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GENERALIZED KROPINA METRIC WITH ISOTROPIC MEAN BERWALD CURVATURE

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Abstract

In this paper, we study the characterization of the generalized Kropina metric and it is weakly Berwald satisfying the curvature properties of (α, β)-metrics which is of isotropic mean Berwald curvature.

1. Introduction

Finsler spaces with (α, β)-metrics have been studied by the many authors. But it is a very important aspect of Finsler geometry and its application to physics and biology. A Finsler metric on a manifold is a family

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of norms in tangent spaces, which vary smoothly with the base point. Every Finsler metric determines a spray by its systems of geodesic equations. We know that if a Finsler metric is affinely equivalent to a Riemannian metric, then it is a Berwald metric. Since every Berwald metric can be constructed from a Riemannian metric. The S-curvature is the rate of change of the distortion along geodesics. In this paper, we investigate the characterization of the generalized Kropina metric would satisfies the necessary and sufficient conditions in which Finsler space with (α, β)-metric is of isotropic mean Berwald curvature.

2. Preliminaries

Let M be an n-dimensional smooth manifold. We denote by TM the tangent bundle of M and by $(x, y) = (x^i, y^i)$ the local coordinates on the tangent bundle TM. A Finsler manifold (M, F) is a smooth manifold equipped with a function $F : TM \to [0, \infty)$, which has the following properties:

- Regularity: F is smooth in $TM \setminus \{0\}$;
- Positively homogeneity: $F(x, \lambda y) = \lambda F(x, y)$, for $\lambda > 0$;
- Strong convexity: the Hessian matrix of F^2, $g_{ij}(x, y) = \frac{1}{2}\left(\frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}\right)$, is positive definite on $TM \setminus \{0\}$. We call F and the tensor g_{ij} the Finsler metric and fundamental tensor of M respectively.

For a Finsler metric $F = F(x, y)$, its geodesics curves are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients and given by following

$$G^i = \frac{1}{4} g^{il}\left\{\frac{\partial^2[F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial[F^2]}{\partial x^l}\right\}, \; y \in T_x M.$$

Definition 2.1. Let F be a Finsler metric on an n-dimensional manifold M.

(1) F is of isotropic mean Berwald curvature if

$$E = \frac{(n+1)}{2} cF^{-1} h;$$

(2) F is of isotropic S-curvature if

$$S = (n+1)cF,$$

where $c = c(x)$ is a scalar function on M and h denotes the angular metric tensor of F which is defined by $h_{ij} = FF_{y^i y^j}$.

Definition 2.2. A Finsler metric is called a Berwald metric if the geodesic coefficients of F are quadratic in y, $G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^i y^k$. Equivalently, a Finsler metric F is a Berwald metric if and only if there exists a Riemann metric α such that F and α have same geodesic coefficients, $G^i = G^i_\alpha$.

A Finsler metric is called a Berwald metric if the Berwald curvature $B = 0$. A Finsler metric is called a weakly-Berwald metric if the mean Berwald curvature $E = 0$.

Lemma 2.1[7]**.** *The geodesic coefficients G^i are related to G^i_α by*

$$G^i = G^i_\alpha + \Theta\{-2Q\alpha s_0 + r_{00}\}\frac{y^i}{\alpha} + \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\} b^i, \tag{2.1}$$

where G^i_α denote the spray coefficients of α and

$$\Theta = \frac{(\phi - s\phi')\phi' - s\phi\phi''}{2\phi((\phi - s\phi' + (b^2 - s^2)\phi'')},$$

$$Q = \frac{\phi'}{\phi - s\phi'},$$

$$\Psi = \frac{\phi''}{2\phi((\phi - s\phi' + (b^2 - s^2)\phi'')},$$

where $s = \beta/\alpha$, $b = \| \beta_x \|_\alpha$.

It is well known that the condition for a Finsler metric to be weakly-

Berwald metric is $B^r_{jkr} = 0$. This is equivalent to that $N^r_r = \partial G^r / \partial y^r$ is a 1-form. By Lemma 2.1 and (2), we have the following.

Lemma 2.2[8]. *An* (α, β)*-metric* $F = \alpha\phi(\beta/\alpha)$ *is a weakly-Berwald metric if and only if* N^r_r *is a* 1*-form.*

By Lemma 2.1, *we get*

$$N^r_r = Lr_{00} + 2Mr_0 + Ns_0, \tag{2.2}$$

where

$$L = (n+1)\alpha^{-1}\Theta + \frac{\partial \Psi}{\partial y^r} b^r,$$

$$M = \Psi,$$

$$N = -2(n+1)Q\Theta + \frac{\partial(2\Psi Q\alpha)}{\partial y^r} b^r + \frac{\partial(\alpha Q)}{\partial y^r} s_0 s_0^r.$$

