Advances and Applications in Mathematical Sciences
Volume 22, Issue 3, January 2023, Pages 717-734

ESTIMATES FOR HIGHER ORDER COEFFICIENTS AND SECOND ORDER HANKEL DETERMINANT OF CERTAIN BI-UNIVALENT FUNCTIONS

M. P. JEYARAMAN and S. PADMAPRIYA

Department of Mathematics
Presidency College
Chennai – 600005, India
E-mail: jeyaraman_mp@yahoo.co.in

Department of Mathematics
SRM Institute of Science and Technology
Ramapuram, Campus, Chennai 600 089, India
E-mail: padmapriya.14@gmail.com

Abstract

Estimates for the initial coefficients $|a_2|, |a_3|$ and higher order coefficients $|a_4|$ and $|a_5|$ of bi-univalent functions belonging to certain classes of analytic functions are obtained. Second order Hankel determinant is also obtained. Improvement of the earlier known estimates are also pointed out.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions defined on the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Suppose that $\mathcal{S}$ is the subclass of $\mathcal{A}$ consisting of univalent functions.

2020 Mathematics Subject Classification: 30C45, 30C80.
Keywords: Bi-univalent functions, Subordination, Coefficient estimates, Second Hankel determinant.
Received January 19, 2022; Accepted April 24, 2022

Being univalent, the functions in the class $\mathcal{S}$ are invertible; however, the inverse need not be defined on entire unit disc. The Koebe's one quarter theorem ensures that the image of the unit disc under every univalent function contains a disc of radius $1/4$. Thus, a function $f \in \mathcal{S}$ has an inverse defined on a disc contains $|w| < 1/4$.

It can be noted that

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3, \ldots, \tag{1.2}$$

in some disc of radius at least $1/4$. A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$, if both f and f^{-1} are univalent in $\mathbb{D}$, and is denoted by σ.

Lewin [3] investigated the class σ of bi-univalent functions and obtained the bound for the second coefficient. Several authors have subsequently studied similar problems in this direction (see [2, 5, 8]). Brannan and Taha [2] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike and bi-convex function and obtained bounds for initial coefficients. Serap Bulut in [1] investigated the subclass $B_{\Sigma}^{h,p}$ of analytic bi-univalent function and obtain estimates on the first two coefficients $|a_2|$ and $|a_3|$. The class $B_{\Sigma}^{h,p}$ generalize familier classes of bi-starlike, strongly bi-starlike. It should be remarked that, only very few articles that deal with higher order coefficients (See [12, 13, 15]).

Motivated by the aforementioned works, in this paper, we introduce and investigate an interesting subclass $R_{\sigma}(\alpha, h, p)$ of analytic and bi-univalent function and obtain initial coefficients $|a_2|$ and $|a_3|$ and higher order coefficients $|a_4|$ and $|a_5|$. Our results would generalize and improve the results obtained in [1, 5].

For any two analytic functions f and ϕ in $\mathbb{D}$, we say that f is subordinate to ϕ written as $f \prec \phi$, if there exists a Schwarz function w analytic in $\mathbb{D}$ with

$w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = \phi(w(z))(z \in \mathbb{D})$. In particular, if the function ϕ is univalent in $\mathbb{D}$, the above subordination is equivalent to $f(0) = \phi(0)$ and $f(\mathbb{D}) \subset \phi(\mathbb{D})$.

Definition 1.1. Let the functions $h, p : \mathbb{D} \to \mathbb{C}$ be constrained that

$$\min\{\mathbb{R}(h(z)), \mathbb{R}(p(z))\} > 0 \ (z \in \mathbb{D}) \text{ and } h(0) = p(0) = 1 \tag{1.3}$$

A function $f \in \sigma$ given by (1.1) is said to be in the class $R_\sigma(\alpha, h, p)$, if it satisfies

$$\left.\begin{array}{l} \dfrac{\alpha z^2 f''(z) + zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)} \in h(\mathbb{D}), \ (0 \le \alpha \le 1) \\ \text{and} \\ \dfrac{aw^2 g''(w) + wg'(w)}{\alpha wg'(w) + (1-\alpha)g(w)} \in p(\mathbb{D}), \ (0 \le \alpha \le 1) \end{array}\right\}. \tag{1.4}$$

We note that, by choosing appropriate values for α, h and p, the class $R_\sigma(\alpha, h, p)$ reduces to several earlier known subclasses of biunivalent function.

