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AN INITIAL VALUE PROBLEM FOR A COUPLED SYSTEM OF SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS WITH ROBIN INITIAL CONDITIONS AND DIFFERENT PERTURBATION PARAMETERS

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Abstract

In this paper, a coupled system of singularly perturbed delay differential equations with Robin Initial Conditions is considered. A Shishkin piecewise uniform mesh is implemented and combined with a classical finite difference method to create a numerical method for solving this problem. The numerical approximations obtained are essentially first order convergence uniformly with respect to the singular perturbation parameters. Numerical result is provided in support of the theory.

1. Introduction

In this section, a coupled system of singularly perturbed delay differential equations with Robin Initial Conditions is considered

$$\vec{L}\vec{u}(x) = E\vec{u}'(x) + A(x)\vec{u}(x) + B(x)\vec{u}(x-1) = \vec{f}(x) \text{ on } (0, 2] \tag{1.1}$$

$$\vec{\beta}\vec{u}(x) = \vec{u}(x) - E\vec{u}'(x) = \vec{l}(x) \text{ where } x \in \Omega^* = [-1, 0] \tag{1.2}$$

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For all $x \in [0, 2]$, $\vec{u}(x) = (u_1(x), u_2(x))^T$ and $\vec{f}(x) = (f_1(x), f_2(x))^T \cdot E$, $A(x)$ and $B(x)$ are 2×2 matrices. $E = diag(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ with $0 < \varepsilon_1 \leq \varepsilon_2 < 1$, $B(x) = diag(\vec{b})$, $\vec{b} = (b_1(x), b_2(x))$. For all $x \in [0, 2]$, the components $a_{ij}(x)$ and $b_j(x)$ of $A(x)$ and $B(x)$ are expected to satisfy

$$b_j(x),\ a_{ij}(x) \leq 0 \text{ for } 1 \leq i \neq j \leq 2 \text{ and } a_{ii}(x) > \sum_{i \neq j} | a_{ij}(x) + b_i(x) | \tag{1.3}$$

$$\text{and } 0 < a < \min_{x \in [0, 2]} \left(\sum_{j=1}^{2} a_{ij}(x) + b_i(x)\right) \tag{1.4}$$

Furthermore, the functions $f_i(x)$, $a_{ij}(x)$, $b_i(x)$, $\leq i$, $j \leq 2$ are considered to be in $C^2([0, 2])$.

Based on the foregoing assumptions, $\vec{u} \in C^2$ where $C = C^0([-1, 2]) \cap C^1([0, 2]) \cap C^2((0, 1) \cup (1, 2])$. The problems (1) and (2) can be rephrased as follows;

$$\vec{L}\vec{u}(x) = \begin{cases} \vec{L}_1\vec{u}(x) = \vec{E}\vec{u}'(x) + A(x)\vec{u}(x) = \vec{g}(x) \text{ on } (0, 1] \\ \vec{L}_2\vec{u}(x) = \vec{E}\vec{u}'(x) + A(x)\vec{u}(x) + B(x)\vec{u}(x-1) = \vec{g}(x) \text{ on } (1, 2] \end{cases} \tag{1.5}$$

where $\vec{g}(x) = \vec{f}(x) - B(x)\vec{\phi}(x-1)$ and $\vec{u}(0) = \vec{\phi}(0)$. The reduced problem that corresponds to (1.5) is

$$\begin{cases} A(x)\vec{u}_0(x) = \vec{f}(x) - B(x)\vec{\phi}(x-1) \text{ on } (0, 1] \\ A(x)\vec{u}_0(x) + B(x)\vec{\phi}(x-1) = \vec{f}(x) \text{ on } (1, 2]. \end{cases}$$

2. Analytical Results

Lemma 2.1 Maximum Principle. *If* $\vec{\zeta}$ *is any function in the domain of* $\vec{L}$ *such that* $\vec{\beta}\vec{\zeta}(v) \geq \vec{0}$, *then* $\vec{L}\vec{\zeta}(x) \geq \vec{0}$ *on* $(0, 2]$ *implies that* $\vec{\zeta}(x) \geq \vec{0}$ *on* $[0, 2]$.

Proof. Let $\zeta_{i^*}(x^*) = \min_{i, x} \zeta_i(x)$, $i = 1, 2$ and assume that $\zeta_{i^*}(x^*) < 0$. Without loss of generality, let $i^* = 1$. By the hypothesis, $x^* \neq 0$ and note that $\zeta_{1'}(x^*) = 0$.

