equation (1.1). Also, it is easy to prove that

$$0 < \delta < 1, \; 0 < \rho < 1.$$

Because δ satisfies the following equation

$$\mu_r\delta^2 - (\lambda + \alpha + \mu_r)\delta + \lambda = 0.$$

Equivalently,

$$\frac{\alpha}{1 - \delta} + \mu_r = \frac{\lambda}{\delta}. \tag{1.3}$$

Theorem 2. *The quasi birth death process* $\{Q(t), J(t)\}$ *is positive recurrent if and only if* $\rho < 1$.

Proof. By the result established by Neuts, to show any QBD process is positive recurrent, it is sufficient to show that spectral radius of state transition rate matrix R is less than 1. Also, the set of equations $xB[R] = 0$ has positive solution, where

$$B[R] = \begin{bmatrix} A_{00} & A_{01} \\ B_{10} & RB + A \end{bmatrix}$$

$$= \begin{bmatrix} -(\lambda+\alpha) & \alpha & \lambda & 0 \\ 0 & -\lambda & 0 & 0 \\ \mu_r & 0 & \dfrac{\lambda}{\delta(1-\delta)} & \dfrac{\alpha}{1-\delta} \\ 0 & \mu_b & 0 & \mu_b - \dfrac{\lambda}{\delta-1} \end{bmatrix}.$$

Thus, $xB[R] = 0$ has positive solution as $B[R]$ is a periodic generator with finite state as well as irreducible. Hence, the necessary and sufficient condition for any QBD process $\{Q(t), J(t)\}$ to be positive recurrent is that

$$SP(R) = \max(\delta, \rho) < 1.$$

As we know that $0 < r < 1$, hence the above relation implies that $\rho < 1$.

3. Distribution of Steady Queue Length

If $\rho < 1, \mu_r < \mu_b$, let the steady limit of the QBD process $\{Q(t), J(t)\}$ be denoted by (Q, J). Let

$$\pi_{kj} = P(Q = k, J = j), (k, j) \in \Omega$$

$$\pi_k = \begin{cases} \pi_{00}, & k = 0 \\ (\pi_{k0}, \pi_{k1}), & k \geq 1. \end{cases}$$

Theorem 3. *If* $\rho < \mu_r, \mu_b$, *the steady state probability distribution of* (Q, J) *is*

$$\pi_{k0} = \delta^k(\delta - 1)\pi_{00}$$

$$\pi_{k1} = \left[-\frac{\alpha\delta^2}{\mu_b} \sum_{j=0}^{k-1} \delta^j \rho^{k-1-j} + \frac{\alpha\delta(\delta-1)\rho^{k-1}}{\mu_b(\delta-1)-\lambda} \right] \pi_{00} \tag{3.1}$$

Where

$$\pi_{00} = \frac{(1-\delta)(1-\rho)}{\left(1+\dfrac{\alpha}{\lambda}\right)(1-\delta)(1-\rho) - \dfrac{\alpha\delta(1-\delta)^2(1-\rho)}{\lambda(\delta-1-\rho)} - (1-\delta)(1-\rho) - \dfrac{\alpha\delta^2}{\mu_b} - \dfrac{\alpha\delta(1-\delta)^2}{(\mu_b(\delta-1)-\lambda)}}. \tag{3.2}$$

Proof. Using matrix-geometric method, we have

$$\pi_k = (\pi_{k0}, \pi_{k1}) = (\pi_{10}, \pi_{11}) R^{k-1} = 0 \tag{3.3}$$

and π_0, π_1 satisfies the following set of equations

$$[\pi_{00},\ \pi_{01}\pi_{10}\ \ \pi_{11}]\, B[R] = 0$$

$$\Rightarrow [\pi_{00},\ \pi_{01}\pi_{10}\ \ \pi_{11}] \begin{bmatrix} -(\lambda + \alpha) & \alpha & \lambda & 0 \\ 0 & -\lambda & 0 & 0 \\ \mu_r & 0 & \frac{\lambda}{\delta(1-\delta)} & \frac{\alpha}{1-\delta} \\ 0 & \mu_b & 0 & \mu_b - \frac{\lambda}{\delta - 1} \end{bmatrix} = 0.$$

