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SCHUR CONVEXITY OF COMPLEMENTARY MEANS WITH RESPECT TO CENTROIDAL MEAN

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Abstract

In this paper, Schur, Schur geometric and Schur harmonic convexities of complementary means of well known classical means with respect to centroidal mean are discussed.

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1. Introduction

The well-known classical means namely; arithmetic, geometric and harmonic means are discussed by Pappus of Alexandria [1]. In Pythagorean school, ten Greek means including the classical means are defined based on proportions. For two positive real numbers α and β; $\mathcal{A}.\mathcal{M} = \mathcal{A}(\alpha, \beta) = \frac{\alpha+\beta}{2}$; $\mathcal{G}.\mathcal{M} = \mathcal{G}(\alpha, \beta) = \sqrt{\alpha\beta}$; $\mathcal{H}.\mathcal{M} = \mathcal{H}(\alpha, \beta) = \frac{2\alpha\beta}{\alpha+\beta}$ and $\mathcal{C}.\mathcal{H}.\mathcal{M} = \mathcal{C}(\alpha, \beta) = \frac{\alpha^2+\beta^2}{\alpha+\beta}$, are respectively called arithmetic, geometric, harmonic and contraharmonic means.

In 1923, Issai Schur introduced the concept of Schur convexity, which has applications in linear regression, analytic inequalities, gamma functions, graph theory and matrices, stochastic orderings, combinatorial optimization, reliability and related fields. In 1958, C. Gini introduced complementary means and G. Toader in 1991 proposed a generalization of complementariness and inversion [17]. In 2003, the concept of "Schur geometrically convex function" was introduced by X. M. Zhang which is nothing but an extension of "Schur convexity function" ([2]-[3]). The Schur geometric convexity for Difference of means, different types of Schur convexity and Schur concavity of generalized Heron means, Schur harmonic and geometric Convexity of Stolarsky's extended mean values are presented in ([4]-[6], [9]). The Schur geometric and Schur harmonic convexity of Gnan mean, invariant contra harmonic mean $(\mathcal{C}.\mathcal{H}.\mathcal{M})$ with respect to $(\mathcal{G}.\mathcal{M})$, Stolarsky and Gini means, the Schur- convexity of generalized Heronian mean, the Schur convex function related to Hadamardtype inequalities and the Schur convexity of extended mean values are discussed in ([7], [8], [10]-[16]). The Schur geometric convexity of an integral $\mathcal{A}.\mathcal{M}$ is given in [19].

2. Definitions and Lemmas

In this section, recalled some definitions and lemmas which are essential to develop the main results of this paper and proved few propositions.

Definition 2.1 [1]**.** For two real numbers $\alpha, \beta \in (1, \infty)$, then well known centroidal mean is given by $\mathcal{CT} = \frac{2(\alpha^2+\alpha\beta+\beta^2)}{3(\alpha+\beta)}$.

Definition 2.2 [18]**.** A mean Q is called complementary to P with respect to S or S-complementary to P, if it verifies $S(P.Q) = S$.

Definition 2.3. Complementary of arithmetic mean with respect to centroidal mean is denoted by $A^{(CT)}$ and is given by $A^{(CT)} = \frac{l_1^2 + l_2^2}{4(l_1 + l_2)} + \frac{\sqrt{(l_1^2 + l_2^2)(5l_1^2 + 5l_2^2 + 8l_1 l_2)}}{l_1 + l_2}$.

Lemma 2.1. *For* $0 < x < y$, (i) $(3x^2 + 4xy - y^2) < \sqrt{(x^2 + y^2)(5x^2 + 5y^2 + 8xy)}$ (ii) $(3y^2 + 4xy - x^2) > \sqrt{(x^2 + y^2)(5x^2 + 5y^2 + 8xy)}$.

