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COMPLEMENT IN FUZZY LABELING GRAPH

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Abstract

In this manuscript, the complement of a fuzzy labeling graph (FLG) is defined. A necessary condition for a fuzzy labeling graph G and $\overline{G}$ to be a fuzzy labeling tree is given. It has been proved that every fuzzy labeling graph G and $\overline{G}$ have $(n-1)$ bridges. If G^* is complete then it has been proved that $\overline{G}$ is connected. Some relation between the connectedness of G and $\overline{G}$ is derived. And also self complementary in FLG is defined and it is proved that $\overline{\overline{G}}$ is equal to G, if G^* is a tree.

1. Introduction

The concept of fuzziness is introduced by L. A. Zadeh in his remarkable paper, which has defined the conditions of membership in a set of objects whose elements have an ambiguous status. The complement of a fuzzy graph G is defined by Moderson (1994), and the concept is further developed by M. S. Sunitha and A. Vijayakumar [1]. Various aspects of fuzzy graph has been analysed in [2, 3, 4]. The analysis of complement of fuzzy labeling graphs explores the structure of a fuzzy graph so that the concepts like fuzzy cutnode, fuzzy bridge, block, fuzzy trees, complete fuzzy graph etc. can be studied in detail.

The complement of a fuzzy graph G is defined as $G : (\sigma, \mu)$ as a fuzzy

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graph $G^c : (\sigma^c, \mu^c)$ where $\sigma^c = \sigma$ and $\mu^c(u, v) = 0$ if $\mu(u, v) = 0$ and $\mu^c(u, v) = \sigma(u) \wedge \sigma(v)$ otherwise. From the definition it is understand that the G^c is a fuzzy graph even if G is not a fuzzy graph and also, the $(G^c)^c = G$ if and only if G is a strong fuzzy graph. A modified definition has given for the complement of a fuzzy graph as $G : (\sigma, \mu)$ is the fuzzy graph $\overline{G} = (\overline{\sigma}, \overline{\mu})$ where $\overline{\sigma} = \sigma$ and $\overline{\mu}(u, v) = \sigma(u) \wedge \sigma(v) - \mu(u, v)$ for all u, v in V.

2. Fuzzy Labeling

2.1 Complement in fuzzy labeling graph

Definition 2.1.1. The complement of a fuzzy labeling graph $G = (\sigma, \mu)$ is a fuzzy labeling graph $\overline{G} = (\overline{\sigma}, \overline{\mu})$ where $\overline{\sigma} = \sigma$ and

$$\overline{\mu} = (u, v) = \begin{cases} \sigma(u) \wedge \sigma(v) - \mu(u, v) & \text{if } \overline{\mu}_{i+1}(u, v) \neq \overline{\mu}_i(u, v) \neq \overline{\sigma} \text{ for all } u, v \in V, 1 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

In this definition the choice $\mu(u, v)$ should be in ascending order, in order to get a unique $\overline{G}$.

Example 2.1.2.

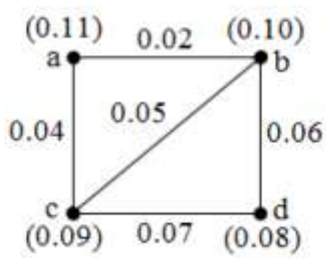

Figure 2.1

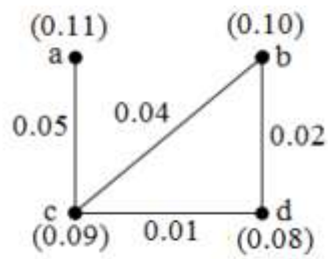

Figure 2.2

Figure 3.1 is a fuzzy labeling graph whose complement is given in figure 2.2.

Remark 2.1.3. By the definition of complement of a fuzzy labeling graph, the cardinality of G is greater than or equal to $\overline{G}$, i.e. $|\overline{G}| \leq |G|$

Proposition 2.1.4. *If* $G = (\sigma, \mu)$ *is a fuzzy labeling graph such that* G^* *is complete, then G and* $\overline{G} = (\overline{\sigma}, \overline{\mu})$ *have* $n-1$ *bridges.*

Proof. For our convenience, let us assume that $\mu_1 > \mu_2 > \mu_3 > \ldots > \mu_n$ and $\sigma_1 > \sigma_2 > \sigma_3 > \ldots > \sigma_n$.