(1) $R_\sigma(0, h, p) = B_\Sigma^{h,p}$ [1, Definition 3]

(2) $R_\sigma(1, h, p) = K_\sigma(p)$ [15]

(3) $R_\sigma\left(0, \dfrac{1+(1-2\beta)z}{1-z}, \dfrac{1-(1-2\beta)z}{1+z}\right) = S_\sigma^*(\beta)(0 \le \beta < 1)$ [2, Definition 3.1]

(4) $R_\sigma\left(0, \left(\dfrac{1+z}{1-z}\right)^\beta, \left(\dfrac{1-z}{1+z}\right)^\beta\right) = SS_\sigma^*(\beta)(0 \le \beta < 1)$ [2, Definition 2.1]

(5) $R_\sigma\left(1, \dfrac{1+(1-2\beta)z}{1-z}, \dfrac{1-(1-2\beta)z}{1+z}\right) = C_\sigma^*(\beta)(0 \le \beta < 1)$ [2, Definition 4.1]

(6) $R_\sigma\left(1, \left(\dfrac{1+z}{1-z}\right)^\beta, \left(\dfrac{1-z}{1+z}\right)^\beta\right) = SC_\sigma^*(\beta)(0 \le \beta < 1)$ [15]

2. Coefficient Estimates

Theorem 2.1. *Let f given by* (1.1) *be in the class* $R_\sigma(\alpha, h, p)$. *Then*

$$|a_2| \le \min\left\{\sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{2(1+\alpha)^2}}, \sqrt{\frac{|h''(0)| + |p''(0)|}{2[4(1+2\alpha) - 2(1+\alpha)^2]}}\right\},$$

$$|a_3| \le \min\left\{\left[\frac{|h'(0)|^2 + |p'(0)|^2}{2(1+\alpha)^2} + \frac{1}{8}\frac{|h''(0)| + |p''(0)|}{(1+2\alpha)}\right],\right.$$

$$\left.\left[\frac{|h''(0)|[8(1+2\alpha) - 2(1+\alpha)^2] + p''(0)2(1+\alpha)^2}{4[(1+2\alpha) - 2(1+\alpha)^2][4(1+2\alpha)]}\right]\right\}.$$

Proof. Let $f \in R_\sigma(\alpha, h, p)$ and g be the analytic extension of f^{-1} to $\mathbb{D}$. It follows from (1.4) that

$$\frac{\alpha z^2 f''(z) + zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)} = h(z) \tag{2.1}$$

and

$$\frac{\alpha w^2 g''(w) + wg'(w)}{\alpha wg'(w) + (1-\alpha)g(w)} = p(w), \tag{2.2}$$

where $h(z)$ and $p(w)$ satisfy the conditions of Definition 1.1.

Furthermore the functions $h(z)$ and $p(w)$ have the following Taylor series expansions

$$h(z) = 1 + h_1 z + h_2 z^2 + \ldots,$$

$$p(w) = 1 + p_1 w + p_2 w^2 + \ldots,$$

respectively.

Now from (2.1), we have

$$a_2(1+\alpha) = h_1 \tag{2.3}$$

$$2a_3(1+2\alpha) = a_2 h_1(1+\alpha) + h_2 \tag{2.4}$$

$$3a_4(1+3\alpha) = a_3h_1(1+2\alpha) + a_2h_2(1+\alpha) + h_3 \tag{2.5}$$

$$4a_5(1+4\alpha) = a_4h_1(1+3\alpha) + a_3h_2(1+2\alpha) + a_2h_3(1+\alpha) + h_4. \tag{2.6}$$

From (2.2), we have

$$a_2(1+\alpha) = -p_1 \tag{2.7}$$

$$2(2a_2^2 - a_3)(1+2\alpha) = -a_2p_1(1+\alpha) + p_2 \tag{2.8}$$

$$-3(5a_2^3 - 5a_2a_3 + a_4)(1+3\alpha) = (2a_2^2 - a_3)p_1(1+2\alpha) - a_2p_2(1+\alpha) + p_3 \tag{2.9}$$

$$4(14a_2^4 - 21a_2^2a_3 + 6a_2a_4 + 3a_3^2 - a_5)(1+4\alpha) = (-1)[5a_2^3 - 5a_2a_3 + a_4]$$

$$p_1(1+3\alpha)$$

$$+ (2a_2^2 - a_3)p_2(1+2\alpha) - a_2p_3(1+\alpha) + p_4. \tag{2.10}$$

From (2.3) and (2.7), we obtain

$$h_1 = -p_1 \tag{2.11}$$

and

$$2a_2^2(1+\alpha)^2 = h_1^2 + p_1^2. \tag{2.12}$$

From (2.4) and (2.8), we get

$$a_2^2 = \frac{h_2 + p_2}{[4(1+2\alpha) - 2(1+\alpha)^2]}. \tag{2.13}$$

Therefore, from (2.12) and (2.13) we find that

$$|a_2| \le \sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{2(1+\alpha)^2}}$$

and

$$|a_2| \le \sqrt{\frac{|h''(0)| + |p''(0)|}{2[4(1+2\alpha) - 2(1+\alpha)^2]}}.$$