Proof: By data $0 < c < y$

Case 1. $y^2 - x^2 > 0, \Rightarrow 4(y^2 - x^2)(y^2 + x^2 + 4xy) > 0$

$$\Rightarrow 4(y^4 - x^4) + 16xy^3 - 16x^3 y > 0$$

$$\Rightarrow 5(x^2 + 5y^2)^2 + 8xy(x^2 + y^2) > (3x^2 + 4xy - y^2)^2$$

$$\Rightarrow (3x^2 + 4xy - y^2) < \sqrt{5(x^2 + 5y^2)^2 + 8xy(x^2 + y^2)}$$

Case 2. By Case 1, $4(y^2 - x^2)(y^2 + x^2 + 4xy) > 0$

$$\Rightarrow (3y^2 + 4xy - x^2)^2 > (x^2 + y^2)(5x^2 + 5y^2 + 8xy)$$

$$\Rightarrow (3y^2 + 4xy - x^2) > \sqrt{(x^2 + y^2)(5x^2 + 5y^2 + 8xy)}.$$

Proposition 2.1. *For* $0 < l_1 < l_2$. *Complementary of arithmetic mean with respect to centroidal mean* $A^{(CT)} = \frac{l_1^2 + l_2^2}{4(l_1 + l_2)} + \frac{\sqrt{(l_1^2 + l_2^2)(5l_1^2 + 5l_2^2 + 8l_1 l_2)}}{l_1 + l_2}$ *is a mean.*

Proof.

Case 1. By the definition

$$l_1 - A^{(CT)} = l_1 - \frac{1}{4}\left[\frac{\{l_1^2 + l_2^2\}}{4\{l_1 + l_2\}} + \frac{\sqrt{(l_1^2 + l_2^2)(5l_1^2 + 5l_2^2 + 8l_1 l_2)}}{l_1 + l_2}\right]$$

$$= \frac{[4l_1(l_1 + l_2) - (l_1^2 + l_2^2) - \sqrt{(l_1^2 + l_2^2)(5l_1^2 + 5l_2^2 + 8l_1 l_2)}]}{4(l_1 + l_2)}$$

$$= \frac{1}{4(l_1 + l_2)} \times F_1(l_1 + l_2).$$

Where $F_1(l_1, l_2) = (3l_1^2 + 4l_1 l_2 - l_2^2) - \sqrt{(l_1^2 + l_2^2)(5l_1^2 + 5l_2^2 + 8l_1 l_2)}$. By lemma 2.1 (i), $(3l_1^2 + 4l_1 l_2 - l_2^2) < \sqrt{(l_1^2 + l_2^2)(5l_1^2 + 5l_2^2 + 8l_1 l_2)}\, F_1(l_1, l_2) < 0$, hence, $l_1 - A^{(CT)} < 0$ or $l_1 < A^{(CT)}$.

Case 2. By the definition

$$A^{(CT)} - l_2 = \frac{1}{4}\left[\frac{l_1^2 + l_2^2}{l_1 + l_2} + \frac{\sqrt{(l_1^2 + l_2^2)(5l_1^2 + 5l_2^2 + 8l_1 l_2)}}{l_1 + l_2}\right] - l_2$$

$$= \frac{[(l_1^2 + l_2^2) + \sqrt{(l_1^2 + l_2^2)(5l_1^2 + 5l_2^2 + 8l_1 l_2)} - 4l_2(l_1 + l_2)}{4(l_1 + l_2)}$$

$$= \frac{1}{4(l_1 + l_2)} \times F_2(l_1 + l_2),$$

where $F_2(l_1, l_2) = \sqrt{(l_1^2 + l_2^2)(5l_1^2 + 5l_2^2 + 8l_1 l_2)} - (3l_1^2 + 4l_1 l_2 - l_1^2)$. By lemma 2.1 (ii), $(3l_2^2 + 4l_1 l_2 - l_1^2) < \sqrt{(l_1^2 + l_2^2)(5l_1^2 + 5l_2^2 + 8l_1 l_2)}\, F_2(l_1, l_2) < 0$. Hence $A^{(CT)} - l_2 < 0$ or $A^{(CT)} < l_2$. By combining the Case 1 and Case 2, verifies $l_1 < A^{(CT)} < l_2, \therefore A^{(CT)}$ is a mean.

Definition 2.4. Complementary of contra harmonic mean with respect to centroidal mean is denoted by $C^{(CT)}$ and is defined by $C^{(CT)} = \frac{1}{2}\left[\frac{l_1 l_2}{l_1 + l_2} + \frac{\sqrt{l_1 l_2(4l_1^2 + 4l_2^2 + l_1 l_2)}}{l_1 + l_2}\right].$

Lemma 2.2. *For* $0 < x < y$ (i) $(2x^2 + xy) < \sqrt{xy(4x^2 + 4y^2 + xy)}$ (ii) $(xy + 2y^2) > \sqrt{xy(4x^2 + 4y^2 + xy)}$.

Proof. By data $x < y$

Case 1. $y^3 - x^3 > 0 \Rightarrow 4xy^3 - 4x^4 > 0 \Rightarrow xy(4x^2 + 4y^2 + xy) > 4x^4 + x^2 y^2 + 4x^3 y (2x^2 + xy) < \sqrt{xy(4x^2 + 4y^2 + xy)}$.