Case (i) if $n = 2$, then there exist only one edge (say e_1). Therefore it is trivial that e_1 is a fuzzy bridge of G and also $|\overline{G}| \leq |G|$. Hence G and $\overline{G}$ have $(n-1)$ bridges.

Case (ii) if $n = 3$, then G^* is a cycle with 3 vertices. If G is FLG such that G^* is a cycle then it will have $(n-1)$ bridges. Hence G has $n-1$ bridges. Here either $|\overline{G}| \leq |G|$ or $|G| = 2$. If $|\overline{G}| \leq |G|$ then $\overline{G}$ will have $n-1$ bridges, otherwise $\overline{G}^*$ is a tree. Hence it is trivial that it has $n-1$ bridges.

Case (iii) if $n = 4$, let V_1, V_2, V_3, V_4 and $V_1V_2, V_2V_4, V_3V_1, V_1V_4, V_2V_3$ be the vertices and edges of G such that $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6$ be the membership values of the vertices and edges respectively.

Then $\mu^{\infty}(V_1, V_2) = \{\mu_1, \mu_5, \mu_6, \mu_4, \mu_6\} = \mu_1$

$\mu^{\infty}(V_1, V_3) = \{\mu_4, \mu_5, \mu_6, \mu_5, \mu_6\} = \mu_3$

$\mu^{\infty}(V_1, V_4) = \{\mu_6, \mu_5, \mu_4, \mu_2, \mu_5\} = \mu_2$

$\mu^{\infty}(V_2, V_3) = \{\mu_5, \mu_6, \mu_4, \mu_5, \mu_5\} = \mu_3$

$\mu^{\infty}(V_2, V_4) = \{\mu_2, \mu_5, \mu_6, \mu_4, \mu_6\} = \mu_2$

$\mu^{\infty}(V_3, V_4) = \{\mu_5, \mu_6, \mu_5, \mu_4, \mu_6\} = \mu_3$

Therefore μ_1, μ_2, μ_3 are the fuzzy bridges of G. hence it has $n-1$ bridges.

Here either $|\overline{G}| \leq |G|$ or $|\overline{G}| = 4$ or $|\overline{G}| = 5$ or $|\overline{G}| = 3$.

Case (a). $|\overline{G}| \leq |G|$, then repeat case (iii) and get $n-1$ bridges.

Case (b). $|\overline{G}| = 3$, then G^* is a tree. Therefore it is trivial that it has $n-1$ bridges.

Case (c). $|\overline{G}| = 4$, then $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and $\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3, \overline{\mu}_4$ be the values of vertices and edges.

Then $\mu^{\infty}(V_1, V_2) = \{\overline{\mu}_4, \overline{\mu}_3\} = \overline{\mu}_3$

$\mu^{\infty}(V_1, V_3) = \{\overline{\mu}_3, \overline{\mu}_4\} = \overline{\mu}_3$

$\mu^{\infty}(V_1, V_4) = \{\overline{\mu}_1\} = \overline{\mu}_1$

$\mu^{\infty}(V_2, V_3) = \{\overline{\mu}_2, \overline{\mu}_4\} = \overline{\mu}_2$

$\mu^{\infty}(V_2, V_3) = \{\overline{\mu}_2, \overline{\mu}_4\} = \overline{\mu}_2$

$\mu^{\infty}(V_2, V_4) = \{\overline{\mu}_4, \overline{\mu}_3\} = \overline{\mu}_3$

$\mu^{\infty}(V_3, V_4) = \{\overline{\mu}_3, \overline{\mu}_4\} = \overline{\mu}_3$

Therefore $\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3$ are the bridges of $\overline{G}$.

$|\overline{G}| = 5$, then $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and $\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3, \overline{\mu}_4, \overline{\mu}_5$ be the values of vertices and edges.

Then $\mu^{\infty}(V_1, V_2) = \{\overline{\mu}_4, \overline{\mu}_3, \overline{\mu}_4\} = \overline{\mu}_3$

$\mu^{\infty}(V_1, V_3) = \{\overline{\mu}_3, \overline{\mu}_4, \overline{\mu}_5\} = \overline{\mu}_3$

$\mu^{\infty}(V_1, V_4) = \{\overline{\mu}_1, \overline{\mu}_4, \overline{\mu}_5\} = \overline{\mu}_1$

$\mu^{\infty}(V_2, V_3) = \{\overline{\mu}_2, \overline{\mu}_5, \overline{\mu}_5\} = \overline{\mu}_2$

$\mu^{\infty}(V_2, V_4) = \{\overline{\mu}_5, \overline{\mu}_4, \overline{\mu}_5, \overline{\mu}_3\} = \overline{\mu}_3$

$\mu^{\infty}(V_3, V_4) = \{\bar{\mu}_4, \bar{\mu}_3, \bar{\mu}_5\} = \bar{\mu}_3$

Therefore $\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3$ are the bridges of $\overline{G}$. Hence it is true for all n.