If $x^* = 0$,

$$(\vec{\beta}\vec{\zeta})_1(x^*) = \zeta_1(x^*) - \varepsilon_i \zeta_1'(x^*)$$

< 0, a contradiction.

$$\therefore x^* \neq 0.$$

Suppose $x^* \in \Omega = (0, 1]$, then

$$\begin{aligned}(\vec{L}\vec{\zeta})_1(x^*) = (\vec{L}_1\vec{\zeta})_1(x^*) &= \varepsilon_1 \zeta_{1'}(x^*) + a_{11}(x^*)\zeta_1(x^*) + a_{12}(x^*)\zeta_2(x^*) \\ &\leq (a_{11} + a_{12})(x^*)\zeta_1(x^*) \\ &< 0,\end{aligned}$$

which is a contradiction.

Suppose $x^* \in \Omega^+ = (1, 2]$,

$$\begin{aligned}(\vec{L}\vec{\zeta})_1(x^*) &= (\vec{L}_2\vec{\zeta})_1(x^*) \\ &= \varepsilon_1 \zeta_{1'}(x^*) + a_{11}(x^*)\zeta_1(x^*) + a_{12}(x^*)\zeta_2(x^*) + b_1(x^*)\zeta_1(x^* - 1) \\ &\leq (a_{11} + a_{12})(x^*)\zeta_1(x^*) + b_1(x^*)\zeta_1(x^*) < 0,\end{aligned}$$

which is a contradiction. As a result, the inference is incorrect. Therefore, $\zeta_{i^*}(x^*) \geq 0$, which establishes the lemma.

As a direct result of the above lemma, the stability result is defined as follows.

Lemma 2.2. *Let* $\vec{\zeta}$ *is any function in the domain of* $\vec{L}$, *such that for each* $x \in [0, 2]$, *then for each* i, $1 \leq i \leq 2$,

$$|\vec{\zeta}(x)| \leq C \max\{\|\vec{\beta}\vec{\zeta}(0)\|, \frac{1}{\alpha}\|\vec{L}\vec{\zeta}\|\}.$$

Proof. Consider the barrier functions

$$\vec{\Lambda}^{\pm}(x) = CM \pm \vec{\zeta}$$

where $M = \max\{\| \vec{\beta}\vec{\zeta}(0) \|, \frac{1}{\alpha}\| \vec{L}\vec{\zeta} \|\}$. Then it is not difficult to show that $\vec{\beta}\vec{\Lambda}^{\pm}(0) \geq \vec{0}$ and $\vec{L}\vec{\Lambda}^{\pm}(0) \geq \vec{0}$ on Ω. From lemma (2.1), it follows that $\vec{\Lambda}^{\pm}(0) \geq \vec{0}$ on $\overline{\Omega}$. Hence $|\vec{\zeta}(x)| \leq C \max\{\| \vec{\beta}\vec{\zeta}(0) \|, \frac{1}{\alpha}\| \vec{L}\vec{\zeta} \|\}$.

Lemma 2.3. *Let* $\vec{u}$ *be the solution of* (1.1), (1.2). *Then, there exists a constant C such that for each* $i = 1, 2$, $x \in (0, 2]$, *we have*

$$| u_i(x) | \leq C\{\| \vec{l} \| + \| \vec{f} \|\}$$

$$| u_i'(x) | \leq C\varepsilon_i^{-1}\{\| \vec{l} \| + \| \vec{f} \|\}$$

$$| u_i''(x) | \leq C\varepsilon_i^{-2}\{\| \vec{l} \| + \| \vec{f} \| + \| \vec{f}' \|\}$$

Proof. The proof is analogous to [9].

3. Bounds on the Solution and its Derivatives

The decomposition of $\vec{u}$ by Shishkin is given by $\vec{u} = \vec{v} + \vec{w}$, where $\vec{v} = (v_1, v_2)^T$ is the solution of

$$\vec{L}_1\vec{v}(x) = E\vec{v}'(x) + A(x)\vec{v}(x) = \vec{g}(x) \text{ on } (0, 1] \tag{3.1}$$

$$\vec{L}_2\vec{v}(x) = E\vec{v}'(x) + A(x)\vec{v}(x) + B(x)\vec{v}(x-1) = \vec{f}(x) \text{ on } (0, 2] \tag{3.2}$$

with $\vec{\beta}\vec{v}(0) = \vec{u}_0(0) - E\vec{u}_0'(0)$ and $\vec{w} = (w_1, w_2)^T$ satisfies $\vec{L}_1\vec{w}(x) = \vec{0}$ for $x \in (0, 1]$ and $\vec{L}_2\vec{w}(x) = \vec{0}$ for $x \in (1, 2]$ with $\vec{\beta}\vec{w}(0) = \vec{\beta}(\vec{u}(0) - \vec{v}(0))$. Here, $\vec{v}$ and $\vec{w}$, are called the smooth and the singular component of $\vec{u}$.