By using (2.4) and (2.8), we obtain

$$a_3 = a_2^2 + \frac{1}{4}\frac{(h_2 - p_2)}{(1+2\alpha)}. \tag{2.14}$$

Using (2.12) and (2.13) in (2.14), we have

$$a_3 = \frac{h_1^2 + p_1^2}{2(1+\alpha)^2} + \frac{1}{4}\frac{(h_2 - p_2)}{(1+2\alpha)}, \tag{2.15}$$

and

$$a_3 = \frac{h_2[8(1+2\alpha) - 2(1+\alpha)^2] + 2(1+\alpha)^2 p_2}{4(1+2\alpha) - 2(1+\alpha)^2][4(1+2\alpha)]}. \tag{2.16}$$

We thus find that

$$|a_3| \le \frac{|h'(0)|^2 + |p'(0)|^2}{2(1+\alpha)^2} + \frac{1}{8}\frac{(|h''(0)| + |p''(0)|)}{(1+2\alpha)}$$

and

$$|a_3| \le \frac{|h''(0)|[8(1+2\alpha) - 2(1+\alpha)^2] + |p''(0)|2(1+\alpha)^2}{2[4(1+2\alpha) - 2(1+\alpha)^2][4(1+2\alpha)]}.$$

This completes the proof of theorem. □

Remark 2.1. For $\alpha = 0$ and $\alpha = 1$, Theorem 2.1 gives the estimates for starlike and convex function which is given in [1] and [15] respectively.

Remark 2.2.

(i) For $\alpha = 0$, $h(z) = \frac{1+(1-2\beta)z}{1-z}$ and $p(z) = \frac{1-(1-2\beta)z}{1+z}$, Theorem 2.1 gives the estimates for starlike function of order β, obtained in [2].

(ii) For $\alpha = 0$, $h(z) = \left(\frac{1+z}{1-z}\right)^{\beta}$ and $p(z) = \left(\frac{1-z}{1+z}\right)^{\beta}$, Theorem 2.1 gives the estimates for strongly starlike function, obtained in [2].

Remark 2.3.

(i) For the choice of $\alpha = 1$, $h(z) = \frac{1+(1-2\beta)z}{1-z}$ and $p(z) = \frac{1-(1-2\beta)z}{1+z}$ in Theorem 2.1, reduces to the estimates for convex function of order α obtained by Brannan and Taha [2].

(ii) By taking $\alpha = 1$, $h(z) = \left(\frac{1+z}{1-z}\right)^{\beta}$ and $p(z) = \left(\frac{1-z}{1+z}\right)^{\beta}$ in Theorem 2.1, we have the result obtained in [15].

Theorem 2.2. *If the function* $f \in R_{\sigma}(\alpha, h, p)$, *then the coefficients an* $(n = 4, 5)$ *of f satisfy*

$$| a_4 | \leq \min\left\{\left(\frac{[| h'(0)^2 + p'(0)^2 |]^{1/2}}{2\sqrt{2}(1+\alpha)} [(R_1(\alpha))| h''(0)| + (R_2(\alpha))| p''(0)|]\right.\right.$$

$$\left. + \frac{1}{36}\frac{| h'''(0)| + | p'''(0)|}{(1+3\alpha)} + \frac{1}{6\sqrt{2}}\frac{| h'(0)^2 + p'(0)^2 |^{3/2} | (1+2\alpha)}{(1+\alpha)^2(1+3\alpha)}\right),$$

$$\left(\frac{| h''(0)^2 + p''(0) |^{1/2}}{\sqrt{[4(1+2\alpha) - 2(1+\alpha)^2]^{1/2}}} [(R_1(\alpha))| h''(0)| + (R_2(\alpha))| p''(0)|]\right.$$

$$\left.\left. + \frac{1}{36}\frac{| h'''(0)| + | p'''(0)|}{(1+3\alpha)} + \frac{1}{6\sqrt{2}}\frac{| h''(0) + p''(0) |^{3/2}(1+\alpha)(1+2\alpha)}{[4(1+2\alpha) - 2(1+\alpha)^2]^{3/2}(1+3\alpha)}\right)\right\},$$