Corollary 2.1.5. *Every connected fuzzy labeling graph G and $\overline{G}$ have $n-1$ bridges.*

Corollary 2.1.6. *If G is a fuzzy labeling graph and if $\overline{G}$ is disconnected, then each component of $\overline{G}$ will have $n-1$ bridges.*

Proposition 2.1.7. *If G is a fuzzy labeling graph such that G^* is a cycle, then G and $\overline{G}$ will have $n-1$ bridges.*

Proof. If G is FLG such that G^* is a cycle then it will have $(n-1)$ bridges. Therefore G has $n-1$ bridges. Now, either $|\overline{G}| = |G|$ or $|\overline{G}| < |G|$. If $|\overline{G}| = |G|$, then $\overline{G}$ will also have $n-1$ bridge. If $|\overline{G}| < |G|$, then $\overline{G}$ is either a connected path or a disconnected path. If $\overline{G}$ is a connected path, then each edge of $\overline{G}$ is a fuzzy bridge. Therefore it is trivial that $\overline{G}$ will have $n-1$ bridges. If $\overline{G}$ is a disconnected path, then each component is a path. Therefore each component will have $n-1$ bridges. Hence G and $\overline{G}$ will have $n-1$ bridges, if G^* is a cycle.

Remark 2.1.8. Complement of a fuzzy labeling tree need not be a fuzzy labeling tree.

Example 2.1.9. Consider the graph given in figure 2.3 which is a fuzzy labeling tree, but its complement in figure 2.4 is not a fuzzy labeling tree.

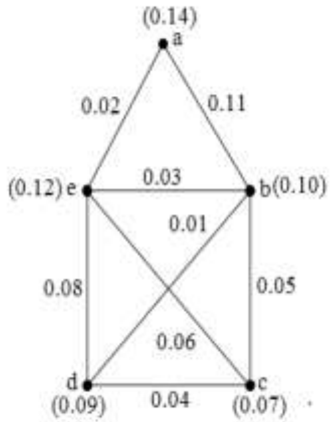

Figure 2.3

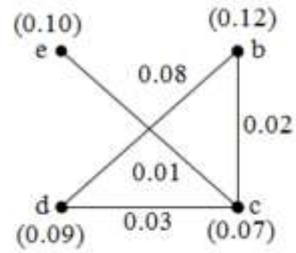

Figure 2.4

Therefore the necessary condition for a graph G and $\overline{G}$ to be a fuzzy labeling tree is given below.

Proposition 2.1.10. *Let G be a fuzzy labeling tree then its complement* $\overline{G}$ *is also a fuzzy labeling tree iff* $\overline{G}$ *is connected.*

Proof. If $\overline{G}$ is a fuzzy labeling tree, by the definition of fuzzy labeling tree, $\mu^{\infty}(x, y) > 0$ for all $x, y \in \overline{G}$. Conversely, assume that $\mu^{\infty}(x, y) = 0$ for some $x, y \in \overline{G}$, then in its spanning sub graph $F = (\sigma, \upsilon)$, $\mu(x, y) = \nu^{\infty}(x, y)$, which is a contradiction.

Proposition 2.1.11. *Let G be a Fuzzy labeling graph, then* $\overline{\overline{G}} = G$, *if* $|G| = |\overline{G}|$.

Proof. Suppose $|G| \neq |\overline{G}|$, then by remark 2.1.3, $|G| > |\overline{G}|$.

Therefore, Let $(x, y) \in |G|$ and $\notin |\overline{G}|$, then

$$\overline{\overline{\mu}}(x, y) = \wedge\{\sigma(x), \sigma(y)\} - \overline{\mu}(x, y)$$

$$= \wedge\{\sigma(x), \sigma(y)\}$$

$$= 0 \text{ (By definition)}$$

Which implies that $(x, y) \notin \overline{\overline{G}}$. Therefore, $\overline{\overline{G}} \neq G$.

