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ON COMPLETENESS, SELF CENTEREDNESS AND HEREDITARY PROPERTIES OF μ^n – FUZZY GRAPHS

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Abstract

To avoid some limitations of fuzzy graphs, we introduced an advanced type of fuzzy graph named μ^n – fuzzy graphs (MFG) in [7]. Molecules with Covalent bond can easily converted into μ^n – fuzzy graphs, the model is introduced, completeness, self centeredness and hereditary properties of μ^n – fuzzy graphs are defined and various properties and theorems related to it are discussed in this paper.

1. Introduction

Graph theory is now a key component of applications in maths and is considered as combinatoric division. A graph is generally utilized device for fixing complex troubles in various fields in all major branches of mathematics. The essential aspect to notice is, whilst we have doubt about vertex/edge/vertex-edge, representation suit with fuzzy-graph. To overcome some limitations of fuzzy graph, authors already established a latest version of fuzzy-graph, namely μ^n – fuzzy graphs (MFG). In this paper, authors introduced various properties and theorems of μ^n – fuzzy graphs. Figure shows a concrete specimen of a fuzzy-graph.

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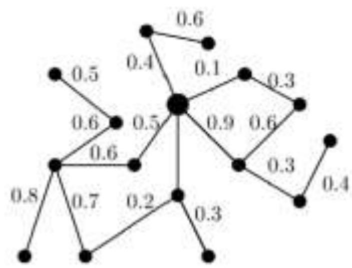
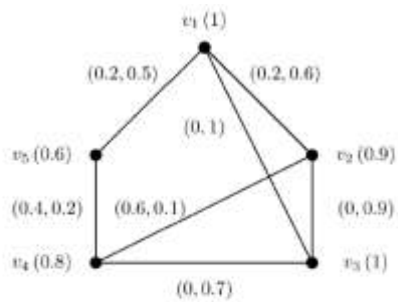


People are represented as vertices in the instance, and particular values had been assigned to them based totally on their friendship. If there is no relationship, we presume the value is 0, and if the friendship is extraordinarily tight, the value will be near to one i.e., relying on the degree of the friendship, all values might be between zero and 1. But one of the drawbacks we experience with this graph is that we aren't able to perceive the traits based on which the value is given. In brief, we are not capable of examine the procedure of assessment i.e., a single real number isn't enough to define a relation between two persons. This dilemma may be overcome with the introduction of a new concept namely MFG. On this model edge membership value is given as n-tuple rather than single real number. So that the above said drawback of fuzzy graph is overcomes the usage of this. Many authors, including Rosenfeld [6], Bhutani [1], Sunil Mathew [2], and Sunitha [9], established many connectivity notions in fuzzy graphs as a result of Zadeh [12] [13], Yeh and Bang's [11] work. More related work can be seen in [10], [8], [5], [3], [4].

A fuzzy-graph is a 3-tuple $K : (M, \sigma, \mu)$, M is set of vertices, σ is fuzzy subset of M and μ is a fuzzy relation on σ such that $\mu(l, m) \leq \sigma(l) \wedge \sigma(m)$ for all $l, m \in M$, where $\wedge$ represent the minimum. We take M is finite, non-empty, μ is reflexive and symmetric. In the examples σ is pick preferably. Next, we recollect the basic definitions in MFGs.

Definition 1.1. A μ^n – fuzzy-graph $K = (M, \sigma, \mu)$ is an algebraic model of non void set M with set of two functions $\sigma : M \to [0, 1]$ and $\mu : M \times M \to [0, 1]^n$: for all $l, m \in M, \mu_i(l, m) \leq \sigma(l) \wedge \sigma(m)$ for $i = 1, 2, \ldots, n,$ where μ_i is the i-th co-ordinate value or projection value of μ, i.e., if $\mu(l, m) = (u_1, u_2, \ldots, u_n), 0 \leq u_i \leq 1, \forall i$ varies from one to n, then $\mu_i(l, m) = u_i$ for $i = 1, 2, \ldots, n$, μ is a symmetric fuzzy relation on σ and $\wedge$ represent the minimum. Here $\sigma(l)$ and $u(l, m)$ denote membership values of the vertex l and of the edge (in n tuple) (l, m) in K appropriately.