where

$$R_1(\alpha) = \frac{(1+\alpha)}{6(1+3\alpha)} + \frac{5}{8(1+2\alpha)},$$

$$R_2(\alpha) = \frac{(1+\alpha)}{6(1+3\alpha)} - \frac{5}{8(1+2\alpha)}$$

and

$$| a_5 | \leq \min\left\{\left(\frac{| h'(0)^2 + p'(0)^2 |^2}{4(1+\alpha)^4} + \frac{| h'(0)^2 + p'(0)^2 |}{4(1+\alpha)^2} [K_2(\alpha)| h''(0)|\right.\right.$$

$$+ K_3(\alpha)| p''(0)|] + \frac{\sqrt{h'(0)^2 + p'(0)^2}}{6\sqrt{2}(1+\alpha)} K_4(\alpha)[| h'''(0)| + | p'''(0)|]$$

$$\left. + \frac{K_5(\alpha)}{4}| h''(0) + p''(0)|^2 + \frac{K_6(\alpha)}{24}| h^{iv}(0)|\right),$$

$$\left(\frac{|h''(0)+p''(0)|^2}{4[K_7(\alpha)]^2}K_1(\alpha)+\frac{|h''(0)|+|p''(0)|}{4[K_7(\alpha)]^2}[K_2(\alpha)|h''(0)|+K_3(\alpha)|p''(0)|]\right.$$

$$+\frac{(h''(0)+p''(0))^{1/2}}{6\sqrt{2}\sqrt{K_7(\alpha)}}K_4(\alpha)[|h'''(0)|+|p'''(0)|]+\frac{K_5(\alpha)}{4}|h''(0)+p''(0)|^2$$

$$\left.+\frac{K_6(\alpha)}{24}|h'^{v}(0)|\right)\Bigg\},$$

where

$$K_1(\alpha)=\frac{1}{3}\frac{(1+\alpha)(1+3\alpha)}{(1+4\alpha)}+\frac{1}{2}\frac{(1+2\alpha)^2}{(1+4\alpha)}-\frac{(1+\alpha)^2}{(1+4\alpha)}+\frac{1}{4}\frac{(1+\alpha)^4}{(1+4\alpha)}$$

$$K_2(\alpha)=\frac{1}{6}\frac{(1+\alpha)^2}{(1+4\alpha)}+\frac{5}{8}\frac{(1+\alpha)(1+3\alpha)}{(1+2\alpha)(1+4\alpha)}+\frac{1}{4}\frac{(1+2\alpha)}{(1+4\alpha)}-\frac{1}{4}\frac{(1+\alpha)^2}{(1+2\alpha)(1+4\alpha)}$$

$$K_3(\alpha)=\frac{1}{6}\frac{(1+\alpha)^2}{(1+4\alpha)}-\frac{5}{8}\frac{(1+\alpha)(1+3\alpha)}{(1+2\alpha)(1+4\alpha)}-\frac{1}{4}\frac{(1+2\alpha)}{(1+4\alpha)}+\frac{1}{4}\frac{(1+\alpha)^2}{(1+2\alpha)(1+4\alpha)}$$

$$K_4(\alpha)=\frac{1}{6}\frac{(1+\alpha)}{(1+4\alpha)}$$

$$K_5(\alpha)=\frac{1}{32}\frac{(1+2\alpha)}{(1+4\alpha)}$$

$$K_6(\alpha)=\frac{1}{4(1+4\alpha)}$$

$$K_7(\alpha)=4(1+2\alpha)-2(1+\alpha)^2.$$

Proof. From (2.5) and (2.9) we have

$$a_4=\frac{a_2}{6}\frac{(1+\alpha)}{(1+3\alpha)}(h_2+p_2)+\frac{1}{6}\frac{(h_3-p_3)}{(1+3\alpha)}+\frac{1}{3}a_2^3\frac{(1+\alpha)(1+2\alpha)}{(1+3\alpha)}$$

$$+\frac{5}{8}a_2\frac{(h_2-p_2)}{(1+2\alpha)}. \tag{2.17}$$

Using (2.12) and (2.13) in (2.17), we get

$$a_4 = \sqrt{\frac{h_1^2 + p_1^2}{2(1+\alpha)^2}}\left[\frac{(1+\alpha)}{6(1+3\alpha)}(h_2 + p_2) + \frac{5}{8}\frac{(h_2 - p_2)}{(1+2\alpha)}\right]$$