Now, Let $\overline{\overline{G}} = G$

$$\overline{\overline{\mu}}(x, y) = \wedge\{\sigma(x), \sigma(y)\} - \overline{\mu}(x, y)$$

$$= \wedge\{\sigma(x), \sigma(y)\} - [\wedge\{\sigma(x), \sigma(y)\} - \mu(x, y)]$$

$$= \mu(x, y)], \text{ for all } x, y \in \overline{\overline{G}}.$$

Therefore, $|G| = |\overline{G}|$.

Thus the theorem is proved.

Proposition 2.1.12. *Let G be a fuzzy labeling tree such that G^* is a cycle. If $(\overline{G})^*$ is a tree and the weakest arc of G does not belongs to $\overline{G}$, then $\overline{G} = \overline{F}$.*

Proof. If G^* is a cycle then the fuzzy labelling cycle G^{ω} has exactly only one weakest arc. Since G^* is a cycle G has exactly only one weakest arc, therefore there exist only one weakest arc say (u, v). Which does not belong to the maximum spanning sub graph F of G. since the arc of F are fuzzy bridges of G. Also F^* is a tree, therefore $F = G - (u, v)$. Which implies $\overline{G} = \overline{F}$. (since (u, v) is not in $\overline{G}$ and $(\overline{G})^*$ is a tree).

2.2 Some relations between the connectedness of FLG and its complement:

Proposition 2.2.1. *If G is a fuzzy labeling graph such that G^* is complete then its complement $\overline{G}$ is connected.*

Proof. Let us prove this proposition by the method of induction and assume that $\mu_1 > \mu_2 > \mu_3 > \ldots > \mu_n$ and $\sigma_1 > \sigma_2 > \sigma_3 > \ldots > \sigma_n$.

Case (i) let $n = 2$, since μ and σ are bijective, $\overline{G}$ is connected.

Case (ii) let $n = 3$, then either $|\overline{G}| = |G|$ or $|\overline{G}| < |G|$ if $|\overline{G}| = |G|$ i.e. $|\overline{G}| = 3 = |G|$, then all $\mu_i(x, y) > 0$ for all $x, y \in V$ otherwise

$|\overline{G}| < |G|$ i.e. either $|\overline{G}| = 2$ or $|\overline{G}| = 1$. If $|\overline{G}| = 2$ then all $\mu_i(x, y) > 0$, for all $x, y \in V$. And $|\overline{G}| = 1$ is not possible, $\overline{G}$ must have $(n-1)$ bridges.

Case (iii) let $n = 4$, then either $|\overline{G}| = |G|$ or $|\overline{G}| < |G|$ if $|\overline{G}| = |G|$, then all $\mu_i(x, y) > 0$ for all $x, y \in V$ otherwise $|\overline{G}| < |G|$ i.e. either $|\overline{G}| = 5$ or 4 or 3, from the proof of proposition 2.1.4 (case b, case c, case d) it is evident that all $\mu_{i,s} > 0$. And $|\overline{G}| = 2$ or 1 is not possible. $\overline{G}$ must have $(n-1)$ bridges, since the FLG have $n-1$ bridges. Hence it can be generalized for all n.

Proposition 2.2.2. *If an arc $(x, y) \in G$ and $\overline{G}$ in a fuzzy bridge then* $\mu^{\infty}(x, y) + \overline{\mu}^{\infty}(x, y) = \wedge\{\sigma(x), \sigma(y)\}$ *for all $x, y \in V$.*

Proof. Let an arc (x, y) be a fuzzy bridge of G and which is not a fuzzy bridge of $\overline{G}$, then $\overline{\mu}^{\infty}(x, y) = \mu(x, y)$ and $\overline{\mu}^{\infty}(x, y) \neq \mu(x, y)$.

Therefore, $\mu^{\infty}(x, y) + \overline{\mu}^{\infty}(x, y) \neq \mu(x, y) + \overline{\mu}(x, y)$.

$$\neq \mu(x, y) + \sigma(x) \wedge \sigma(y) - \mu(x, y).$$

$$\neq \sigma(x) \wedge \sigma(y)$$

Similarly, If (x, y) is not a bridge of G and which is a bridge of $\overline{G}$ then $\overline{\mu}^{\infty}(x, y) \neq \mu(x, y)$

$$\overline{\mu}(x, y) = \mu(x, y)$$

$$\mu^{\infty}(x, y) + \overline{\mu}^{\infty}(x, y) \neq \mu(x, y) + \overline{\mu}(x, y)$$

$$\neq \mu(x, y) + \sigma(x) \wedge \sigma(y) - \mu(x, y).$$

$$\neq \sigma(x) \wedge \sigma(y)$$

Hence the proof and the converse are not true.