Case 2. By case $1, 4xy^3 - 4x^4 > 0 \Rightarrow x^2 y^2 + 4y^4 + 4xy^3 - 4x^3 y - 4xy^3 - x^2 y^2 > 0 (xy + 2y^2)^2 > 4x^3 y + 4xy^3 + x^2 y^2 \Rightarrow xy + 2y^2 > \sqrt{xy(4x^2 + 4y^2 + xy)}$. Hence the proof.

Proposition 2.2. *For* $0 < l_1 < l_2$, *then* $C^{(CT)} = \frac{1}{2}\left[\frac{l_1 l_2}{l_1 + l_2} + \frac{\sqrt{l_1 l_2 (4l_1^2 + 4l_2^2 + l_1 l_2)}}{l_1 + l_2}\right]$ *is a mean.*

Proof. Case 1. By definition,

$$l_1 - C^{(CT)} = l_1 - \frac{1}{2}\left[\frac{l_1 l_2}{l_1 + l_2} + \frac{\sqrt{l_1 l_2 (4l_1^2 + 4l_2^2 + l_1 l_2)}}{l_1 + l_2}\right]$$

$$= \frac{[2l_1 (l_1 + l_2) - l_1 l_2 - \sqrt{l_1 l_2 (4l_1^2 + 4l_2^2 + l_1 l_2)}]}{2(l_1 + l_2)} = \frac{1}{2(l_1 + l_2)} \times F_3(l_1, l_2).$$

Where $F_3(l_1, l_2) = (2l_1^2 + l_1 l_2) - \sqrt{l_1 l_2 (4l_1^2 + 4l_2^2 + l_1 l_2)}$. By lemma 2.2 (i), $(2l_1^2 + l_1 l_2) < \sqrt{l_1 l_2 (4l_1^2 + 4l_2^2 + l_1 l_2)}$. Hence $F_3(l_1, l_2) < 0 \Rightarrow l_1 - C^{(CT)} < 0$ or $l_1 < C^{(CT)}$.

Case 2. By definition,

$$C^{(CT)} - l_2 = \frac{1}{2}\left[\frac{l_1 l_2}{l_1 + l_2} + \frac{\sqrt{l_1 l_2 (4l_1^2 + 4l_2^2 + l_1 l_2)}}{l_1 + l_2}\right] - l_2$$

$$= \frac{[l_1 l_2 + \sqrt{l_1 l_2 (4l_1^2 + 4l_2^2 + l_1 l_2)} - 2l_2 2(l_1 + l_2)]}{2(l_1 + l_2)}$$

$$= \frac{1}{2(l_1 + l_2)} \times F_4(l_1, l_2),$$

where $F_4(l_1, l_2) = \sqrt{l_1 l_2 (4l_1^2 + 4l_2^2 + l_1 l_2)} - (l_1 l_2 + 2l_2^2)$. By lemma 2.2 (ii), $(l_1 l_2 + 2l_2^2) > \sqrt{l_1 l_2 (4l_1^2 + 4l_2^2 + l_1 l_2)}$. Hence $F_4(l_1, l_2) - l_2 < 0 \Rightarrow C^{(CT)} - l_2 < 0$ or $C^{(CT)} < l_2$. By combining the Case 1 and Case 2, verifies $l_1 < C^{(CT)} < n, \therefore C^{(CT)}$ is a mean.

Definition 2.5. Complementary of harmonic mean with respect to centroidal mean is denoted by $H^{(CT)}$ and is defined by

$$H^{(CT)} = \frac{1}{2}\left[\frac{l_1^2 + l_2^2 - l_1 l_2}{l_1 + l_2} + \frac{\sqrt{(l_1^2 - l_1 l_2 + l_2^2)(l_1^2 + l_2^2 + 7l_1 l_2)}}{l_1 + l_2}\right].$$

Lemma 2.3. *Given* $0 < x < y$,

(i) $(x^2 + 3xy - y^2) < \sqrt{(x^2 + y^2 - xy)(x^2 + y^2 + 7xy)}$

(ii) $(y^2 + 3xy - x^2) < \sqrt{(x^2 + y^2 - xy)(x^2 + y^2 + 7xy)}$.