On solving this, we obtain following set of equations

$$-(\lambda + \alpha)\pi_{00} + \mu_r \pi_{10} = 0$$

$$\alpha\pi_{00} - \lambda\pi_{01} + \mu_b \pi_{11} = 0$$

$$\lambda\pi_{00} + \frac{\lambda}{\delta(1-\delta)} \pi_{10} = 0$$

$$\frac{\alpha}{1-\delta} \pi_{10} + \left(\mu_b - \frac{\lambda}{\delta - 1}\right)\pi_{11} = 0.$$

On solving these set of equations, we get

$$\pi_{01} = \frac{1}{\lambda}\left[\alpha + \frac{\alpha\delta(\delta - 1)}{(\delta - 1 - \rho)}\right]\pi_{00}$$

$$\pi_{10} = \delta(\delta - 1)\pi_{00}$$

$$\pi_{11} = \frac{\alpha\delta(\delta - 1)}{\mu_b(\delta - 1) - \lambda}\, \pi_{00}.$$

Further

$$R^k = \begin{bmatrix} \delta^k & \frac{\alpha\delta}{\mu_b(1-\delta)} \sum_{j=0}^{k-1} \delta^j \rho^{k-1-j} \\ 0 & \rho^k \end{bmatrix},\ k \ge 1.$$

Putting the values of $(\pi_{10}\pi_{11})$ and R^{k-1} in the equation (3.3), we get

$$\pi_{k0} = \delta^k(\delta - 1)\pi_{00}$$

$$\pi_{k1} = \left[-\frac{\alpha\delta^2}{\mu_b} \sum_{j=0}^{k-1} \delta^j \rho^{k-1-j} + \frac{\alpha\delta(\delta-1)\rho^{k-1}}{\mu_b(\delta-1)-\lambda} \right] \pi_{00}.$$

Normalization condition is used to find the value of π_{00} given by (3.2).

In different states, the probabilities of the system can be found using equation (3.1) as follows:

P [The system is in a closed state] $= \pi_{01}$

$$= \frac{1}{\lambda} \left[\alpha + \frac{\alpha\delta(\delta - 1)}{(\delta - 1 - \rho)} \right] \pi_{00}.$$

P [The system is under repair due to breakdown] $= P[J = 0]$.

$$= \sum_{k=1}^{\infty} \pi_{k0} = \sum_{k=1}^{\infty} \delta^k (\delta - 1)\pi_{00} = \frac{(\delta - 1)}{(1 - \delta)} \pi_{00} = -\pi_{00}.$$

P [The system is in regular busy period] =

$$P[J = 1] = \sum_{k=1}^{\infty} \pi_{k1}$$

$$= \sum_{k=1}^{\infty} \left[-\frac{\alpha\,\delta^2}{\mu_b} \sum_{j=0}^{k-1} \delta^j \rho^{k-1-j} + \frac{\alpha\delta(\delta-1)\rho^{k-1}}{\mu_b(\delta-1)-\lambda} \right] \pi_{00}$$

$$= \left[-\frac{\alpha\delta^2}{\mu_b} \frac{1}{(1-\delta)} \frac{1}{(1-\rho)} + \frac{\alpha\delta(\delta - 1)}{(\mu_b(\delta - 1) - \lambda)(1 - \rho)} \right] \pi_{00}.$$

4. Stationary Average Queue Size and Average Sojourn Time

Theorem 4. *If* $\rho < \mu_r < \mu_b$, *in stationary state* (*steady state*) *the average queue size is given by*

$$E(L) = \left[-k - \frac{\alpha\delta^2}{\mu_b}(4\rho\delta - 3\delta - 3\rho + 2) + \frac{\alpha\delta(\delta - 1)}{(\mu_b(\delta - 1) - \lambda)(1 - \rho)^2} \right] \pi_{00}. \quad (4.1)$$