$$+ \frac{1}{6}\frac{(h_3 - p_3)}{(1+3\alpha)} + \frac{1}{3}\frac{(h_1^2 + p_1^2)^{3/2}}{2\sqrt{2}}\frac{(1+2\alpha)}{(1+\alpha)^2(1+3\alpha)} \tag{2.18}$$

and

$$a_4 = \sqrt{\frac{h_2 + p_2}{[4(1+2\alpha) - 2(1+\alpha)^2]}}\left[\frac{(1+\alpha)}{6(1+3\alpha)}(h_2 + p_2) + \frac{5}{8}\frac{(h_2 - p_2)}{(1+2\alpha)}\right]$$

$$+ \frac{1}{6}\frac{(h_3 - p_3)}{(1+3\alpha)} + \frac{1}{3}\frac{(h_2 + p_2)^{3/2}}{[4(1+2\alpha) - 2(1+\alpha)^2]^{3/2}}\frac{(1+\alpha)(1+2\alpha)}{(1+3\alpha)}. \tag{2.19}$$

We thus find that

$$|a_4| \leq \frac{[| h'(0)^2 + p'(0)^2 |]^{1/2}}{2\sqrt{2}(1+\alpha)}[(R_1(\alpha))|h''(0)| + (R_2(\alpha))|p''(0)|]$$

$$+ \frac{1}{36}\frac{|h'''(0)| + |p'''(0)|}{(1+3\alpha)} + \frac{1}{6\sqrt{2}}\frac{|h'(0)^2 + p'(0)^2|^{3/2} \ |(1+2\alpha)}{(1+\alpha)^2(1+3\alpha)}$$

and

$$|a_4| \leq \frac{|h''(0) + p''(0)|^{1/2}}{\sqrt{[4(1+2\alpha) - 2(1+\alpha)^2]^{1/2}}}[(R_1(\alpha))|h''(0)| + (R_2(\alpha))|p''(0)|]$$

$$+ \frac{1}{36}\frac{|h'''(0)| + |p'''(0)|}{(1+3\alpha)} + \frac{1}{6\sqrt{2}}\frac{|h''(0) + p''(0)|^{3/2}(1+\alpha)(1+2\alpha)}{[4(1+2\alpha) - 2(1+\alpha)^2]^{3/2}(1+3\alpha)}.$$

Using (2.6) and (2.10) we obtain

$$a_5 = (1+\alpha)(1+3\alpha)a_2a_4 + \frac{1}{2}a_3^2(1+2\alpha)^2 - a_2^2a_3\frac{(1+\alpha)^2}{(1+4\alpha)} + \frac{1}{4}\frac{(1+\alpha)^4}{(1+4\alpha)}a_2^4$$

$$+ \frac{1}{4}h_4 \tag{2.20}$$

and

$$a_5 = a_2^4K_1(\alpha) + a_2^2[K_2(\alpha)h_2 + K_3(\alpha)p_2] + a_2[K_4(\alpha)](h_3 - p_3)$$

$$+ K_5(\alpha)(h_2 - p_2)^2 + K_6(\alpha)h_4. \tag{2.21}$$

Using (2.12) and (2.13) we get

$$| a_5 | \leq \frac{| h'(0)^2 + p'(0)^2 |^2}{4(1+\alpha)^4} + \frac{| h'(0)^2 + p'(0)^2 |}{4(1+\alpha)^2}[K_2(\alpha)| h''(0) | + K_3(\alpha)| p''(0) |]$$

$$+ \frac{\sqrt{h'(0)^2 + p'(0)^2}}{6\sqrt{2}(1+\alpha)} K_4(\alpha)[| h'''(0) | + | p'''(0) |] + \frac{K_5(\alpha)}{4}| h''(0) + p''(0) |^2$$

$$+ \frac{K_6(\alpha)}{24}| h'^{v}(0) |$$

and

$$| a_5 | \leq \frac{| h''(0) + p''(0) |^2}{4[K_7(\alpha)]^2} K_1(\alpha) + \frac{| h''(0) | + | p''(0) |}{4[K_7(\alpha)]^2}[K_2(\alpha)| h''(0) |$$

$$+ K_3(\alpha)| p''(0) |]$$

$$+ \frac{(h''(0) + p''(0))^{1/2}}{6\sqrt{2}K_7(\alpha)} K_4(\alpha)[| h'''(0) | + | p'''(0) |] + \frac{K_5(\alpha)}{4}| h''(0) + p''(0) |^2$$

$$+ \frac{K_6(\alpha)}{24}| h'^{v}(0) |,$$

which completes the proof of the theorem. □

For $\alpha = 0$, $h(z) = \frac{1 + (1 - 2\beta)z}{1 - z}$ and $p(z) = \frac{1 - (1 - 2\beta)z}{1 + z}$, Theorem 2.2 gives the following estimates for starlike function of order β.

