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GEODETIC HOP DOMINATION NUMBER IN JOIN AND CORONA OF GRAPHS

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Abstract

A subset S of vertices in a connected graph is G called a geodetic hop dominating set of G if S is both a geodetic set and a hop dominating set of G. The minimum cardinality of a geodetic hop dominating set of is its geodetic hop domination number and is denoted by $\gamma_{hg}(G)$. In this paper we studied the concept of geodetic hop domination number in join and corona of graphs.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and n respectively. For basic graph theoretic terminology, we refer to [4]. For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in G/uv \in E(G)\}$. The degree of a vertex $v \in V$ is $\deg(v) = |N(v)|$. If

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$e = \{u, v\}$ is an edge of a graph G with $\deg(u) = 1$ and $\deg(v) > 1$, then we call e a pendant edge or end edge, u a leaf or end vertex and v a support. A vertex of degree $n - 1$ is called a universal vertex. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. A vertex x is said to lie on a $u - v$ geodesic P if x is a vertex of P including the vertices u and v. For two vertices u and v, the closed interval $I[u, v]$ consists of u and v together with all vertices lying on some $u - v$ geodesic. For a set $S \subseteq V(G)$, in the interval $I_G[S]$ is the union of all $I_G[S]$ for $u, v \in S$. If $I_G[S] = V(G)$, then S is a geodetic set of G. The cardinality $g(G)$ of a minimum geodetic set of G is called the geodetic number of G. A geodetic set of cardinality $g(G)$ is called a g-set of G. The geodetic number of a graph was studied in [5, 8, 14, 15].

A set $D \subset V$ is a dominating set of G if every vertex $v \in V - D$ is adjacent to some vertex in D. A dominating set D is said to be minimal if no subset of D is a dominating set of G. The minimum cardinality of a minimal dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. The domination number of a graph was studied in [7]. A set $S \subseteq V$ of a graph G is a hop dominating set (hd-set, in short) of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hd-set of G is called the hop domination number and is denoted by $\gamma_h(G)$. Any hd-set of order $\gamma_h(G)$ is called γ_h-set of G. The hop domination number of a graph was studied in [1, 10-12]. A geodetic dominating set S is both a geodetic and a dominating set. The geodetic domination number $\gamma_g(G)$ of G is the minimum cardinality among all geodetic dominating sets in G. The geodetic domination number of a graph was studied in [6]. A subset S of vertices in a connected graph G is called a geodetic hop dominating set of G if S is both a geodetic set and a hop dominating set of G. The minimum cardinality of a geodetic hop dominating set of G is its geodetic hop domination number and is denoted by $\gamma_{hg}(G)$. The geodetic hop domination number of a graph was studied in [2, 3]. Let H and K be two graphs. The join $G + H$ of two graphs G and H is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G)$

$\cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The join concept was studied in [13]. The corona product $K \odot H$ is defined as the graph obtained from K and H by taking one copy of K and $|V(K)|$ copies of H and then joining by an edge, all the vertices from the i^{th}-copy of H to the i^{th}-vertex of K, where $i = 1, 2, \ldots, |V(H)|$. The corona concept was studied in [9,13]. The dominating concept have interesting applications in social networks. By applying the geodetic hop dominating concept we can improve the privacy in social networks.

The following theorems are used in sequel.

Theorem 1.1 [2]. *For the complete graph* $G = K_n, (n \geq 3)$, $\gamma_{hg}(K_n) = n$.

Theorem 1.2 [2]. *Let* G *be a graph of order* $n \geq 3$. *Then* $\gamma_{hg}(G) = 2$ if *and only if there exists a geodetic hop dominating set* $S = \{u, v\}$ *of* G *such that* $d(u, v) \leq 3$.

2. Geodetic hop domination in join of graphs

Theorem 2.1. $\gamma_{hg}(K_{n_1} + K_{n_2}) = n_1 + n_2$.