Proposition 2.2.3. *Let G and $\overline{G}$ be a fuzzy labeling graph, then* $\mu^{\infty}(x, y) + \overline{\mu}^{\infty}(x, y) > \mu^{\infty}(x, y) + \overline{\mu}^{\infty}(x, y)$. *If (x, y) in a bridge of G or $\overline{G}$ or both.*

Proof. Let $(x, y) \in G$ and $\overline{G}$ and (x, y) be a bridge of G, but not in $\overline{G}$, then

$$\mu^{\infty}(x, y) > \mu^{\infty}(x, y) \tag{1}$$

$$\overline{\mu}^{\infty}(x, y) = \overline{\mu}^{\infty}(x, y) \tag{2}$$

Adding equations (1) and (2) we get

$$\mu^{\infty}(x, y) + \overline{\mu}^{\infty}(x, y) > \mu^{\infty}(x, y) + \overline{\mu}^{\infty}(x, y)$$

Similarly If (x, y) is not a bridge in G, but (x, y) is a bridge of $\overline{G}$ then

$$\mu^{\infty}(x, y) = \mu^{\infty}(x, y) \tag{3}$$

$$\overline{\mu}^{\infty}(x, y) > \overline{\mu}^{\infty}(x, y) \tag{4}$$

Adding (3) and (4) we get

$$\mu^{\infty}(x, y) + \overline{\mu}^{\infty}(x, y) > \mu^{\infty}(x, y) + \overline{\mu}^{\infty}(x, y)$$

now $(x, y) \in G$ and $\overline{G}$ is a bridge in both G and $\overline{G}$ then

$$\mu^{\infty}(x, y) > \mu^{\infty}(x, y) \tag{5}$$

$$\overline{\mu}^{\infty}(x, y) > \overline{\mu}^{\infty}(x, y) \tag{6}$$

Adding (5) and (6) we get

$$\mu^{\infty}(x, y) + \overline{\mu}^{\infty}(x, y) > \mu^{\infty}(x, y) + \overline{\mu}^{\infty}(x, y)$$

Corollary 2.2.4. *If $(x, y) \in G$ and $\overline{G}$ is not a bridge in both G and $\overline{G}$, then*

$$\mu^{\infty}(x, y) + \overline{\mu}^{\infty}(x, y) = \mu^{\infty}(x, y) + \overline{\mu}^{\infty}(x, y)$$

Remark 2.2.5. The strongest path of G need not be the strongest path of $\overline{G}$.

Example 2.2.6.

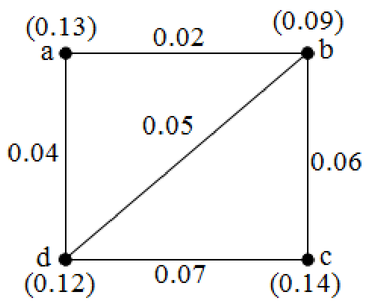

Figure 2.5

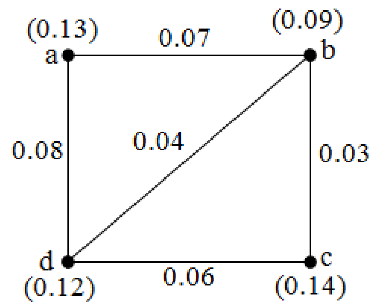

Figure 2.6

In Figure 2.5, (a, b) has two strongest path (a, d, b) and (a, d, c, b). But in figure 2.6 the arc (a, b) itself is the strongest path of (a, b). In the above example 2.2.6, there exist two strongest path between the nodes of a and b, (a, d, b) and (a, d, c, b).

For $\overline{G}$ also between the node a and b (a, d, b) in the strongest path.

Proposition 2.2.7. *If an arc* (x, y) *in a fuzzy bridge of a fuzzy labeling graph G and* $\overline{G}$ *and*

(i) *If* $\mu(x, y) \geq \overline{\mu}(x, y)$ *then* $\mu^{\infty}(x, y) \geq \overline{\mu}^{\infty}(x, y)$

(ii) *If* $\mu(x, y) \leq \overline{\mu}(x, y)$ *then* $\mu^{\infty}(x, y) \leq \overline{\mu}^{\infty}(x, y)$.