Corollary 2.1. *If* $f \in S^*_\sigma(\beta)$, *then*

$$| a_4 | \leq \min\left\{\frac{4}{3}(1-\beta)^2 + \frac{2}{3}(1-\beta) + \frac{8}{3}(1-\beta)^3, \frac{4\sqrt{2}}{3}(1-\beta)^{3/2} + \frac{2}{3}(1-\beta)\right\}$$

$$| a_5 | \leq \min\left\{\left[\frac{8}{3}(1-\beta)^4 + \frac{8}{3}(1-\beta)^3 + \frac{4}{3}(1-\beta)^2 + \frac{1}{8}(1-\beta)^2 + \frac{1}{2}(1-\beta)\right],\right.$$

$$\left[\frac{1}{3}(1-\beta)^2 + \frac{2\sqrt{2}}{3}(1-\beta)^{3/2} + \frac{4}{3}(1-\beta)^2 + \frac{1}{8}(1-\beta)^2 + \frac{1}{2}(1-\beta)\right]\Bigg\}.$$

For the choice of $\alpha = 0$, $h(z) = \left(\frac{1+z}{1-z}\right)^{\beta}$ and $p(z) = \left(\frac{1-z}{1+z}\right)^{\beta}$, Theorem 2.2 gives the following estimates for strongly starlike function of order β.

Corollary 2.2. *If* $f \in SS_{\sigma}^{*}(\beta)$ *then*

$$|a_4| \le \min\left\{\frac{4}{3}\beta^2 + \frac{2}{9}\beta + \frac{4}{9}\beta^3 + \frac{8}{3}\beta^3, \frac{4\sqrt{2}}{3}\beta^3 + \frac{4}{9}\beta^3 + \frac{2}{9}\beta\right\}$$

$$|a_5| \le \min\Bigg\{\left[\frac{8}{3}\beta^2 + \frac{8}{3}\beta^4 + \frac{8}{9}\beta^3 + \frac{4}{9}\beta + \frac{1}{2}\beta^2 + \frac{5}{48}\beta^4 + \frac{19}{48}\beta^2\right],$$

$$\left[\frac{1}{3}\beta^4 + \frac{4}{3}\beta^4 + \frac{4\sqrt{2}}{9}\beta^4 + \frac{2\sqrt{2}}{9}\beta^2 + \frac{1}{2}\beta^4 + \frac{5}{48}\beta^4 + \frac{19}{48}\beta^2\right]\Bigg\}.$$

For $\alpha = 1$, $h(z) = \frac{1+(1-2\beta)z}{1-z}$ and $p(z) = \frac{1-(1-2\beta)z}{1+z}$, Theorem 2.2 gives the following estimates for convex function of order β.

Corollary 2.3. *If* $f \in C_{\sigma}^{*}(\beta)$, *then*

$$|a_4| \le \min\left\{\frac{1}{3}(1-\beta)^2 + \frac{1}{6}(1-\beta) + \frac{1}{2}(1-\beta)^2, \frac{5}{6}(1-\beta)^{3/2} + \frac{1}{3}(1-\beta)\right\}$$

$$|a_5| \le \min\Bigg\{\left[\frac{7}{5}(1-\beta)^4 + \frac{8}{15}(1-\beta)^3 + \frac{8}{15}(1-\beta)^2 + \frac{3}{20}(1-\beta)^2 + \frac{1}{10}(1-\beta)\right],$$

$$\left[\frac{7}{5}(1-\beta)^2 + \frac{8}{15}(1-\beta)^2 + \frac{4}{15}(1-\beta)^{3/2} + \frac{3}{20}(1-\beta)^2 + \frac{1}{10}(1-\beta)\right]\Bigg\}.$$

For the choice of $\alpha = 1$, $h(z) = \left(\frac{1+z}{1-z}\right)^{\beta}$ and $p(z) = \left(\frac{1-z}{1+z}\right)^{\beta}$, Theorem 2.2 gives the following estimates for strongly convex function of order β.