Proof. Since $\gamma_{hg}(K_{n_1} + K_{n_2}) = K_{n_1+n_2}$. The result follows from Theorem 1.1. ∎

Theorem 2.2. *Let* G *be a complete graph of order* $n_1 \geq 2$ *and* H *be a non-complete graph of order* $n_2 \geq 2$. *Then* $V(G)$ *is a subset of every geodetic hop dominating set of* $G + H$.

Proof. Let S be a γ_{hg}-set of $G + H$. Since G is complete and H is non-complete, $G + H$ is non-complete. We have to prove that $V(G) \subseteq S$. On the contrary, suppose $V(G) \subsetneq S$. Then there exists a vertex $z \in G$ such that $z \notin S$. By the definition of $G + H$, $d(x, z) = 1$ for every $x \in H$, which is a contradiction to the definition of hop dominating set of $G + H$. Therefore $V(G) \subseteq S$. ∎

Theorem 2.3. *Let* G *and* H *be two non-empty graphs. Then*

$\gamma_{hg}(G+H)=2$ *if and only if* $G=k_1$ *and* $H=K_1$.

Proof. Let G and H be two non-empty graphs. First assume that $\gamma_{hg}(G+H)=2$ Then by Theorems 2.1 and 1.2, $G+H$ is either K_2 or there exists a γ_{hg}-set $S=\{u, v\}$ of $G+H$ such that $d_{G+H}(u, v)\leq 3$. If $G+H=K_2$, then the result is obvious. Therefore $d_{G+H}(u, v)$ is either 2 or 3. If $d_{G+H}(u, v)=2$, then either $S\subseteq V(G)$ or $S\subseteq V(H)$. Without loss of generality, let us assume that $S\subseteq V(G)$. Since S is a g-set of G, let x be a vertex in $V(G)$ which lies in $u-v$ geodesic in G. Then $xu, xv\in E(G+H)$. Hence it follows that $d_{G+H}(x, u)=d_{G+H}(x, v)=1$, which is a contradiction. Therefore $d_{G+H}(u, v)\neq 2$. Hence it follows that $d_{G+H}(u, v)=3$. Therefore either G or H must be empty, which is not possible. Therefore $G+H=K_2$. Hence it follows that G is K_1 and H is K_1. The converse is clear. ■

Theorem 2.4. *Let G and H be two non-empty graphs. Then $\gamma_{hg}(G+H)=3$ if and only if $G+H=K_3$.*

Proof. Let $G+H=K_3$. Then by Theorem 1.1, $\gamma_{hg}(G+H)=3$. Conversely, assume that $\gamma_{hg}(G+H)=3$. Let S be a γ_{hg}-set of $G+H$. We have to prove that $G+H=K_3$. On the contrary, suppose that $G+H\neq K_3$. Therefore S contains at least four elements. Then at least one of G or H is non complete or both G and H are non-complete. Without loss of generality, let us assume that G is complete and H is non-complete. Then $d(G+H)=2$. By Theorem 2.2, $V(G)\subseteq S$. Since G is non-complete, at least three elements of H belongs to S. Hence it follows that $|S|\geq 4$, which is a contradiction. If G and H are non-complete, then S contains at least two elements from G and at least 2 elements from H. Hence it follows that $|S|\geq 4$, which is a contradiction. Therefore $G+H=K_3$. ■

Theorem 2.5. *Let G and H be two non-trivial connected graphs. Then $\gamma_{hg}(G+H)\geq 4$.*

Proof. Let G and H be two non-trivial connected graphs. Let S be a γ_{hg}-set of $G+H$. We have to prove that $\gamma_{hg}(G+H)\geq 4$. On the contrary, suppose that $|S| \leq 3$. First assume that $|S| = 2$ then by Theorem 2.3, $G + H = K_2$, which implies $G = K_1$ and $H = K_1$, which is a contradiction to G and H are non-trivial. Next assume that $|S| \geq 3$. Since $d(G + H) = 2$, either $S \subseteq V(G)$ or $S \subseteq V(H)$. Without loss of generality, let us assume that $S \subseteq V(G)$. Then for every $x \in V(H)$, $d(x, S) = 1$, which is a contradiction. Hence $\gamma_{hg}(G + H) \geq 4$. ■