For n equals 1, it is simply a fuzzy-graph, for $n = 2$, σ from M to $[0, 1]$ and $\mu : M \times M \to [0, 1] \times [0, 1]$: for all $l, m \in M$, $\mu_1(l, m) \leq \sigma(l) \wedge \sigma(m)$ and $\mu_2(l, m) \leq \sigma(l) \wedge \sigma(m)$. For example, consider the graph $K = (M, \sigma, \mu)$, $M = \{a_1, a_2, a_3, a_4, a_5\}$, $\sigma(a_1) = 1$, $\sigma(a_2) = 0.9$, $\sigma(a_3) = 1$, $\sigma(a_4) = 0.8$, $\sigma(a_5) = 0.6$ and the edge membership values are $\mu(a_1, a_2) = (0.2, 0.6)$, $\mu(a_2, a_3) = (0.0, 0.9)$, $\mu(a_3, a_4) = (0.0, 0.7)$, $\mu(a_4, a_5) = (0.4, 0.2)$, $\mu(a_5, a_1) = (0.2, 0.5)$, $\mu(a_1, a_3) = (0.0, 1.0)$, $\mu(a_4, a_2) = (0.6, 0.1)$.



Definition 1.2. The weight of an edge in a MFG, $K = (M, \sigma, \mu)$ is defined as the norm of the membership value of that edge. i.e., weight of the edge (l, m), $w(l, m) = \| \mu(l, m) \|$, where $\| \cdot \|$ is a norm on $\mathbb{R}^n$, we can choose suitable norms according to our requirements.

Definition 1.3. Given a MFG, $K = (M, \sigma, \mu)$, the underlying crisp graph is defined as $K^* : (\sigma^*, \mu^*)$ where $\sigma^* = \{l \in M : \sigma(l) > 0\}$ and $\mu^* = \{(l, m) \in M \times M : w(l, m) > 0\}$.

Definition 1.4. A path in K is a sequence of vertices $x_0, x_1, \ldots, x_m$, such that $w(x_{i-1}, x_i) > 0$ for i varies over 1 to m, having length m. Nodes which can be joined by way of a route are said to be connected.

Definition 1.5. Let $K = (M, \sigma, \mu)$ is a MFG, then the $\mu^n -$ strength of the path $P : u_1 - u_2 - \ldots - u_n = e_1 - e_2 - \ldots - e_{n-1}$ is defined by $S_\mu(P) = \sum_{i=1}^{n-1} \| \mu(u_i, u_{i+1}) \|$ or $S_\mu(P) = \sum_{i=1}^{n-1} \| \mu(e_i) \|$ where $u_1, u_2, \ldots, u_n$ are the vertices of the path P and $e_1, e_2, \ldots, e_{n-1}$ are the corresponding edges and $\| \cdot \|$ is a norm on $\mathbb{R}^n$, we can choose suitable norms according to our requirements.

Definition 1.6. Let $K : (M, \sigma, \mu)$ is a MFG, $\mu^n -$fuzzy distance (MFD) among pair of vertex l and m in K is defined to be $d_\mu(l, m) = \wedge_P \{S_\mu(P)/P$ is a $l - m$ path$\}$, where $S_\mu(P)$ be $\mu^n -$strength of the path $P, \wedge$ represents minimum.

Remark 1.1. For a path $P : u_1 - u_2 - \ldots - u_n$ in MFG, $K : (M, \sigma, \mu)$, $\| \mu(u_i, u_{i+1}) \| = 0$ for some i iff $u_1 = u_n$.

Definition 1.7. The $\mu^n -$fuzzy-graph $\chi = (\tau, \nu)$ is said to be a $\mu^n -$fuzzy subgraph of $\xi = (\sigma, \mu)$ if $\tau(l) = \sigma(l) \; \forall \, l \in \tau^*$ and $\nu_i(l, m) = \mu_i(l, m)$, $i = 1, 2, \ldots, n$ for all edges $(l, m) \in \nu^*, \, l, m \in M$.