Lemma 3.1. *For* $i = 1, 2$ *there exists a constant C such that* $\| v_i^{(k)} \| \leq C$ *for* $k = 0, 1$ *and* $\| v_i'' \| \leq C\varepsilon_i^{(-1)}$.

Proof. The proof is analogous to [7].

The solution components u_i, $i = 1, 2$ have exponential layers represented by $e^{\alpha x/\varepsilon_i}$ and $e^{\alpha(x-1)/e_i}$. The following layer functions are defined as

$$\mathfrak{B}_{(p,i)}(x) = e^{-(x-p)\alpha/\varepsilon_i},\ p = 0, 1,\ i = 1, 2 \text{ on } [0, 2].$$

The bounds on the singular component $\vec{w}$, in terms of these layer functions, are contained in the following lemma.

Lemma 3.2. *Let* $A(x)$, $B(x)$ *satisfies* (1.3) *and* (1.4). *Then there exist a constant* C, *such that for each* $x \in [0, 1)$,

$$|w_i(x)| \le C\mathfrak{B}_{0,2}(x),\ |w_i'(x)| \le C\sum_{q=1}^{2} \frac{\mathfrak{B}_{0,q}(x)}{\varepsilon_q},\ |\varepsilon_i w_i''(x)| \le C\sum_{q=1}^{2} \frac{\mathfrak{B}_{0,q}(x)}{\varepsilon_q},$$

and for $x \in [1, 2]$

$$|w_i(x)| \le C\mathfrak{B}_{1,2}(x),\ |w_i'(x)| \le C\sum_{q=1}^{2} \frac{\mathfrak{B}_{1,q}(x)}{\varepsilon_q},\ |\varepsilon_i w_i''(x)| \le C\sum_{q=1}^{2} \frac{\mathfrak{B}_{1,q}(x)}{\varepsilon_q}.$$

4. The Shishkin Mesh

A piecewise uniform Shishkin mesh $\overline{\Omega}^N = \overline{\Omega}^{-N} \cup \overline{\Omega}^{+N}$ where $\overline{\Omega}^{-N} = \{x_j\}_0^{N/2}$ and $\overline{\Omega}^{+N} = \{x_j\}_{\frac{N}{2}+1}^{N}$ with N mesh intervals is now constructed on $\overline{\Omega} = [0, 2]$ as follows for the case $\varepsilon_1 < \varepsilon_2$. In the case $\varepsilon_1 = \varepsilon_2$ a simpler construction requiring just one parameter τ suffices. The interval $[0, 1]$ is subdivided into 3 subintervals, $[0, \tau_1] \cup (\tau_1, \tau_2] \cup (\tau_2, 1]$. The parameters τ_r, $r = 1, 2$. The points dividing the uniform meshes are determined by

$$\tau_2 = \min\left\{\frac{1}{2}, \frac{\varepsilon_2}{a} \ln N\right\} \tag{4.1}$$

and $$\tau_1 = \min\left\{\frac{\tau_2}{2}, \frac{\varepsilon_1}{a} \ln N\right\} \tag{4.2}$$

Clearly $0 < \tau_1 < \tau_2 \le \frac{1}{2}$

Then on the subinterval $(\tau_2, 1]$ a uniform mesh with $\frac{N}{4}$ mesh points is placed and on each of the subintervals $(0, \tau_1]$ and $(\tau_1, \tau_2]$, a uniform mesh of $\frac{N}{8}$ mesh points is placed. Similarly, the interval $[1, 2]$ is also divided into 3 subintervals $[1, 1+\tau_1), (1+\tau_1, 1+\tau_2], (1+\tau_2, 2]$ having the same number of mesh intervals as in the interval $[0, \tau_1] \cup (\tau_1, \tau_2] \cup (\tau_2, 1]$ respectively.