In 2009, Cheng and Z. Shen have obtained the formula for the S-curvature of an (α, β)-metric on an n-dimensional manifold M as follows

Lemma 2.3[2]. *The S-curvature of an* (α, β)*-metric is given by*

$$S = \mu(r_0 + s_0) + 2(\Psi + QC)s_0 - 2\Psi r_0 + \alpha^{-1}[(b^2 - s^2)\Psi' + (n+1)\Theta]r_{00}, \tag{2.3}$$

where $\mu = \dfrac{f'(b)}{[bf(b)]}$ *is a scalar function on M and* $C = -(b^2 - s^2)\Psi' - (n+1)\Theta.$

3. Characterization of Weakly Berwald Generalized Kropina Metric

Theorem 3.1. *Let* $F = \dfrac{\alpha^{m+1}}{\beta^m}$ *be a generalized Kropina metric on an n-dimensional manifold M, where m is a real number with* $m \neq -1, 0$. *Then F is weakly-Berwald metric if and only if* $r_{ij} = 0, s_i = 0$.

Proof. First assume that F is weakly-Berwald metric.

For $F = \frac{\alpha^{m+1}}{\beta^m}$, by Lemmas 2.1 and 2.2 we can get the following

$$L = -\frac{(n+1)m\beta}{[mb^2\alpha^2 - \beta^2(m-1)]} + \frac{m(m+1)\beta(\beta^2 - \beta^2\alpha^2)}{[mb^2\alpha^2 - \beta^2(m-1)]^2},$$

$$M = \frac{m}{[mb^2\alpha^2 - \beta^2(m-1)]},$$

$$N = -\frac{2(n+1)m^2\alpha^2}{[mb^2\alpha^2 - \beta^2(m-1)](m+1)} + \frac{-2m^2(m-1)(\alpha^2 b^2 - \beta)\alpha^2}{[mb^2\alpha^2 - \beta^2(m-1)]^2(m+1)}$$

$$+ \frac{m^2\alpha^2(\alpha^2 b^2 - \beta)}{[mb^2\alpha^2 - \beta^2(m-1)]\beta^2(m+1)} + \frac{m\alpha^2 - 2m}{\beta(m+1)}. \tag{3.1}$$

Plugging (3.1) and (2.2) yields the following equation

$$A\alpha^6 + B\alpha^4 + C\alpha^2 + D = 0, \tag{3.2}$$

where

$$A = m^3 b^4 (1 + \beta) s_0,$$

$$B = 2m^3(b^4 + (n+1)b^2)s_0 + m^2(m+1)b^4\beta^2 N_r^r + 3m^2(m-1)b^2\beta^2 s_0$$
$$+ m^3 b^2 \beta s_0 - 2m^2(m-1)b^2\beta^3 s_0,$$

$$C = 2m(m^2-1)b^2\beta^4 N_r^r + m(m-1)\beta^5 s_0 + m^2(m+1)b^2\beta^2 r_0$$
$$+ 2m^2(m-1)(n+1)\beta^2 s_0 - m(m+1)[m(n+1) + (m+1)]b^2\beta^3 r_{00}$$
$$+ m^2(m-1)(3 + 4b^2)\beta^3 s_0,$$

$$D = m[(m+1)^2 + (m^2-1)(n+1)]\beta^5 r_{00} - (m+1)(m-1)^2\beta^6 N_r^r$$
$$- 2m(m-1)^2\beta^5 s_0 - m(m^2-1)\beta^4 r_0.$$

Assume that N_r^r is a 1-form. Note that the coefficients of α in (3.2) must be zero (because α^{even} is a polynomial in y^i).

$$A\alpha^6 + B\alpha^4 + C\alpha^2 + m[(m+1)^2 + (m^2-1)(n+1)]\beta^5 r_{00}$$

$$-(m+1)(m-1)^2\beta^6 N_{r\cdot}^{r} - 2m(m-1)^2\beta^5 s_0 - m(m^2-1)\beta^4 r_0 = 0. \tag{3.3}$$

Note that $m \neq -1, 0$. From (3.3) that $\beta^5 r_{00}$ can be divided by α^2. Because β^5 and α^2 are relatively prime polynomials of (y^i), there is a scalar function $\rho(x)$ on M such that

$$r_{00} = \rho(x)\alpha^2. \tag{3.4}$$

Substituting (3.4) into (3.3), we get the following

$$A\alpha^6 + B\alpha^4 + C\alpha^2 + m[(m+1)^2 + (m^2-1)(n+1)]\beta^5\rho(x)\alpha^2$$

$$= (m+1)(m-1)^2\beta^6 N_{r\cdot}^{r} - 2m(m-1)^2\beta^5 s_0 - m(m^2-1)\beta^4 r_0. \tag{3.5}$$