Proof. By data $x < y$

Case 1. $12xy^2(y - x) > 0 \Rightarrow 12xy^3 - 12x^2y^2 > 0$

$$(x^2 + y^2 - xt)(x^2 + y^2 + 7xy) > (x^2 + 3xy - y^2)^2$$

$$(x^2 + 3xy - y^2) < \sqrt{(x^2 + y^2 - xy)(x^2 + y^2 + 7xy)}.$$

Case 2. Let $x < y \Rightarrow 12x^2y(y - x) > 0$

$$y^4+9x^2y^2+x^4+6xy^3-6x^3y-2x^2y^2>x^4+x^2y^2-x^3y$$

$$+7x^3y+7xy^3-7x^2y^2+x^2y^2+y^4-xy^3$$

$(y^2+3xy-x^2)>\sqrt{(x^2+y^2-xy)(x^2+y^2+7xy)}$. Hence the proof.

Proposition 2.3. *For* $0<l_1<l_2, H^{(CT)}=\frac{1}{2}\left[\frac{l_1^2+l_2^2-l_1l_2}{l_1+l_2}\right.$ $\left.+\frac{\sqrt{(l_1^2-l_1l_2+l_2^2)(l_1^2+l_2^2+7l_1l_2)}}{l_1+l_2}\right]$ *is a mean.*

Proof. Case 1. By definition,

$$l_1-H^{(CT)}=l_1-\frac{1}{2}\left[\frac{l_1^2+l_2^2-l_1l_2}{l_1+l_2}+\frac{\sqrt{(l_1^2-l_1l_2+l_2^2)(l_1^2+l_2^2+7l_1l_2)}}{l_1+l_2}\right]$$

$$=\frac{1}{2(l_1+l_2)}\times F_5(l_1,l_2).$$

Where $F_5(l_1,l_2)=(l_1^2+3l_1l_2)-l_2^2-\sqrt{(l_1^2+l_2^2-l_1l_2)(l_1^2+l_2^2+7l_1l_2)}$. By lemma 2.3 (i), $(l_1^2+3l_1l_2-l_2^2)<\sqrt{(l_1^2+l_2^2-l_1l_2)(l_1^2+l_2^2+7l_1l_2)}$. Hence $F_5(l_1,l_2)<0\Rightarrow l_1<H^{(CT)}$.

Case 2. Consider

$$H^{(CT)}-l_2=\frac{1}{2}\left[\frac{(l_1^2+l_2^2-l_1l_2)}{l_1+l_2}+\frac{\sqrt{(l_1^2-l_1l_2+l_2^2)(l_1^2+l_2^2+7l_1l_2)}}{l_1+l_2}\right]-l_2$$

$$=\frac{F_6(l_1,l_2)}{2(l_1+l_2)},$$

where $F_6(l_1,l_2)=(l_1^2-3l_1l_2-l_2^2)+\sqrt{(l_1^2+l_2^2-l_1l_2)(l_1^2+l_2^2+7l_1l_2)}$. By lemma 2.3 (ii), $\sqrt{(l_1^2+l_2^2-l_1l_2)(l_1^2+l_2^2+7l_1l_2)}>(l_1^2-3l_1l_2-l_2^2)$. Hence $F_6(l_1,l_2)-l_2<0\Rightarrow H^{(CT)}<l_2$. By combining the Case 1 and Case 2, verifies $l_1<H^{(CT)}<l_2$, $\therefore H^{(CT)}$ is a mean.

Definition 2.6 [19, 20]. Let $p = \{(p_1, p_2, \ldots, p_n)\}$ and $q = \{(q_1, q_2, \ldots, q_n)\} \in \mathbb{R}^n$, $\Phi \subseteq \mathbb{R}^n$ the function $\psi : \Phi \to \mathbb{R}$ be said to be a Schur convex function on Φ if $x \leq y$ on Φ implies $\psi(p) \leq \psi(q)$. φ is said to be a Schur concave function on Ω if and only if $-\psi$ is Schur convex.

Definition 2.7 [20]. Let $p = \{(p_1, p_2, \ldots, p_n)\}$ and $q = \{(q_1, q_2, \ldots, q_n)\}$ $\in \mathbb{R}^n_+, \Phi \subseteq \mathbb{R}^n$ is called geometrically convex set if $p_1^\alpha q_1^\beta, \ldots, p_n^\alpha q_n^\beta \in \Phi$ for all p and $q \in \Phi$ where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. Let $\Phi \subseteq \mathbb{R}^n_+$, the function $\psi : \Phi \to \mathbb{R}_+$ is said to be Schur geometrically convex function on Φ if $(\ln p_1, \ldots, \ln p_n) \prec (\ln q_1, \ldots, \ln q_n)$ on Φ implies $\varphi(x) \leq \varphi(y)$. Then φ is said to be a Schur geometrically concave function on Φ if and only if $-\varphi$ is Schur geometrically convex.