Proof. Let (x, y) be a fuzzy bridge of G and its component $\overline{G}$ then

$$\text{If } \mu^{\infty}(x, y) = \mu(x, y) \tag{7}$$

$$\overline{\mu}^{\infty}(x, y) = \overline{\mu}(x, y) \tag{8}$$

If $\mu(x, y) \geq \overline{\mu}(x, y)$

$\mu^{\infty}(x, y) \geq \overline{\mu}^{\infty}(x, y)$ {by (7) and (8)}

Similarly If $\mu(x, y) \leq \overline{\mu}(x, y)$

$\mu^{\infty}(x, y) \leq \overline{\mu}^{\infty}(x, y)$ {by (7) and (8)}

Proposition 2.2.8. *If an arc* (x, y) *is a fuzzy bridge in* G, *but not in* $\overline{G}$ *and if* $\mu(x, y) \leq \overline{\mu}(x, y)$ *then* $\mu^{\infty}(x, y) \leq \overline{\mu}^{\infty}(x, y)$.

Proof. Let (x, y) be a fuzzy bridge in G, then $\mu^{\infty}(x, y) \leq \mu(x, y)$ (9)

Now Let (x, y) is not a fuzzy bridge in G, then $\overline{\mu}^{\infty}(x, y) \geq \overline{\mu}(x, y)$ (10)

$\therefore$ If $\mu(x, y) \leq \overline{\mu}(x, y)$

$\Rightarrow \mu^{\infty}(x, y) \leq \overline{\mu}(x, y)$ {By (9)}

$\Rightarrow \mu^{\infty}(x, y) \leq \overline{\mu}(x, y) < \overline{\mu}^{\infty}(x, y)$ {By (10)}

$\Rightarrow \mu^{\infty}(x, y) \leq \overline{\mu}^{\infty}(x, y)$.

Proposition 2.2.9. *If* (x, y) *is a fuzzy bridge in* $\overline{G}$, *but not in* G *and if* $\mu(x, y) \geq \overline{\mu}(x, y)$ *then* $\mu^{\infty}(x, y) \leq \overline{\mu}^{\infty}(x, y)$.

Proof. Let (x, y) be a fuzzy bridge in $\overline{G}$, then $\overline{\mu}^{\infty}(x, y) \leq \overline{\mu}(x, y)$ (11)

Now Let (x, y) is a fuzzy bridge in G, then $\mu(x, y) < \mu^{\infty}(x, y)$ (12)

If $\mu(x, y) \geq \overline{\mu}(x, y)$

$\Rightarrow \mu(x, y) \geq \overline{\mu}^{\infty}(x, y)$ {By (11)}

$\Rightarrow \mu^{\infty}(x, y) > \mu(x, y)$

$\Rightarrow \mu^{\infty}(x, y) \leq \overline{\mu}(x, y) \geq \overline{\mu}^{\infty}(x, y)$ {By (12)}

$\Rightarrow \mu^{\infty}(x, y) \geq \overline{\mu}^{\infty}(x, y)$

Remarks 2.2.10. If (x, y) is a bridge in G, but not in $\overline{G}$, and $\mu(x, y) \geq \overline{\mu}(x, y)$ then its connectedness need not follow the same inequality.

Example 2.2.11. Let $G : (\alpha, \mu)$ be with $\sigma^{*} = \{a, b, c, d\}$ $\sigma(a) = 0.14$ $\sigma(b) = 0.09$, $\sigma(c) = 0.16$, $\sigma(d) = 0.15$ and $\mu(a, b) = 0.05$, $\mu(b, c) = 0.06$, $\mu(c, d) = 0.07$, $\mu(d, a) = 0.04$, $\mu(b, d) = 0.02$. Then $\overline{G} = (\overline{\sigma}, \overline{\mu})$ be with

$\overline{\mu}(a, b) = 0.04$, $\overline{\mu}(b, c) = 0.03$, $\overline{\mu}(c, d) = 0.08$, $\overline{\mu}(d, a) = 0.10$, $\overline{\mu}(b, d) = 0.07$.

$\mu(b, c) \geq \overline{\mu}(b, c)$, but $\mu^{\infty}(b, c) = 0.06 < \overline{\mu}^{\infty}(b, c) = 0.07$.