Corollary 2.6. *Let G and H be two non-trivial connected graphs. Then every geodetic hop dominating set of $G + H$ contains at least two vertices from G and at least two vertices from H.*

Proof. This follows from Theorem 2.5. ■

Theorem 2.7. $$\gamma_{hg}(C_n + P_n) = \begin{cases} 6 & \text{if } n = 3 \text{ or } 4 \\ 7 & \text{if } n = 5 \\ 8r & \text{if } n = 6r \\ 8r + 2 & \text{if } n = 6r + 1 \\ 8r + 3 & \text{if } n = 6r + 2 \\ 8r + 4 & \text{if } n = 6r + 3 \\ 8r + 6 & \text{if } n = 6r + 4 \\ 8r + 7 & \text{if } n = 6r + 5 \end{cases}$$

Proof. $V(C_n + P_n) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$

Case 1: $3 \leq n \leq 5$

Case 1a: $n = 3$

Then $S = \{v_1, v_2, v_3, u_1, u_2, u_3\}$ is the only γ_{hg}-set of $C_n + P_n$ so that $\gamma_{hg}(C_n + P_n) = 6$.

Case 1b: $n = 4$

Then $S = \{v_1, v_3, v_4, u_1, u_3, u_4\}$ is the only γ_{hg}-set of $C_n + P_n$ so that $\gamma_{hg}(C_n + P_n) = 6$.

Case 1c: $n = 5$

Then $S = \{v_1, v_3, v_4, v_5, u_1, u_3, u_4\}$ is the only γ_{hg}-set of $C_n + P_n$ so that $\gamma_{hg}(C_n + P_n) = 7$.

Case 2: $n \geq 6$

Case 2a: $n = 6r$

Let $S = \{v_1, v_4, \ldots, v_{6r-2}, v_3, v_6, \ldots, v_{6r}, u_1, u_4, \ldots, u_{6r-2}, u_3, u_6, \ldots, u_{6r}\}$.

Then S is a geodetic hop dominating set of $C_n + P_n$ and so $\gamma_{hg}(C_n + P_n) \leq |S| = 8r$. We have to prove that $\gamma_{hg}(C_n + P_n) = 8r$. On the contrary, suppose that $\gamma_{hg}(C_n + P_n) < 8r$. Then there exists a γ_{hg}-set S' of $C_n + P_n$ such that $|S'| < 8r$. Let x be a vertex of $C_n + P_n$ such that $x \in S$ and $x \notin S'$. First assume that $x \in v_1, v_4, \ldots, v_{6r-2}, u_1, u_4, u_{6r-2}\}$. Without loss of generality, let us assume that $x = v_1$. Then $x \notin I_{C_{n+P_n}}[S']$. Next assume that $x \in \{v_3, v_6, \ldots, v_{6r}, u_3, u_6, \ldots, u_{6r}\}$. Without loss of generality, let us assume that $x = v_3$. Then $x \notin I_{C_n+P_n}[S']$. Hence S' is not a geodetic hop dominating set of $C_n + P_n$, which is a contradiction. Therefore $\gamma_{hg}(C_n + P_n) = 8r$.

Case 2b: $n = 6r + 1$

Let $S = \{v_1, v_4, \ldots, v_{6+1}, v_3, v_6, \ldots v_{6r}, u_1, u_4, \ldots, u_{6r+1}, u_3, u_6, \ldots, u_{6r}\}$. Then as in Case 2a, we get $\gamma_{hg}(C_n + P_n) = 8r + 2$.