Lemma 2.4 [19, 20]. *Let $\Phi \subseteq R^n$ be a symmetric and non empty geometrically interior convex set. Let $\psi : \Phi \to R_+$ be continuous on and differentiable in Φ^0. If ψ is symmetric on Φ and* $(s_1 - s_2)\left(\frac{\partial \psi}{\partial s_1} - \frac{\partial \psi}{\partial s_2}\right) \geq 0 (\leq 0)$,

$(\ln s_1 - \ln s_2)\left(s_1 \frac{\partial \psi}{\partial s_1} - s_2 \frac{\partial \psi}{\partial s_2}\right) \geq 0 (\leq 0)$ *and* $(s_1 - s_2)\left(s_1^2 \frac{\partial \psi}{\partial s_1} - s_2^2 \frac{\partial \psi}{\partial s_2}\right) \geq 0 (\leq 0)$,

holds for any $p = \{(p_1, p_2, \ldots, p_n)\} \in \Phi^0$, then ψ is a Schur convex (*concave*), *Schur-geometrically convex* (*concave*) *and Schur harmonically convex* (*concave*) *function respectively*.

3. Main Results

In this section, various Schur convexities of the complementary of arithmetic mean, contra harmonic and harmonic means with respect to centroidal mean are discussed.

Theorem 3.1. *For $0 < l < m$, complementary of arithmetic mean with respect to centroidal mean is* (i) *Schur convex* (ii) *Schur geometrically convex and* (iii) *Schur harmonically convex*.

Proof. Complementary of arithmetic mean with respect to centroidal mean is denoted by $A^{(CT)}$ and is given by $A^{(CT)} = \frac{1}{4}$ $\left[\frac{l^2+m^2}{l+m} + \frac{\sqrt{(l^2+m^2)(5l^2+5m^2+8lm)}}{l+m}\right]$. The partial derivatives of $A^{(CT)}$ with respect to x and y are given by

$$\frac{\partial A^{(CT)}}{\partial l} = \frac{1}{4}\left[\frac{(l+m)(2l)-(l^2+m^2)}{(l+m)^2} - \frac{\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}}{(l+m)^2}\right]$$

$$+\frac{1}{4}\left[\frac{(10l+8m)(l^2+m^2)+2l(5l^2+8lm+5m^2)}{2(l+m)\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}}\right]$$

and

$$\frac{\partial A^{(CT)}}{\partial m} = \frac{1}{4}\left[\frac{(l+m)(2m)-(l^2+m^2)}{(l+m)^2} - \frac{\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}}{(l+m)^2}\right]$$

$$+\frac{1}{4}\left[\frac{(10l+8m)(l^2+m^2)+2m(5l^2+8lm+5m^2)}{2(l+m)\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}}\right]$$

(i) $$\frac{\partial A^{(CT)}}{\partial l} - \frac{\partial A^{(CT)}}{\partial m} = \frac{(l-m)[2\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}+6l^2+8lm+6m^2]}{4(l+m)\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}}$$

and

$$(l-m)\left(\frac{\partial A^{(CT)}}{\partial l} - \frac{\partial A^{(CT)}}{\partial m}\right)$$

$$= \frac{(l-m)^2\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}+3l^2+4lm+3m^2}{2(l+m)\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}}$$

$$\therefore (l-m)\left(\frac{\partial A^{(CT)}}{\partial l} - \frac{\partial A^{(CT)}}{\partial m}\right) > 0, \text{ for } l,\ m > 0.$$

Hence the complementary of arithmetic mean with respect to centroidal mean is Schur convex.

(ii) $$l\frac{\partial A^{(CT)}}{\partial l} - l\frac{\partial A^{(CT)}}{\partial m} = \frac{(l-m)}{4}\left[\frac{l^2+m^2+4lm-\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}}{(l+m)^2}\right]$$

$$+\frac{(l-y)}{4}\left[\frac{10l^2+8lm+10m^2}{\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}}\right]$$

and $$(\ln l - \ln m)\left(l\frac{\partial A^{(CT)}}{\partial l} - m\frac{\partial A^{(CT)}}{\partial m}\right) = \frac{(\ln l - \ln m)(l-m)\times\phi_1(l, m)}{4(l+m)^2\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}}$$

where $$\phi_1(l, m) = 5(l^4+m^4) + 20lm(l^2+m^2) + 26l^2m^2 + (l^2+m^2+4lm)$$

$\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)} > 0$ $\therefore$ $(\ln l - \ln m)\left(l\frac{\partial A^{(CT)}}{\partial l} - m\frac{\partial A^{(CT)}}{\partial l}\right) > 0$ for $l, m > 0$. Hence the complementary of arithmetic mean with respect to centroidal mean is Schur geometrically convex.