Further, average sojourn time in stationary state (steady state) is given by

$$E(W) = \frac{1}{\lambda}\left[-k - \frac{\alpha\delta^2}{\mu_b}(4\rho\delta - 3\delta - 3\rho + 2) + \frac{\alpha\delta(\delta - 1)}{(\mu_b(\delta - 1) - \lambda)(1 - \rho)^2}\right]\pi_{00}. \quad (4.2)$$

Proof. The probability generating function (p.g.f.) $Q(z)$ with the help of (3.1) can be written as

$$Q(z) = \sum_{k=0}^{\infty}(\pi_{k0} + \pi_{k1})z^k$$

$$= \left[1 + \frac{1}{\lambda}\left\{\alpha + \frac{\alpha\delta(\delta-1)}{(\delta-1-\rho)}\right\} - z^k - \frac{\alpha\delta^2}{\mu_b}\frac{z}{(1-\delta z)}\frac{z}{(1-\rho z)} + \frac{\alpha\delta(\delta-1)}{(\mu_b(\delta-1)-\lambda)}\frac{z}{(1-\rho z)}\right]\pi_{00}$$

$$E(L) = [Q'(z)]_{z=1}$$

$$E(L) = \left[-k - \frac{\alpha\delta^2}{\mu_b}(4\rho\delta - 3\delta - 3\rho + 2) + \frac{\alpha\delta(\delta-1)}{(\mu_b(\delta-1)-\lambda)(1-\rho)^2}\right]\pi_{00}.$$

Further, the average sojourn time in steady state is

$$E(W) = \frac{1}{\lambda}E(L).$$

This implies,

$$E(W) = \frac{1}{\lambda}\left[-k - \frac{\alpha\delta^2}{\mu_b}(4\rho\delta - 3\delta - 3\rho + 2) + \frac{\alpha\delta(\delta-1)}{(\mu_b(\delta-1)-\lambda)(1-\rho)^2}\right]\pi_{00}.$$

5. Conclusion

In this paper, we have studied the $M/M/1$ queue under repair using quasi birth death process and matrix geometric method. When the system is under repair due to some technical breakdown, then in this case the server is providing service manually at a slower rate than the regular busy period. We have derived the distribution of number of customers in the system, the average queue size and average sojourn time in stationary state.

References

[1] Chun Xiu, Naishuo Tian and Yazhen Liu, The $M/M/1$ queue with single working vacation serving at a slower rate during the start-up period, Journal of Mathematics Research 2(1) (2010), 98-102.

[2] B. Doshi, Queueing systems with vacations-a survey, Queueing Syst. 1(1) (1986), 29-66.

[3] G. Ayyappan and S. Shyamala, $M^x/G_1, G_2/1$ with setup time, Bernoulli vacation, breakdown and delayed repair, International Journal of Stochastic Analysis, 2014.

[4] Madhu Jain, Chandra Shekhar and Shalini Shukla, Queueing analysis of machine repair problem with controlled rates and working vacation under F-policy, National Academy of Sciences, India (2016), 21-31.

[5] M. Neuts, Matrix-Geometric Solution in Stochastic Models, Johns Hopking University Press, Baltimore, 1981.

[6] S. Hanumantha Rao, V. Vasanta Kumar, B. Srinivasakumar and T. Srinivasa Rao, Analysis of two phase queueing system with impatient customers, server breakdowns and delayed repair, International Journal of Pure and Applied Mathematics 115(4) (2017), 651-663.

[7] E. M. Shoukry, Salwa M. Essar and Boshra A. Shehata, Matrix Geometric Method for $M/M/1$ queueing model with and without breakdown ATM Machines, American Journal of Engineering Research 7(1) (2018), 246-252.

[8] Srinivas R. Chakravarthy, Shruti and Rakhee Kulshrestha, A queueing model with server breakdowns, repairs, vacations and backup server, Operations Research Perspectives, Elsevier 7, 2020.

[9] Yutaka Baba, The $M^x/M/1$ queue with multiple working vacation, American Journal of Operations Research (2012), 217-224.

[10] Zhong-jun Zhang and Xiu-li Xu, Analysis for the $M/M/1$ queue with Multiple working vacations and N-policy, Information of Management Sciences 19(3) (2008), 495-506.