It is clear that left hand side of the (3.5) can be divided by α^2. Hence $N_{r\cdot}^{r}$ can be divided by α^2. However, $N_{r\cdot}^{r}$ is a 1-form. So we obtain

$$N_{r\cdot}^{r} = 0. \tag{3.6}$$

By (3.4), we have

$$r_0 = \rho(x)\beta. \tag{3.7}$$

Plugging (3.4), (3.6), (3.7) into (3.3) yields

$$\{A\alpha^6 + B\alpha^4 + C\alpha^2 + m[(m+1)^2 + (m^2-1)(n+1)]\beta^5\rho(x)\}\alpha^2$$

$$= \{[m(m^2-1)\rho(x) + 2m(m-1)^2 s_0]\beta\}\beta^4. \tag{3.8}$$

Since α^2 is not divided by β^4, from the equation (3.8), we get

$$m(m^2-1)\rho(x) + 2m(m-1)^2 s_0 = 0.$$

contracting the above equation by b^i yields

$$m(m^2-1)\rho(x)b_i + 2m(m-1)^2 s_i = 0. \tag{3.9}$$

Contract (3.9) with b^i yields $m(m^2 - 1)\rho(x)b^2 = 0$. Since $m \neq -1, 0$ we obtain $\rho(x) = 0$. Therefore, from (3.5), (3.8) and (3.9), we obtain

$$r_{00} = r_0 = s_0 = 0. \tag{3.10}$$

Conversely, suppose that the equation $r_{ij} = s_i = 0$ hold. Then from (2.2) we have $N_r^r = 0$. This completes the proof.

Theorem 3.2. *Let* $F = \frac{\alpha^{m+1}}{\beta^m}$ *be an* (α, β)*-metric on an n-dimensional manifold M, where m is a real number* $m \neq -1, 0$. *Then the following conditions are equivalent*:

(1) *F is of isotropic S-curvature,* $S = (n+1)cF$;

(2) *F is of isotropic mean Berwald curvature,* $E = \frac{n+1}{2} cF^{-1}h$;

(3) β *is killing* 1*-form with* $b =$*constant with respect to* α, *that is,* $r_{ij} = 0, s_i = 0$;

(4) $S = 0$;

(5) *F is weakly-Berwald metric i.e.,* $E = 0$,

where $c = c(x)$ *is scalar function on M.*

Proof. For $F = \frac{\alpha^{m+1}}{\beta^m}$, by Lemma 2.1 we have the following

$$Q = \frac{-m\alpha}{\beta(m+1)},$$

$$\Theta = \frac{-m\beta\alpha}{m\alpha^2 b^2 - \beta^2(m-1)},$$

$$\Psi = \frac{m\alpha^2}{2[m\alpha^2 b^2 - \beta^2(m-1)]},$$

$$\Psi' = \frac{m\beta\alpha^3(m-1)}{[m\alpha^2 b^2 - \beta^2(m-1)]^2}. \tag{3.11}$$

Step 1. In fact, it is clearly true that if F is isotropic S-curvature then it implies that F is of isotropic mean Berwald curvature. We assume that (2) holds, which is equivalent to

$$S = (n+1)(cF + \eta), \tag{3.12}$$

where η is a 1-form on M. So (1) is equivalent to (2) if and only if $\eta = 0$. Plugging (3.11) and (3.12) into (2.3) yields

$$X_1\alpha^6 + X_2\alpha^4 + X_3\alpha^2 + X_4 = (n+1)c[Y_1\alpha^4 + Y_2\alpha^2 + Y_3]\frac{\alpha^{m+1}}{\beta^m}, \tag{3.13}$$

where

$$X_1 = -m^3(m+1)b^4 r_0,$$

$$X_2 = \mu(r_0 + s_0)m^2(m+1)b^4 + m^2(m+1)b^2 + 2m^2(m-1)b^2 + 2m^3(n+1)b^2 s_0$$
$$+ 2m^2(m^2-1)b^2\beta^2 r_0 - m^2(m+1)(n+1)\eta b^4,$$

$$X_3 = 2m(m^2-1)\mu(r_0 + s_0)b^2\beta^2 - m(m^2-1)\beta^2 - 2m^2(m-1)[1 + (n+1)s_0]\beta^2$$
$$- m(m^2-1)\beta^4 r_0 - m(m^2-1)b^2\beta r_{00} - 2m(m^2-1)(n+1)b^2\eta\beta^2,$$

$$X_4 = (m^2-1)\mu(r_0 + s_0)\beta^4 + m(m^2-1)\beta^2 r_{00} - m(m^2-1)(n+1)\beta^3 r_{00}$$
$$- (n+1)(m^2-1)\eta\beta^2,$$

$$Y_1 = m^2(m+1)b^4,$$

$$Y_2 = -2m(m^2-1)b^2\beta^2,$$

$$Y_3 = (m^2-1)\beta^2.$$

We can rewrite (3.13) as in the following form

$$\beta^m[X_1\alpha^6 + X_2\alpha^4 + X_3\alpha^2] - X_4\beta^m - (n+1)c[Y_1\alpha^4 + Y_2\alpha^2 + Y_3]\alpha^{m+1} = 0. \tag{3.14}$$