Definition 1.8. A $\mu^n -$fuzzy-graph $I : (\tau, \nu)$ is said to be a partial $\mu^n -$fuzzy subgraph of $K : (\sigma, \mu)$ if $\tau(l) = \sigma(l) \; \forall \, l \in \tau^*$ and $\nu_i(l, m) \le \mu_i(u, v)$ $\forall (l, m) \in \nu^*$ for $i = 1, 2, \ldots, n$ and in addition $\tau^* = \sigma^*$, then I is called a spanning $\mu^n -$fuzzy subgraph of K.

Definition 1.9. The strength of connectedness across two nodes l and m is defined as the largest of the $\mu^n -$strengths of all paths connecting l and m and is denoted by $\mu^\infty(l, m)$ or $CONNK(l, m)$. All over, we suppose K is connected. An $l - m$ path P is called a strongest $l - m$ path if its strength equals $CONNK(l, m)$.

Definition 1.10. Let $I - (l, m)$ is MFG get from I by replacing $\mu_i(l, m)$

by 0 for $i = 1, 2, \ldots, n$. Arc (l, m) strong in I if $\mu_i(l, m) > 0$ for some $i = 1, 2, \ldots, n$ and $\| \mu(l, m) \| \geq CONN_{I-(l, m)}(l, m)$, otherwise the arc (l, m) is called weak.

Definition 1.11. Let $K = (M, \sigma, \mu)$ is a connected MFG, the μ^n – fuzzy eccentricity (MFE) of a vertex $l \in M$ is defined, and denoted by $e_\mu(l) = \vee_{m \in M} \{d_\mu(l, m)\}$, where $\vee$ represents larger. The set of all μ^n – fuzzy eccentric (MFE) nodes of a node l is denoted by l^*. K is a unique eccentric node (u.e.n) MFG if each node in K has a unique eccentric node.

Definition 1.12. The smallest of the MFE's of all vertices is said to be the μ^n – radius of the graph K. It is denoted as $r_\mu(K)$. Thus $r_\mu(K) = \wedge_{u \in M} \{e_\mu(u)\}$.

Definition 1.13. Largest of the μ^n – fuzzy eccentricities of all the vertex is said to be μ^n – fuzzy diameter of the graph K. It is denoted as $d_\mu(K)$. i.e., $d_\mu(K) = \wedge_{u \in M} \{e_\mu(u)\}$.

Definition 1.14. A vertex $l \in M(G)$ is said to be a MFE node of different vertex m if $e_\mu(l) = d_\mu(l, m)$.

Definition 1.15. A vertex l is a central vertex or μ^n – fuzzy radial vertex if $e_\mu(l) = r_\mu(K)$ (vertex with minimum MFE), and $C_\mu(K)$ is the collection of central vertices. The μ^n – fuzzy sub graph induced by $C_\mu(K)$ indicated by $\langle C_\mu(K) \rangle = I : (V, \tau, \nu)$ is said to be μ^n – centre of K. A non-disconnected MFG K is self-centered if every vertex is μ^n – central vertex i.e. $K \approx I$. A node u is a peripheral-vertex or μ^n – fuzzy diametral nodes if $e_\mu(l) = d_\mu(K)$ (nodes with maximum MFE).

2. μ^n – fuzzy self centered graphs

We talk about the properties of μ^n – self-centered (FSC) graphs in this section.

Definition 2.1. A MFG, K is said to be fuzzy self-centered (FSC), if it is isomorphic with its μ^n – fuzzy center.

Next, we discuss a necessary condition for a MFG, K to be FSC.

Theorem 2.1. *A MFG, $K = (M, \sigma, \mu)$, which is connected is FSC, then all vertex of K is μ^n – fuzzy-eccentric (MFE).*

Proof of Theorem 2.1. Anticipate that the MFG, K is FSC. Need to prove, every vertex of K is MFE l be any random vertex of K and let $m \in l^*$. By denition of a MFE node, $e_\mu(l) = d_\mu(K)$. Yet we know K is FSC, $e_\mu(l) = e_\mu(m)$ and so $e_\mu(l) = d_\mu(l, m) = d_\mu(l, m)$. So, l is MFE vertex of m. So, all vertex of K is MFE.

We introduce another necessary condition in the following theorem.