Note that, when both the parameters τ_r, $r = 1, 2$, take on their hand value, the Shishkin mesh becomes a classical uniform mesh on $[0, 2]$. This construction leads to a class of eight possible Shishkin piecewise uniform meshes $M_{\vec{b}}$ where $\vec{b} = (b_1, b_2)$ with $b_i = 0$ if $\tau_i = \frac{\tau_{i+1}}{2}$ and $b_i = 1$ otherwise.

5. The Discrete Problem

The backward Euler technique is applied to the Piecewise uniform fitted mesh $\overline{\Omega}^N$ to discretize the initial value problem (1.1), (1.2). The discrete problem is as follows:

$$\vec{L}^N \vec{U}(x_j) = ED^- \vec{U}(x_j) + A(x_j)\vec{U}(x_j) + B(x_j)\vec{U}(x_j - 1) = \vec{f}(x_j),\ x_j \in \Omega^N \tag{5.1}$$

$$\vec{\beta}^N \vec{U}(x_j) = \vec{U}(x_j) - ED^+ \vec{U}(x_j),\ x_j \in \Omega^{*N}, \tag{5.2}$$

where

$$D^-\vec{U}(x_j) = \frac{\vec{U}(x_j) - \vec{U}(x_j - 1)}{x_j - x_{j-1}},\ D^+\vec{U}(x_j) = \frac{\vec{U}(x_{j+1}) - \vec{U}(x_j)}{x_{j+1} - x_j},\ j = 1, 2, \ldots, N$$

The problem (5.1)-(5.2) can be rephrased as follows.

$$\begin{cases} (\vec{L}_1^N \vec{U})(x_j) = ED^-\vec{U}(x_j) + A(x_j)\vec{U}(x_j) = \vec{f}(x_j) - B(x_j)\vec{\varphi}(x_j - 1),\ x_j \in \Omega^{-N} \\ (\vec{L}_1^N \vec{U})(x_j) = ED^-\vec{U}(x_j) + A(x_j)\vec{U}(x_j) + B(x_j)\vec{U}(x_j - 1) = \vec{f}(x_j),\ x_j \in \Omega^{+N} \end{cases} \tag{5.3}$$

with $\vec{\beta}^N \vec{U}(x_j) = \vec{U}(x_j) - ED^+ \vec{U}(x_j) = \vec{l}(x_j),\ x_j \in \Omega^{*N}$.

Lemma 5.1. *If* $\vec{u}(x_j)$ *is any mesh function such that* $\vec{\beta}\vec{\gamma}(0) \vec{\geq}$ *and* $\vec{L}^N \vec{\gamma}(x_j) \geq \vec{0}$ *for* $1 \leq k \leq N$, *then* $\vec{\gamma}(x_j) \geq \vec{0}$ *for all* $1 \leq j \leq N$.

Proof. Let $i^*,\ k$ be such that $\gamma_{i^*}(x_k) = \min_{i,\ j} \{\gamma_i(x_j)\}$, for $1 \leq j \leq N$.

If $x_k = 0$, $(\vec{\beta}^N \vec{\gamma})_{i^*}(0) = \gamma_{i^*}(x_k) - \varepsilon_{i^*} D^+ \gamma_{i^*}(x_k)$

$$= \gamma_{i^*}(x_k) - \varepsilon_{i^*}\left(\frac{\gamma_{i^*}(x_{k+1}) - \gamma_{i^*}(x_k)}{h_{k+1}}\right) < 0,$$

which is a contradiction.

Suppose $j^* \in \Omega^{-N}$

$$(\vec{L}^N \vec{\gamma})_{i^*}(x_k) = (\vec{L}_1^N \vec{\gamma})_{i^*}(x_k) = \varepsilon_{i^*} D^- \gamma_{i^*}(x_k) + \sum_{j=1}^{2} a_{i^*}(x_k) \gamma_{i^*}(x_k)$$

$$= \varepsilon_{i^*} D^- \gamma_{i^*}(x_k) + \sum_{j=1}^{2} a_{i^*}(x_k) \gamma_{i^*}(x_k)$$

which is false and for $x_k \in \Omega^{+N}$

$$(\vec{L}^N \vec{\gamma})_{i^*}(x_k) = (\vec{L}_2^N \vec{\gamma})_{i^*}(x_k) = \varepsilon_{i^*} D^- \gamma_{i^*}(x_k) + \sum_{j=1}^{2} a_{i^* j}(x_k) \gamma_j(x_k) + b_{i^*}$$

$$= \varepsilon_{i^*} D^- \gamma_{i^*}(x_k) + \sum_{j=1}^{2} a_{i^* j}(x_k) \gamma_i(x_k) + b_{i^*}(x_k) \gamma_j(x_k - 1) < 0,$$

which contradicts the given statement and proves the lemma.