When m is a positive integer, it is easy to see that the term which does not include α in (3.14) is just $-(n+1)cY_3$. Because α^2 is not divided by β, we get $c = 0$. So

$$X_1\alpha^6 + X_2\alpha^4 + X_3\alpha^2 + X_4 = 0.$$

When m is a non-zero real number but not a positive integer, we know that the left-hand side of (3.14) is a polynomial in α, but the term $\frac{\alpha^{m+1}}{\beta^m}$ is not a polynomial in α.

Therefore, we also have

$$(X_1\alpha^2 + X_2)\alpha^4 + X_3\alpha^2 + X_4 = 0 \tag{3.15}$$

In homogeneous of degree one the coefficients of α in (3.15) must be zero (because α^{even} is polynomial in y^i). Then (3.15) is equivalent to the following two equations

$$X_1\alpha^2 + X_2 = 0, \tag{3.16}$$

$$X_3\alpha^2 + X_4 = 0. \tag{3.17}$$

This implies that

$$X\alpha^2 + (X_2 - X_4) = 0,$$

where $X = (X_1 - X_3)$.

Then

$$X\alpha^2 + (m^2-1)[2m^2b^2r_0 + (n+1)\eta - mr_{00}]\beta^2 + \mu(r_0 + s_0)[m^2(m+1)b^4 - (m^2-1)\beta^4] + m(m^2-1)(n+1)\beta^3 r_{00} + m^2b^2(3m-1) = 0. \tag{3.18}$$

Note $m \neq -1, 0$. β^2 and α^2 are relatively prime polynomials of (y^i), we know that r_{00} can be divided by α^2. That is, there is a scalar function $\tau(x)$ on M such that

$$r_{00} = \tau(x)\alpha^2. \tag{3.19}$$

Substituting (3.19) in (3.17), we get

$$X_3\alpha^2 + m(m^2-1)[\beta^2(x) - (n+1)\beta^3\tau(x)]\alpha^2 - (m^2-1)[\mu(r_0+s_0)\beta^2$$
$$-(n+1)\eta]\beta^2 = 0. \tag{3.20}$$

This implies that $\mu(r_0 + s_0)\beta^2 - (n+1)\eta = 0$ can be divided by α^2. Because β^4 and α^2 are relatively polynomial in (y^i), we know that $\mu(r_0 + s_0)\beta^2 - (n+1)\eta$ can be divided by α^2, which is impossible unless

$$\mu(r_0 + s_0)\beta^2 - (n+1)\eta = 0. \tag{3.21}$$

From (3.19) we have

$$r_0 = \tau(x)\beta. \tag{3.22}$$

Since $m \neq -1, 0$ and so $\tau(x)b_i + s_i = 0$ is contracting by b^i yields

$$\tau(x)b^2 = 0. \tag{3.23}$$

Because $b^2 \neq 0$ and $m \neq 0, \tau(x) = 0.$ Finally we obtain

$$r_{00} = 0, s_0 = 0, r_0 = 0. \tag{3.24}$$

Hence, by (3.21) finally, we obtain $\eta = 0$.

Step 2. If F is isotropic mean Berwald curvature then β is killing 1-form with b w.r.t α. So the proof as same in the step 1.

Step 3. (3) $\Rightarrow$ (4). By Lemma 2.2 we have $S = 0$ (when $r_{00} = 0, s_0 = 0$).

Step 4. If F has vanishing S-curvature then F is weakly Berwald metric i.e., $E = 0$. It implies that F is of isotropic S-curvature with $c = 0$. It implies that F is of isotropic S-curvature with $c = 0$. Therefore we obtain $E = 0$. Therefore we obtain $E = 0$.

Step 5. If F is weakly-Berwald metric then F is of isotropic S-curvature. $E = 0$ i.e., F is of isotropic mean Berwald curvature with $c = 0$. By the equivalence of (1) and (2), we know that F has isotropic S-curvature with $c = 0$. This completes the proof.

4. Conclusion

We concluded that generalized Kropina metric can be weakly Berwald if it satisfies the Lemma 2.2 [8]. In n-dimensional manifold generalized Kropina metric is weakly-Berwald if and only if $r_{ij} = 0$ and $s_{ij} = 0$. Then, generalized Kropina metric F which satisfied the conditions that F is of isotropic S-curvature, isotropic mean Berwald curvature $S = 0$ and F is weakly Berwald metric.

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