Theorem 2.2. *A MFG $G : (V, \sigma, \mu)$, which is connected is FSC, then for all couple of vertices l, m : whenever l is a MFE vertex of m, then m should be one of the MFE nodes of l.*

Proof of Theorem 2.2. Suppose $K = (M, \sigma, \mu)$ is a FSC. And suppose l is MFE node of m. That is, $e_\mu(m) = d_\mu(m, l)$. We know K is FSC, every vertex have the same μ^n – fuzzy eccentricity. Therefore $e_\mu(m) = e_\mu(l)$. From the above two equations, $e_\mu(l) = d_\mu(m, l) = d_\mu(l, m)$. Thus $e_\mu(u) = d_\mu(l, m)$. i.e., m is a MFE node of l.

This is not a sufficient result.

3. Completeness and Hereditary Properties of MFG

Bhutani develop idea of complete fuzzy graph in 1989. A complete fuzzy graph (CFG) is a fuzzy graph $G = (\sigma, \mu)$ such that $\mu(uv) = \sigma(u) \wedge \sigma(v)$ $\forall u, v \in \sigma^*$. If the graph so, $\mu^{\infty} = \mu$ and it has no fuzzy-cut vertices.

Definition 3.1. A norm complete MFG (NCMFG) is $\mu^n -$fuzzy graph $K = (M, \sigma, \mu) : \| \mu(l, m) \| = \sigma(l) \wedge \sigma(m) \ \forall l, m \in \sigma^*$.

Theorem 3.1. *If $K = (M, \sigma, \mu)$ is NCMFG, for every edge $lm \in \mu^*$, $\mu^{\infty}(l, m) = \| \mu(lm) \|$.*

Proof of Theorem 3.1. We have, $\mu^2(l, m) = \vee_{n \in \sigma^*} \{\| \mu(lm) \| \wedge \| \mu(nm) \|\}$ $= \vee_{n \in \sigma^*} \{\sigma(l), \sigma(m), \sigma(n)\}$. There are two cases, $\sigma(n) \leq \sigma(l) \wedge \sigma(m)$ and $\sigma(n) > \sigma(l) \wedge \sigma(m)$.

Case (i). $\sigma(n) \leq \sigma(l) \wedge \sigma(m)$. $\mu^2(l, m) = \vee_{n \in \sigma^*} \{\sigma(n) \wedge \sigma(l) \wedge \sigma(m)\}$ $= \vee_{n \in \sigma^*} \{\sigma(n)\} \leq \sigma(l) \wedge \sigma(m) = \| \mu(lm) \|$ i.e., $\mu^2(l, m) \leq \| \mu(lm) \|$.

Case (ii). $\sigma(n) \leq \sigma(l) \wedge \sigma(m)$. $\mu^2(l, m) = \vee_{n \in \sigma^*} \{\sigma(n) \wedge \sigma(l) \wedge \sigma(m)\}$ $= \vee_{n \in \sigma^*} \{\sigma(n)\} \leq \sigma(l) \wedge \sigma(m) = \| \mu(lm) \|$ i.e., $\mu^2(l, m) \leq \| \mu(lm) \|$ from these two cases, we can conclude that $\mu^2(l, m) \leq \| \mu(lm) \|$. In the same way, $\mu^3(l, m) \leq \| \mu(lm) \|$ and in general $\mu^k(l, m) \leq \| \mu(lm) \| \ \forall k \in \mathbb{Z}$ and k is positive. So, $\mu^{\infty}(l, m) = \sup \mu^k(l, m) : \forall$ integers $k \geq 1 = \| \mu(lm) \|$.

Corollary 3.2. *A norm complete MFG has no fuzzy cut vertices*.

Definition 3.2. A property P of a MFG is hereditary provided that if $K : (M, \sigma, \mu)$ has property P, then every coordinate MFG, $G' = (V, \sigma, \mu_i)$, $i = 1, 2, \ldots, n$ has property P.

Definition 3.3. A complete-MFG is a MFG, $K : (\sigma, \mu) : \mu_i(lm)$ $= \sigma(l) \wedge \sigma(m)$ for all $l, m \in \sigma^*$ and $i = 1, 2, \ldots, n$.

Theorem 3.3. *Being complete is hereditary in a MFG*, $K : (M, \sigma, \mu)$.