(iii) $$(l-m)\left(l^2\frac{\partial A^{(CT)}}{\partial l} - m^2\frac{\partial A^{(CT)}}{\partial m}\right) = \frac{1}{4}\left[\frac{2(l^3-m^3)(l+m)-(l^2-m^2)(l+m)}{(l+m)^2}\right.$$

$$\left.+\frac{10(l^3-m^3)(l^2+m^2)+8lm(l-m)(l^2+m^2)}{2(l+m)\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}}\right]$$

$$+\frac{1}{4}\left[\frac{2(l^3-m^3)(5l^2+8lm+5m^2)}{2(l+m)\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}} - \frac{(l^3-m^3)\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}}{(l+m)^2}\right]$$

$$= \frac{(l-m)^2\times\phi_2(l, m)}{4(l+m)\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)}},$$

where

$$\phi_2(l, m) = (l+m)^2\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)} - (l^2+m^2)$$

$$(5l^2+8lm+5m^2) + 10l^4 + 10m^4 + 22lm(l^2+m^2) + 14l^2m^2$$

$$\phi_2(l, m) = (l+m)^2\sqrt{(l^2+m^2)(5l^2+8lm+5m^2)} + 5l^4$$

$$+4l^3m + 4l^2m^2 + 4lm^3 + 5m^4 > 0$$

$$\therefore (l-m)\left(l^2\frac{\partial A^{(CT)}}{\partial l} - m^2\frac{\partial A^{(CT)}}{\partial m}\right) > 0, \text{ for } l, m > 0.$$

Hence complementary of arithmetic mean with respect to centroidal mean is Schur harmonically convex.

Theorem 3.2 *For* $0 < l < m$, *complementary of contra harmonic mean with respect to centroidal mean is* (i) *Schur convex* (ii) *Schur geometrically convex and* (iii) *Schur harmonically convex.*

Proof. Complementary of contra harmonic mean with respect to centroidal mean is given in the definition (2.4), and $\frac{\partial C^{(CT)}}{\partial l}$ and $\frac{\partial C^{(CT)}}{\partial m}$ are

$$\frac{\partial C^{(CT)}}{\partial l} = \frac{1}{2}\left[\frac{(l+m)m - lm}{(l+m)^2} + \frac{m(2l^3 + 6l^2m - lm^2 + 2m^3)}{(l+m)^2\sqrt{4l^3m + l^2m^2 + 4lm^3}}\right]$$

and

$$\frac{\partial C^{(CT)}}{\partial l} = \frac{1}{2}\left[\frac{(l+m)l - lm}{(l+m)^2} + \frac{l(2l^3 + 6lm^2 - l^2m + 2m^3)}{(l+m)^2\sqrt{4l^3y + l^2y^2 + 4lm^3}}\right]$$

$$\text{(i)}\ (l-m)\left(\frac{\partial C^{(CT)}}{\partial l} - \frac{\partial C^{(CT)}}{\partial m}\right) = \frac{(l-m)^2}{2(l+m)\sqrt{4l^3m + l^2m^2 + 4lm^3}} \times \phi_3(l, m)$$

where $\phi_3(l, m) = 3lm - 2l^2 - 2m^2 - \sqrt{4l^3m + l^2m^2 + 4lm^3}$.

But $(l-m)^4 > 0 \Rightarrow l^4 - 4l^3m + 4l^2m^2 - 4lm^3 + m^4 > 0$

$$9l^2m^2 + 4l^4 + 4m^4 - 12l^3m + 8l^2m^2 - 12lm^3 > 4l^3m + l^2m^2 + 4lm^3$$

$$(3lm - 2l^2 - 2m^2)^2 > 4l^3m + l^2m^2 + 4lm^3.$$

On taking square root, $3lm - 2l^2 - 2m^2 > \sqrt{4l^3m + l^2m^2 + 4lm^3}$

$$3lm - 2l^2 - 2m^2 - \sqrt{4l^3m + l^2m^2 + 4lm^3} > 0 \Rightarrow \phi_3(l, m) > 0$$

$$\therefore (l-m)\left(\frac{\partial A^{(CT)}}{\partial l} - \frac{\partial A^{(CT)}}{\partial m}\right) > 0.$$

Hence complementary of contra harmonic mean with respect to centroidal mean is Schur convex.