Case 2c: $n = 6r + 2$

Let $S = \{v_1, v_4, \ldots, v_{6r+1}, v_3, v_6, \ldots, v_{6r+3}, u_1, u_4, \ldots, u_{6r+1}, u_3, u_6, \ldots, u_{6r}, u_{6r+2}\}$. Then as in Case 2a, we get $\gamma_{hg}(C_n + P_n) = 8r + 3$.

Case 2d: $n = 6r + 3$

Let $S = \{v_1, v_4, \ldots v_{6r+1}, v_3, v_6, \ldots, v_{6r+3}, u_1, u_4, \ldots, u_{6r+1}, u_3, u_6, \ldots, u_{6r+3}\}$. Then as in Case 2a, we get $\gamma_{hg}(C_n + P_n) = 8r + 4$.

Case 2e: $n = 6r + 4$

Let $S = \{v_1, v_4, \ldots, v_{6r+4}, v_3, v_6, \ldots, v_{6r+3}, u_1, u_4, \ldots, u_{6r+4}, u_3, u_6, \ldots, u_{6r+3}\}$. Then as in Case 2a, we get $\gamma_{hg}(C_n + P_n) = 8r + 6$.

Case 2f: $n = 6r + 5$

Let $S = \{v_1, v_4, \ldots v_{6r+4}, v_3, v_6, \ldots, v_{6r+3}, u_1, u_4, \ldots, u_{6r+4}, u_3, u_6, \ldots, u_{6r+3}, u_{6r+5}\}$. Then as in Case 2a, we get $\gamma_{hg}(C_n + P_n) = 8r + 7$.

Theorem 2.8. $\gamma_{hg}(C_n + K_2) = \begin{cases} 5 & if\ n = 3\ or\ 4\ or\ 5 \\ 4r + 2 & if\ n = 6r \\ 4r + 3 & if\ n = 6r + 1\ or\ 6r + 2 \\ 4r + 4 & if\ n = 6r + 3 \\ 4r + 3 & if\ n = 6r + 4\ or\ 6r + 5 \end{cases}$

Proof. The proof of this Theorem is similar to the proof of Theorem 2.7. ■

Theorem 2.9. $\gamma_{hg}(K_{n_1} + C_{n_2}) = \begin{cases} n_1 + 3 & if\ n_2 = 3\ or\ 4\ or\ 5 \\ n_1 + 4r & if\ n_2 = 6r \\ n_1 + 4r + 1 & if\ n_2 = 6r + 1\ or\ 6 + 2 \\ n_1 + 4r + 2 & if\ n_2 = 6r + 3 \\ n_1 + 4r + 3 & if\ n_2 = 6r + 4\ or\ 6r + 5 \end{cases}$

Proof. The proof of this Theorem is similar to the proof of Theorem 2.7. ■

3. Corona of graphs

Theorem 3.1. *Let G and H be two connected graphs of order* n_1 *and* n_2 *respectively. Let S be a* γ_{hg}*-set of* $G \circ H$ *and* $H_1, H_2, \ldots, H_{n_2}$ *be the copies of H Then*

(i) $S \cap V(H_i) \neq \emptyset$ *for all* $i, 1 \leq i \leq n_2$.

(ii) $S \cap V(G) = \emptyset$.

Proof. (i) We have to prove that $S \cap V(H_i) = \emptyset$, for all $i, 1 \leq i \leq n_2$. On the contrary, suppose that $S \cap V(H_i) = \emptyset$, for some $i, 1 \leq i \leq n_2$. By the definition of $G \circ H$, every vertex of $H_i (1 \leq i \leq n_2)$ is adjacent to exactly one vertex of G Hence it follows that S is not a γ_{hg}-set of $G \circ H$, which is a contradiction.

(ii) We have to prove that $S \cap V(G) = \emptyset$. On the contrary, suppose that $S \cap V(G) \neq \emptyset$. Let u be a vertex of $S \cap V(G) = \emptyset$. Hence $u \in S$ and $u \in V(G)$. By Theorem 2.9 (i), $S \cap V(H_i) \neq \emptyset, 1 \leq i \leq n_2$. Then $d(u, v) = 1$ for $v \in V(H_i)$ for $1 \leq i \leq n_2$, which is contradiction. ■

Theorem 3.2. $\gamma_{hg}(W_{n_1} \circ K_{n_2}) = n_1 n_2,\ n_1 \geq 4,\ n_2 \geq 2$.