Corollary 2.4. *If* $f \in SC_{\sigma}^{*}(\beta)$ *then*

$$|a_4| \le \left\{\frac{1}{3}\beta^2 + \frac{1}{2}\beta^3 + \frac{1}{9}\beta^3 + \frac{1}{18}\beta\right\}$$

$$|a_5| \leq \left[\frac{7}{5}\beta^4 + \frac{8}{15}\beta^4 + \frac{8}{45}\beta^4 + \frac{4}{45}\beta^2 + \frac{3}{10}\beta^4 + \frac{1}{48}\beta^4 + \frac{19}{240}\beta^2\right].$$

3. Second Hankel Determinant

The q^{th} Hankel determinant (denoted by $H_q(n)$) for $q = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$ of the function f is the $q \times q$ determinant given by $H_q(n) = \det(a_{n+i+j-2})$. Here $a_{n+i+j-2}$ denotes the entry for the i^{th} row and j^{th} column of the matrix. The second Hankel determinant $H_2(2) = a_2a_4 - a_3^2$ for the class of functions whose derivative has positive real part, the classes of starlike and convex functions with respect to symmetric points have been studied in [3, 4]. The upper bound for the functional $H_2(2)$ for bi-starlike and bi-convex functions of order β obtained in [8].

For the recent works on second Hankel determinant of certain subclass of analytic and bi-univalent function see ([6, 9, 13]). In this section, we obtain second Hankel determinant for function in the class $R_\sigma(\alpha, h, p)$.

To establish our results, we recall the following.

Lemma 3.1 [17]. *If* $p \in \mathcal{P}$, *then* $|P_k| \leq 2$ *for each* $k \in N$, *where* $\mathcal{P}$ *is the family of all functions* p *analytic in* $\mathbb{D}$ *for which* $\operatorname{Re} p(z) > 0$, $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ *for* $z \in \mathbb{D}$.

Lemma 3.2 [18]. *If the function* $p \in \mathcal{P}$, *then*

$$2p_2 = p_1^2 + x(4 - p_1^2)$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1 2)x^2 + 2(4 - p_1^2)(1 - |x|^2)s,$$

for some x, s *with* $|x| \leq 1$ *and* $|s| \leq 1$.

Theorem 3.1. *Let f given by* (1.1) *be in the class* $R_\sigma(\alpha, h, p)$, *then*

$$|a_2a_4 - a_3^2| \leq \begin{cases} R, & Q \leq 0,\ P \leq -\dfrac{Q}{4} \\ 16P + 4Q + R, & Q \geq 0,\ P \geq -\dfrac{Q}{8} \\ & (or) Q \leq 0,\ P \geq -\dfrac{Q}{4} \\ \dfrac{4PR - Q^2}{4P}, & Q > 0,\ P \leq -\dfrac{Q}{8} \end{cases}$$

where

$$P = \left[\frac{(1+2\alpha)}{3(1+\alpha)^3(1+3\alpha)} - \frac{1}{8(1+\alpha)^2(1+2\alpha)} - \frac{1}{3(1+\alpha)(1+3\alpha)} + \frac{1}{(1+\alpha)^4} + \frac{1}{16(1+2\alpha)^2}\right]$$

$$Q = \left[\frac{(1+2\alpha)}{2(1+\alpha)^2(1+2\alpha)} + \frac{7}{3(1+\alpha)(1+3\alpha)} - \frac{1}{2(1+2\alpha)^2}\right]$$

$$R = \frac{1}{(1+2\alpha)^4}.$$

Proof. Let $f \in R_\sigma(\alpha, h, p)$, $0 < \alpha \leq 1$. Then from (2.1), (2.14) and (2.17), we have

$$a_2a_4 - a_3^2 = \frac{1}{3}\frac{h_1^4(1+2\alpha)}{(1+\alpha)^3(1+3\alpha)} + \frac{1}{8}\frac{h_1^2(h_2 - p_2)}{(1+\alpha)^2(1+2\alpha)} + \frac{1}{6}\frac{h_1^2(h_2 + p_2)}{(1+\alpha)(1+3\alpha)} + \frac{1}{6}\frac{h_1(h_3 + p_3)}{(1+\alpha)(1+3\alpha)} - \frac{h_1^4}{(1+\alpha)^4} - \frac{1}{16}\frac{(h_2 - p_2)^2}{(1+2\alpha)^2}. \tag{3.1}$$