Lemma 5.2 (Stability result). *Let* $\vec{Q}$ *be any vector-valued function in the domain of* $\vec{L}^N$. *Then*

$$\| \vec{Q}(x_j) \| \leq \max \{\| \vec{\beta}^N \vec{Q}(0) \|, \frac{1}{\alpha} \| \vec{L}^N \vec{Q}(x_j) \|_{x_j \in \Omega^N} \}.$$

6. The Local Truncation Error

From the discrete stability result, it is seen that in order to bound the error $\vec{U}-\vec{u}$, it suffices to bound $\vec{L}^N(\vec{U}-\vec{u})$. Notice that, for $x_j \in \Omega^N$,

$$\vec{L}(\vec{u}(x_j)-\vec{U}(x_j)) = \vec{L}^N\vec{u}(x_j)-\vec{L}^N\vec{U}(x_j)$$

$$= (\vec{L}-\vec{L}^N)\vec{u}(x_j)$$

and $(\vec{L}-\vec{L}^N)\vec{u}(x_j) = \varepsilon_i(D^- - D)v_i(x_j)+\varepsilon_i(D^- - D)w_i(x_j)$

which is the local truncation of the first derivative. Then, by the triangle inequality,

$$|(\vec{L}-\vec{L}^N)\vec{u}(x_j)| = |\varepsilon_i(D^- - D)v_i(x_j)|+|\varepsilon_i(D^- - D)w_i(x_j)|$$

The proof is similar to [9].

In the same way as the continuous case, the discrete solution $\vec{U}$ can be split into $\vec{V}$ and $\vec{W}$ which are defined as the solutions to the discrete problems listed below

$$(\vec{L}_1^N\vec{U})(x_j^*) = ED^-\vec{U}(x_j)+A(x_j)\vec{V}(x_j) = \vec{f}(x_j)-B(x_j)\vec{\varphi}(x_j-1),\ x_j \in \Omega^{-N} \quad (6.1)$$

$$(\vec{L}_2^N\vec{U})(x_j^*) = ED^-\vec{U}(x_j)+A(x_j)\vec{V}(x_j)+B(x_j)\vec{V}(x_j-1) = \vec{f}(x_j),\ x_j \in \Omega^{+N} \quad (6.2)$$

$$\vec{\beta}^N\vec{V}(x_j) = \vec{\beta}^N\vec{v}(x_j)$$

and

$$(\vec{L}_1^N\vec{W})(x_j) = 0,\ x_j \in \Omega^{-N}$$

$$(\vec{L}_2^N\vec{W})(x_j) = 0,\ x_j \in \Omega^{+N}$$

$$\vec{\beta}^N\vec{W}(x_j) = \vec{\beta}^N\vec{w}(x_j)$$

The error at each point $x_j \in \Omega^{-N}$ is denoted by $\vec{U}(x_j)-\vec{u}(x_j)$. Then the local truncation error $\vec{L}^N e(x_j)$ has the decomposition $\vec{L}^N\vec{e}(x_j)\vec{L}^N(\vec{V}-\vec{v})(x_j)$

$+ \vec{L}^N(\vec{W} - \vec{w})(x_j)$. It is to be noted that for any smooth function σ, the following two distinct estimates of the local truncation of its first derivative hold.

$$|(D^- - D)\vec{\sigma}(x_j)| \leq 2 \max_{s \in I_j} |\vec{\sigma}'(s)| \tag{6.3}$$

and

$$|(D^- - D)\vec{\sigma}(x_j)| \leq \frac{h_j}{2} \max_{s \in I_j} |\vec{\sigma}''(s)| \tag{6.4}$$

where $I_j = x_j - x_{j-1}$.