Proof. Let $V(W_{n_1}) = \{v_1, v_2, \ldots, v_{n_1}\}$ with central vertex v_{n_1} and $v_1, v_2, \ldots, v_{n_1-1}, v_1$ as cycle. Let $V(K_m) = \{u_{11}, u_{12}, \ldots, u_{1n_2}, u_{21}, u_{22}, \ldots, u_{2n_2}, \ldots, u_{n_1 1}u_{n_1 2}, u_{n_1 n_2}\}$ be the vertex set of n_1^{th} copy of K_{n_2} attached with exactly one vertex of $W_{n_1} \cdot V(W_{n_1} \circ K_{n_2}) = \{v_1, v_2, \ldots, v_{n1}, u_{11}, u_{12}, u_{1n_2}, u_{21}, u_{22}, \ldots, u_{2n_2}, u_{n_1 1}, u_{n_1 2}, \ldots, u_{n_1 n_2}\}$. We have to prove that $\gamma_{hg}(W_{n_1} \circ K_{n_2}) = n_1 n_2$. Let $S = \{u_{11}, u_{12}, \ldots, u_{1n_2}, u_{21}, u_{22}, \ldots, u_{2n_2}, u_{n_1 1}, u_{n_1 2}, \ldots, u_{n_1 n_2}\}$. Then S is a γ_{hg}-set of $W_{n_1} \circ K_{n_2}$ and so $\gamma_{hg}(W_{n_1} \circ K_{n_2}) \leq |S| = n_1 n_2$. On the contrary, suppose that $\gamma_{hg}(W_{n_1} \circ K_{n_2}) < n_1 n_2$. Then there exits a γ_{hg}-set S' of $W_{n_1} \circ K_{n_2}$ such that $|S'| < n_1 n_2$. Let y be a vertex of $W_{n_1} \circ K_{n_2}$ such that $y \in S$ and $y \notin S'$. First assume that $y \in \{u_{11}, u_{12}, \ldots, u_{1n_2}\}$. Without loss of generality, let us assume that $y = v_{11}$. Then $y \notin I_{W_{n_1} \circ K_{n_2}}[S']$. Next assume that $y \in \{u_{21}, u_{22}, \ldots, u_{2n_2}, \ldots, u_{n_1 1}, u_{n_1 2}, \ldots, u_{n_1 n_2}\}$. Without loss of generality, let us assume that $y = v_{21}$. Then $y \notin I_{W_{n_1} \circ K_{n_2}}[S']$. Hence S' is not a γ_{hg}-set of $W_{n_1} \circ K_{n_2}$, which is a contradiction. Therefore $\gamma_{hg}(W_{n_1} \circ K_{n_2}) = n_1 n_2$. ■

Theorem 3.3. $$\gamma_{hg}(C_{n_1} \circ P_{n_2}) = \begin{cases} 3n_1 & \text{if } n_2 = 3 \text{ or } 4 \\ 4n_1 & \text{if } n_2 = 5 \\ 4n_{1r} & \text{if } n_2 = 6r \\ n_1(4r+1) & \text{if } n_2 = 6r+1 \\ n_1(4r+2) & \text{if } n_2 = 6r+2 \text{ or } 6r+3 \\ n_1(4r+3) & \text{if } n_2 = 6r+4 \\ n_1(4r+4) & \text{if } n_2 = 6r+5 \end{cases}$$

Proof. $V(C_{n_1} \circ P_{n_2}) = \{v_1, v_2, \ldots, v_{n_1}, u_{11}, u_{12}, \ldots, u_{1n_2}, u_{21}, u_{22}, \ldots, u_{2n_2}, \ldots, u_{n_1 1}, u_{n_1 2}, \ldots, u_{n_1 n_2}\}$.