If (x, y) is a bridge in $\overline{G}$, but not in G and $\mu(x, y) \geq \overline{\mu}(x, y)$ then its connectedness need not follow the same inequality.

Example 2.2.12. Let $G : (\alpha, \mu)$ be with $\sigma^* = \{a, b, c, d\}$ $\sigma(a) = 0.14$ $\sigma(b) = 0.11$, $\sigma(c) = 0.16$, $\sigma(d) = 0.15$ and $\mu(a, b) = 0.09$, $\mu(b, c) = 0.08$, $\mu(c, d) = 0.04$, $\mu(d, a) = 0.05$, $\mu(b, d) = 0.03$, $\mu(b, d) = 0.06$. Then $\overline{G}$ be with $\overline{\mu}(b, c) = 0.02$, $\overline{\mu}(c, d) = 0.06$, $\overline{\mu}(d, a) = 0.08$, $\overline{\mu}(a, c) = 0.07$, $\overline{\mu}(b, d) = 0.05$.

Hence, $\mu(a, c) \geq \overline{\mu}(a, c)$, but $\mu^{\infty}(a, c) = 0.08 < \overline{\mu}^{\infty}(a, c) = 0.07$.

If (x, y) is not a bridge in G and $\overline{G}$, and $\mu(x, y) \geq \overline{\mu}(x, y)$, or $\mu(x, y) \leq \overline{\mu}(x, y)$ then also its connectedness need not follow the same inequality.

Example 2.2.13. Let $G : (\alpha, \mu)$ be with $\sigma^* = \{a, b, c, d\}$ $\sigma(a) = 0.14$ $\sigma(b) = 0.09$, $\sigma(c) = 0.13$, $\sigma(d) = 0.12$ and $\mu(a, b) = 0.02$, $\mu(b, c) = 0.06$, $\mu(c, d) = 0.07$, $\mu(d, a) = 0.04$, $\mu(b, d) = 0.05$. Then $\overline{G} = (\overline{\sigma}, \overline{\mu})$ be with $\overline{\mu}(a, b) = 0.07$, $\overline{\mu}(b, c) = 0.03$, $\overline{\mu}(c, d) = 0.05$, $\overline{\mu}(d, a) = 0.08$, $\overline{\mu}(b, d) = 0.04$.

Then (b, d) is not a bridge in G and $\overline{G}$, and $\mu(b, d) > \overline{\mu}(b, d)$ but, $\mu^{\infty}(b, d) = 0.06 < \overline{\mu}^{\infty}(b, d) = 0.07$.

2.3 Self Complement in Fuzzy labeling graph.

Definition 2.3.1. A Fuzzy labeling graph is itself complementary if $\overline{G} = G$.

Remark 2.3.2. The following results were already proved for fuzzy graph. Therefore, the proofs of the following results are similar to fuzzy graph. So the results are omitted.

Proposition 2.3.3. *If* $G = (\sigma, \mu)$ *is a self complementary fuzzy labeling graph, then* $\sum_{u \neq v} \mu(u, v) = 1/2 \sum_{u \neq v} \sigma(u) \wedge \sigma(v)$.

Proposition 2.3.4. *If* $G = (\sigma, \mu)$ *is a fuzzy labeling graph, such that* $\mu(u, v) = 1/2\{\sigma(u) \wedge \sigma(v)\}$ *for* $u, v \in V$, *then G is a self complementary graph.*

Proposition 2.3.5. *A fuzzy labeling graph G is self complementary, if* G^* *is a tree.*

Proof. Suppose G is self complementary, such that G^* is not a tree. Then, there exist a cycle.

Consider a cycle with three vertices (i.e.) v_1, v_2 and v_3. And assume that $\sigma(v_1) > \sigma(v_2) > \sigma(v_3)$.

Then by proposition 2.3.4,

$$\mu(v_1, v_2) = 1/2[\sigma(v_1) \wedge \sigma(v_2)] = \sigma(v_2)/2$$

$$\mu(v_2, v_3) = 1/2[\sigma(v_2) \wedge \sigma(v_3)] = \sigma(v_3)/2$$

$$\mu(v_3, v_1) = 1/2[\sigma(v_3) \wedge \sigma(v_1)] = \sigma(v_3)/2$$

$\therefore |G| \neq |\overline{G}|$, which is a contradiction to G. Hence, G^* is a tree.

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