7. Error Estimate

Theorem 7.1. *Let $\vec{v}$ denote the smooth component of the solution of the problem* (1.1), (1.2) *and V denote the smooth component of the solution of the problem* (5.1), (5.2). *Then*

$$|\vec{L}^N(\vec{V} - \vec{v})\vec{\sigma}(x_j)| \leq CN^{-1}.$$

Theorem 7.2. *Let $\vec{w}$ denote the singular component of the solution of the problem* (1.1), (1.2) *and $\vec{W}$ denote the singular component of the solution of the problem* (5.1), (5.2). *Then*

$$|\vec{L}^N(\vec{W} - \vec{w})\vec{\sigma}(x_j)| \leq CN^{-1} \ln N.$$

Theorem 7.3. *Let $\vec{u}$ denote the singular component of the solution of the problem* (1.1), (1.2) *and $\vec{U}$ denote the singular component of the solution of the problem* (5.1), (5.2). *Then*

$$\| \vec{U}(x_j) - \vec{u}(x_j) \| \leq CN^{-1} \ln N.$$

Proof. It is clear that, in order to prove the above theorem it suffices to prove that $\| (\vec{L}^N(\vec{U} - \vec{u})) \| \leq CN^{-1} \ln N$. But $\| (\vec{L}^N(\vec{U} - \vec{u})) \| \leq \| (\vec{L}^N(\vec{V} - \vec{v})) \| + \| (\vec{L}^N(\vec{W} - \vec{w})) \|$. Hence using theorem (7.1) and (7.2), the above result is derived.

The numerical method proposed above is illustrated through an example presented in this section.

Example 1. Consider the initial value problem

$$E\vec{u}'(x) + A(x)\vec{u}(x) = \vec{g}(x) \quad \forall x \in (0, 1]$$

$$E\vec{u}'(x) + A(x)\vec{u}(x) + B(x)\vec{u}(x-1) = \vec{f}(x) \ \forall x \in (0, 2]$$

with

$$-u_1(0) - \varepsilon u_1'(0) = 1.$$

$$u_2(0) - \varepsilon u_2'(0) = 1.$$

Where $A(x) = \begin{pmatrix} 3+x & -1 \\ -1 & 5+x \end{pmatrix}$, $B(x) = diag(-1, -1)$, $\vec{f}(x) = (1, 1)^T$, $\vec{g}(x) = (3, 1)^T$, $E = diag\,(\varepsilon_1, \varepsilon_2)$.

The numerical solution obtained by applying the fitted mesh method (6.1) and (6.2) to the Example is shown in Figure 1. The order of convergence and the error constant are calculated and are presented in Table 1.

Table I

Values of D_ε^N, D^N, p^N, p^* and $C_{p^*}^N$ generated for the example.

η	Number of mesh points			
	64	128	256	512
0.100E+01	0.312E-01	0.195E-01	0.111E-01	0.591E-02
0.125E+00	0.427E-01	0.367E-01	0.278E-01	0.189E-01
0.156E-01	0.434E-01	0.374E-01	0.284E-01	0.194E-01
0.195E-02	0.435E-01	0.375E-01	0.285E-01	0.194E-01
0.244E-03	0.435E-01	0.375E-01	0.285E-01	0.194E-01
D^N	0.435E-01	0.375E-01	0.285E-01	0.194E-01
p^N	0.215E+00	0.396E+00	0.552E+00	

C_p^N	0.768E+00	0.768E+00	0.678E+00	0.537E+00
The Order of Convergence = 0.2150E+00				
The Error Constant = 0.7684E+00				

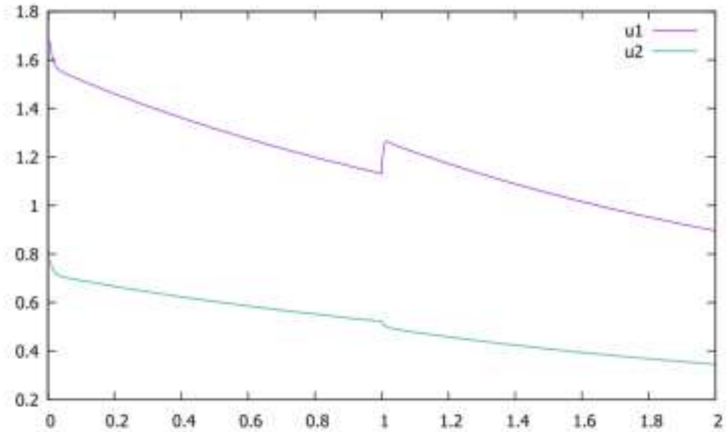

Figure 1. D_ε^N, D^N, p and $C_{p^*}^N$ as an example.

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