Case 1: $3 \leq n_2 \leq 5$

Case 1a: $n_2 = 3$

Then $S = \{u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, \ldots, u_{n_1 1}, u_{n_1 2}, u_{n_1 3}\}$ is the only

γ_{hg} -set of $C_{n_1} \circ P_{n_2}$ so that $\gamma_{hg}(C_{n_1} \circ P_{n_2}) = 3n_1$.

Case 1b: $n_2 = 4$

Then $S = \{u_{11}, u_{13}, u_{14}, u_{21}, u_{23}, u_{24}, \ldots, u_{n_1 1}, u_{n_1 3}, u_{n_1 4}\}$ is the only γ_{hg} -set of $C_{n_1} \circ P_{n_2}$ so that $\gamma_{hg}(C_{n_1} \circ P_{n_2}) = 3n_1$.

Case 1c: $n_2 = 5$

Then $S = \{u_{11}, u_{13}, u_{14}, u_{15}, u_{21}, u_{23}, u_{24}, u_{25}, \ldots, u_{n_1 1}, u_{n_1 3}, u_{n_1 4} u_{n_1 5}\}$ is the only γ_{hg} -set of $C_{n_1} \circ P_{n_2}$ so that $\gamma_{hg}(C_{n_1} \circ P_{n_2}) = 4n_1$.

Case 2: $n_2 \geq 6$.

Case 2a: $n_2 = 6r$

Let $S = \{u_{11}, u_{14}, \ldots, u_{16r-2}, u_{21}, u_{24}, \ldots, u_{26r-2}, u_{n_1 1}, u_{n_1 4}, \ldots, u_{n_1 6r-2}\}$ $\cup \{u_{13}, u_{16}, \ldots, u_{16r}, u_{23}, u_{26}, \ldots, u_{26r}, u_{n_1 3}, u_{n_1 3}, u_{n_1 6}, \ldots, u_{n_1 6r}\}$. Then S is a geodetic hop dominating set of $C_{n_1} \circ P_{n_2}$ and so $\gamma_{hg}(C_{n_1} \circ P_{n_2}) \leq |S| = 4_{n_1} r$. We prove that $\gamma_{hg}(C_{n_1} \circ P_{n_2}) = 4n_1 r$.

On the contrary, suppose that $\gamma_{hg}(C_{n_1} \circ P_{n_2}) = 4n_1 r$. Then there exists a γ_{hg} -set S' of $C_{n_1} \circ P_{n_2}$ such that $| S' | < 4n_1 r$. Let u be a vertex of $C_{n_1} \circ P_{n_2}$ such that $u \in S$ and $u \notin S'$ First assume that $u \in \{u_{11}, u_{14}, \ldots, u_{16r-2}, u_{21}, u_{24}, \ldots, u_{26r-2}, \ldots, u_{n_1 1}, u_{n_1 4}, \ldots, u_{n_1 6r-4}\}$. Without loss of generality, let us assume that $u = u_{11}$. Then $u \notin I_{C_{n_1} \circ P_{n_2}}[S']$.

Next assume that $u \in \{u_{13}, u_{16}, \ldots u_{16r}, u_{23}, u_{26}, \ldots, u_{26r}, u_{n_1 3}, u_{n_1 6}, \ldots, u_{n_1 6r}\}$. Without loss of generality, let us assume that $u = u_{13}$. Then $u \notin I_{C_{n_1} \circ P_{n_2}}[S']$. Hence S' is not γ_{hg} -set of $C_{n_1} \circ P_{n_2}$, which is a contradiction. Therefore $\gamma_{hg}(C_{n_1} \circ P_{n_2}) = 4n_1 r$.