(ii) $(\ln l - \ln m)\left(l\frac{\partial C^{(CT)}}{\partial l} - m\frac{\partial C^{(CT)}}{\partial m}\right) = \frac{lm(\ln l - \ln m)(l-m)}{2(l+m)^2\sqrt{4l^3m + l^2m^2 + 4lm^3}}$

$\times\phi_4(l, m)$ where $\phi_4(l, m) = 7lm - \sqrt{lm(4l^2 + lm + 4m^2)}$.

But $(l-m)^2 < 10lm \Rightarrow l^2 + m^2 - 12lm < 0$.

Multiply $4lm$ on both sides $4l^3m + 4lm^3 - 48l^2m^2 < 0$

$4l^3m + 4lm^3 + l^2m^2 < 49l^2m^2 \Rightarrow \sqrt{lm(4l^2 + lm + 4m^2)} < 7lm$

$7lm - \sqrt{lm(4l^2 + lm + 4m^2)} > 0$ this implies $\phi_4(l, m) > 0$

$$\therefore (\ln l - \ln m)\left(l\frac{\partial C^{(CT)}}{\partial l} - m\frac{\partial C^{(CT)}}{\partial m}\right) > 0 \text{ for } 1, m > 0.$$

Hence the complementary of contra harmonic mean with respect to centroidal mean is Schur geometrically convex.

(iii) $l^2\frac{\partial C^{(CT)}}{\partial l} = \frac{1}{2}\left[\frac{(l+m)l^2m - l^3m}{(l+m)^2} + \frac{l^2m(2l^3 + 6l^2m - lm^2 + 2m^3)}{(l+m)^2\sqrt{4l^3m + l^2m^2 + 4lm^3}}\right]$

and

$$m^2\frac{\partial C^{(CT)}}{\partial l} = \frac{1}{2}\left[\frac{(l+m)lm^2 - lm^3}{(l+m)^2} + \frac{lm^2(2l^3 + 6lm^2 - l^2m + 2m^3)}{(l+m)^2\sqrt{4l^3m + l^2m^2 + 4lm^3}}\right]$$

$$\therefore l^2\frac{\partial C^{(CT)}}{\partial l} - m^2\frac{\partial C^{(CT)}}{\partial m} = \frac{1}{2}\left[\frac{lm \times \phi_5(l, m)}{(l+m)^2\sqrt{4l^3m + l^2m^2 + 4lm^3}}\right] \text{ where}$$

$\phi_5(l, m) = l(2l^3 + 6l^2m - lm^2 + 2m^3) - m(2l^3 + 6lm^2 - l^2m + 2m^3)$

$= 2(l-m)(l+m)^3$

$$\therefore (l-m)\left(l^2\frac{\partial C^{(CT)}}{\partial l} - m^2\frac{\partial C^{(CT)}}{\partial m}\right) = \frac{1}{2}\left[\frac{lm(l+m)(l-m)^2}{\sqrt{4l^3m + l^2m^2 + 4lm^3}}\right] > 0, \text{ for } l, m > 0.$$

Hence complementary of contra harmonic mean with respect to centroidal mean is Schur harmonically convex.

Theorem 3.3. *For* $0 < l < m$, *complementary of harmonic mean with respect to centroidal mean is* (i) *Schur convex* (ii) *Schur geometrically convex and* (iii) *Schur harmonically convex.*

Proof. Complementary of harmonic mean with respect to centroidal mean is given in definition (2.5), the expressions for $\frac{\partial H^{(CT)}}{\partial l}$ and $\frac{\partial H^{(CT)}}{\partial m}$ are

$$\frac{\partial H^{(CT)}}{\partial l} = \frac{2l - m}{l + m} - \frac{l^2 - lm + m^2}{(l + m)^2} - \frac{\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}{(l + m)^2}$$

$$+ \frac{(2l + 7m)(l^2 - lm + m^2) + (2l - m)(l^2 + 7lm + m^2)}{2(l + m)\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}$$

and

$$\frac{\partial H^{(CT)}}{\partial l} = \frac{2m - l}{l + m} - \frac{l^2 - lm + m^2}{(l + m)^2} - \frac{\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}{(l + m)^2}$$

$$+ \frac{(7l + 2m)(l^2 - lm + m^2) + (2m - l)(l^2 + 7l^2m^2 + m^2)}{2(l + m)\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}$$