Proof of Theorem 3.3. We have MFG, $K : (M, \sigma, \mu)$ is complete, then

$\mu_i(l, m) = \sigma(l) \wedge \sigma(m), \forall l, m \in \sigma^*$ for $i = 1, 2, \ldots, n.$ i.e.,

$\mu_1(l, m) = \sigma(l) \wedge \sigma(m), \forall l, m \in \sigma^*$ (1)

$\mu_2(l, m) = \sigma(l) \wedge \sigma(m), \forall l, m \in \sigma^*$ (2)

....

$\mu_n(l, m) = \sigma(l) \wedge \sigma(m), \forall l, m \in \sigma^*$ (n)

(1) $\Rightarrow (M, \sigma, \mu_1)$ is a complete-MFG.

(2) $\Rightarrow (M, \sigma, \mu_2)$ is a complete-MFG.

....

(n) $\Rightarrow (M, \sigma, \mu_n)$ is a complete-MFG.

i.e., completeness is a hereditary property.

Converse also true, i.e., if $(M, \sigma, \mu_i), i = 1, 2, \ldots, n$ are all complete, then (M, σ, μ) is complete.

Theorem 3.4. *If* $(M, \sigma, \mu_i), i = 1, 2, \ldots, n$ *are all complete MFGs, then* $K : (M, \sigma, \mu)$ *is complete MFG.*

Proof of Theorem 3.4. Given, $(M, \sigma, \mu_i),$ where i varies over 1 to n are complete fuzzy graphs, then

$\mu_1(l, m) = \sigma(l) \wedge \sigma(m), \forall l, m \in \sigma^*$ (1)

$\mu_2(l, m) = \sigma(l) \wedge \sigma(m), \forall l, m \in \sigma^*$ (2)

....

$\mu_n(l, m) = \sigma(l) \wedge \sigma(m), \forall l, m \in \sigma^*$ (n)

$\Rightarrow \mu_i(l, m) = \sigma(l) \wedge \sigma(m), \forall l, m \in \sigma^*$ for $i = 1, 2, \ldots, n$

$\Rightarrow$ MFG $K : (M, \sigma, \mu)$ is complete.

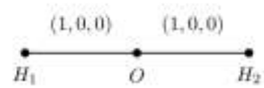
Hence the theorem.

4. Representation of Molecules with Covalent Bonding

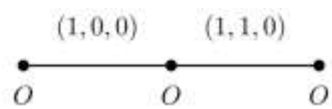
A new representation of molecules with Covalent bonding is introduced in this section. We can represent molecules with covalent bond as μ^n – fuzzy graphs, the representation is given below.

Replace atoms with vertex of the MFGs, if there are covalent bonding between any two atoms, we can add an edge between them and the edge membership value is $(1, 0, 0)$ if there is only a single bond, $(1, 1, 0)$ if there is double bond and $(1, 1, 1)$ if there is triple bond. All vertex membership values fixed to be 1. A single line in molecular representation denote single-bond among a pair of atoms (contains 1 couple of electron), = denote a double bond between two atoms (contains 2 couple of electron), and ≡ denote a triple bond (contains 3 couple of electron).

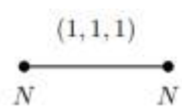
Consider H_2O, the molecular representation is $H - O - H$, the corresponding MFG representation is (for avid confusion, the two hydrogen atoms labelled H_1 and H_2)



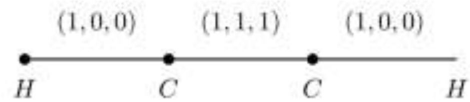
For Ozone (O_3), the structure is $O - O - O$, the corresponding MFG representation is



For nitrogen (N_2), the structure is $N \equiv N$, the corresponding MFG representation is



Consider acetylene (C_2H_2), the structure is $H - C \equiv C - H$, the corresponding MFG representation is



If the edge $e = (v_1, v_2)$ in MFG representation of a molecule, $\| \mu(v_1, v_2) \|^2$ gives the bonding between v_1 and v_2, where $\| \cdot \|$ represent usual norm on $\mathbb{R}^n$.

5. Conclusion

In this paper, the author's discussed some properties and theorems in MFG. The idea of $\mu^n -$fuzzy self-centered graphs, Completeness and hereditary properties of MFG are defined and discussed, a MFG representation of molecules with covalent bonding is introduced.

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