Case 2b: $n_2 = 6r + 1$

Let $S = \{u_{11}, u_{14}, u_{16r+1}, u_{21}, u_{24}, \ldots, u_{26r+1}, u_{n_1 1}, u_{n_1 4}, \ldots, u_{n_1 6r+1}\}$

$\cup\{u_{13}, u_{16}, \ldots, u_{16r}, u_{23}, u_{26}, \ldots, u_{26}r, u_{n_1 3}, u_{n_1 6}, \ldots, u_{n_1 6r}\}$. Then in Case 2a, we get $\gamma_{hg}(C_{n_1} \circ P_{n_2}) = n_1(4r + 2)$.

Case 2c: $n_2 = 6r + 2$

Let $S = \{u_{11}, u_{14}, u_{16r+1}, u_{21}, u_{24}, \ldots, u_{26r+1}, u_{n_1 1}, u_{n_1 4}, \ldots, u_{n_1 6r+1}\}$ $\cup\{u_{13}, u_{16}, \ldots, u_{16r}, u_{23}, u_{26}, \ldots, u_{n_1 3}, u_{n_1 6}, \ldots, u_{n_1 6r}\} \cup \{u_{n_1 6r+2}\}$. Then as in Case 2a, we get $\gamma_{hg}(C_{n_1} \circ P_{n_2}) = n_1(4r + 2)$.

Case 2d: $n_2 = 6r + 3$

Let $S = \{u_{11}, u_{14}, \ldots, u_{16r+4}, u_{21}, u_{24}, \ldots, u_{26r+4}, u_{n_1 1}, u_{n_1 4}, \ldots, u_{n_1 6r+4}\}$ $\cup\{u_{13}, u_{16}, \ldots, u_{16r+3}, u_{23}, u_{26}, \ldots, u_{26r+3}, \ldots, u_{n_1 3}, u_{n_1 6}, \ldots, u_{n_1 6r+3}\}$. Then as in Case 2a, we get $\gamma_{hg}(C_{n_1} \circ P_{n_2}) = n_1(4r + 3)$.

Case 2e: $n_2 = 6r + 4$

Let $S = \{u_{11}, u_{14}, \ldots, u_{16r+4}, u_{21}, u_{24}, \ldots, u_{26r+4}, u_{n_1 1}, u_{n_1 4}, \ldots, u_{n_1 6r+4}\}$ $\cup\{u_{13}, u_{16}, \ldots, u_{16r+3}, u_{23}, u_{26}, \ldots, u_{26r+3}, \ldots, u_{n_1 3,} u_{n_1 6}, \ldots, u_{n_1 6r+3}\}$ Then as in Case 2a, we get $\gamma_{hg}(C_{n_1} \circ P_{n_2}) = n_1(4r + 3)$.

Case 2f: $n_2 = 6r + 5$

Let $S = \{u_{11}, u_{14}, \ldots, u_{16r+4}, u_{21}, u_{24}, \ldots, u_{26r+4}, u_{n_1 1}, u_{n_1 4}, \ldots, u_{n_1 6r+4}\}$ $\cup\{u_{13}, u_{16}, \ldots, u_{16r+3}, u_{23}, u_{26}, \ldots, u_{26r+3}, \ldots, u_{n_1 3}, u_{n_1 6}, \ldots, u_{n_1 6r+3}\} \cup \{u_{n_1 6r+5}\}$.

Then as in Case 2a, we get $\gamma_{hg}(C_{n_1} \circ P_{n_2}) = n_1(4r + 4)$. ∎

Theorem 3.4. $\gamma_{hg}(K_{n_1} \circ C_{n_2}) = \begin{cases} 3n_1 & if\ n_2 = 3\ or\ 4\ or\ 5 \\ 4n_1 r & if\ n_2 = 6r \\ n_1(4r+1) & if\ n_2 = 6r+1\ or\ 6r+2 \\ n_1(4r+2) & if\ n_2 = 6r+3 \\ n_1(4r+3) & if\ n_2 = 6r+4\ or\ 6r+5 \end{cases}$

Proof. The proof of this Theorem is similar to the proof of Theorem 3.3. ∎

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