(i) $$(l-m)\left(\frac{\partial H^{(CT)}}{\partial l} - \frac{\partial H^{(CT)}}{\partial m}\right) = \frac{(l-m)^2 \times \phi_6(l, m)}{(l+m)\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}$$

where $\phi_6(l, m) = 3\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)} - (l^2 - 13lm + m^2)$. For real value of $\phi_6(l, m)$ with $0 < l < m \Rightarrow l^2 + m^2 - lm > 0 \Rightarrow l^2 + m^2 > lm$ $l^2 + m^2 - 13lm < lm - 13lm > -12lm$

$\therefore \phi_6(l, m) = 3\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)} - (l^2 - 13lm + m^2)$
$< 3\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)} + 12lm\, \phi_6(l, m) > 0$ this implies
$(l - m)\left(\frac{\partial A^{(CT)}}{\partial l} - \frac{\partial A^{(CT)}}{\partial m}\right) > 0$, for $l, m > 0$.

Hence the complementary of harmonic mean with respect to centroidal mean is Schur convex.

(ii) $$l\frac{\partial H^{(CT)}}{\partial l} = \frac{2l^2 - lm}{l+m} - \frac{l(l^2 - lm + m^2)}{(l+m)^2} - \frac{l\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}{(l+m)^2}$$

$$+ \frac{(2l^2 + 7lm)(l^2 - lm + m^2) + (2l^2 - lm)(l^2 + 7lm + l^2)}{2(l+m)\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}$$

and

$$m\frac{\partial H^{(CT)}}{\partial l} = \frac{2m^2 - lm}{l+m} - \frac{m(l^2 - lm + m^2)}{(l+m)^2} - \frac{m\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}{(l+m)^2}$$

$$+ \frac{(7lm + 2m^2)(l^2 - lm + m^2) + (2m^2 - lm)(l^2 + 7lm + m^2)}{2(l+m)\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}$$

$$\therefore (\ln l - \ln m)\left(l\frac{\partial H^{(CT)}}{\partial l} - m\frac{\partial H^{(CT)}}{\partial m}\right) = \frac{(\ln l - \ln m)(l-m) \times \phi_7(l, m)}{(l+m)^2\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}$$

where

$$\phi_7(l, m) = (l^2 + m^2 + 5lm)\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}$$

$$+ l^4 + 4l^3m + 21l^2m^2 + 4lm^3 + m^4 > 0$$

$$\therefore (\ln l - \ln m)\left(l\frac{\partial H^{(CT)}}{\partial l} - m\frac{\partial H^{(CT)}}{\partial m}\right) > 0, \text{ for } l, m > 0.$$

Hence the complementary of harmonic mean with respect to centroidal mean is Schur geometrically convex.

(iii)

$$l^2\frac{\partial H^{(CT)}}{\partial l} = \frac{2l^3 - l^2m}{l+m} - \frac{l^2(l^2 - lm + m^2)}{(l+m)^2} - \frac{l^2\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}{(l+m)^2}$$

$$+ \frac{(2l^3 + 7l^2m)(l^2 - lm + m^2) + (2l^3 - l^2m)(l^2 + 7lm + m^2)}{2(l+m)\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}$$

and

$$m^2 \frac{\partial H^{(CT)}}{\partial l} = \frac{2m^3 - lm^2}{l+m} - \frac{m^2(l^2 - lm + m^2)}{(l+m)^2} - \frac{y^2\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}{(l+m)^2}$$

$$+ \frac{(7lm^2 + 2m^3)(l^2 - lm + m^2) + (2m^3 - lm^2)(l^2 + 7lm + m^2)}{2(l+m)\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}$$

$$\therefore l^2 \frac{\partial H^{(CT)}}{\partial l} - m^2 \frac{\partial H^{(CT)}}{\partial m} = \left[\frac{(l-m)\phi_8(l, m)}{(l+m)\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}} \right]$$

$$\therefore (l-m)\left(l^2 \frac{\partial C^{(CT)}}{\partial l} - m^2 \frac{\partial C^{(CT)}}{\partial m} \right) = \frac{(l-m)^2 \times \phi_8(l, m)}{(l+m)\sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}}$$

Where

$$\phi_8(l, m) = (l+m)^2 \sqrt{(l^2 - lm + m^2)(l^2 + 7l^2m^2 + m^2)}$$

$$+ (l^4 + 5l^3m + 8l^2m^2 + 5lm^3 + y^4) > 0$$

$$\therefore (l-m)\left(l^2 \frac{\partial H^{(CT)}}{\partial l} - m^2 \frac{\partial H^{(CT)}}{\partial m} \right) > 0.$$

Hence complementary of harmonic mean with respect to centroidal mean is Schur harmonically convex.

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