According to Lemma 3.2, we write

$$2h_2 = h_1^2 + x(4 - h_1^2)$$

$$2p_2 = p_1^2 + y(4 - p_1^2)$$

$$(h_2 - p_2) = \left(\frac{4 - h_1^2}{2}\right)(x - y) \tag{3.2}$$

and

$$4h_3 = h_1^3 + 2(4 - h_1^2)(h_1 x) - h_1(4 - h_1^2)x^2 + 2(4 - h_1^2)(1 - |x|^2)z$$

$$4p_3 = p_1^3 + 2(4 - h_1^2)(p, y) - p_1(4 - h_1^2)y^2 + 2(4 - h_1^2)(1 - |y|^2)w.$$

Therefore, we have

$$h_3 - p_3 = \frac{h_1^3}{2} + h_1(4 - h_1^2)(x + y) - \frac{h_1(4 - h_1^2)}{4}(x^2 + y^2)$$

$$+ \frac{(4 - h_1^2)}{2}[(1 - |x|^2)z - (1 - |y|^2)w] \tag{3.3}$$

$$h_2 + p_2 = h_1^2 + \left(\frac{(4 - h_1^2)}{2}\right)(x + y) \tag{3.4}$$

for some x, y and z, w with $|x| \leq 1, |y| \leq 1, |w| \leq 1, |z| \leq 1.$

Using (3.2), (3.3) and (3.4), then triangle inequality and letting $|x| = \lambda, |y| = \mu$ from the last equality, we obtain

$$|a_2 a_4 - a_3^2| \leq T_1 + T_2(\lambda + \mu) + T_3(\lambda^2 + \mu^2) + T_4(\lambda + \mu)^2 = F(\lambda, \mu),$$

where

$$T_1 = \left[\frac{1}{4}\frac{1}{(1+\alpha)(1+3\alpha)} + \frac{1}{3}\frac{(1+2\alpha)}{(1+\alpha)^3(1+3\alpha)} + \frac{1}{(1+\alpha)^4}\right]h_1^4$$

$$+ \frac{1}{6}\frac{h_1(4 - h_1^2)}{(1+\alpha)(1+3\alpha)}$$

$$T_2 = \left[\frac{1}{16}\frac{1}{(1+\alpha)^2(1+2\alpha)} + \frac{1}{4}\frac{1}{(1+\alpha)(1+3\alpha)}\right]h_1^2(4 - h_1^2)(|x| + |y|)$$

$$T_3 = \left[\frac{1}{24}\frac{h_1^2(4 - h_1^2)}{(1+\alpha)(1+3\alpha)} - \frac{1}{12}\frac{h_1(4 - h_1^2)}{(1+\alpha)(1+3\alpha)}\right](|x|^2 + |y|^2)$$

$$T_4 = \frac{1}{64}\frac{(4-h_1^2)^2}{(1+2\alpha)^2}(|x|+|y|)^2.$$

We need to maximize the function $F(\lambda, \mu)$ in the closed square $S = \{(\lambda, \mu) : \lambda, \mu \in [0, 1]\}$ for $h \in [0, 2]$. We must investigate the maximum of the function F in the case $h = 0$, $h = 2$ and $h \in (0, 2)$.

Let $h = 0$ then

$$F(\lambda, \mu) = \frac{1}{4(1+2\alpha)^2}(\lambda+\mu)^2 \leq \max\{F(\lambda, \mu : \lambda, \mu \in S)\} = \frac{1}{(1+2\alpha)^2}.$$

For $h = 2$, the function $F(\lambda, \mu)$ is constant as follows

$$F(\lambda, \mu) = \left(\frac{1}{4(1+\alpha)(1+3\alpha)} + \frac{(1+2\alpha)}{3(1+\alpha)^3(1+3\alpha)} + \frac{1}{(1+\alpha)^4}\right)(16)$$

$$= \left(\frac{4}{(1+\alpha)(1+3\alpha)} + \frac{16(1+2\alpha)}{3(1+\alpha)^3(1+3\alpha)} + \frac{16}{(1+\alpha)^4}\right).$$

Now, let $h \in (0, 2)$. In this case, we must investigate the maximum of the function F according to $h \in (0, 2)$ taking into account the sign of $\Delta = F_{\lambda\lambda}F_{\mu\mu} - F_{\lambda\mu}^2$.

Since $\Delta = 4T_3(T_3 + 2T_4)$, $T_3 < 0$ and $T_3 + 2T_4 > 0$ for every $h \in (0, 2)$, $\Delta < 0$, that is, the function $F(\lambda, \mu)$ cannot have a local maximum in the interior of the square S.

Now, we investigate the maximum of F on the boundary of the square S.

For $\lambda = 0$ and $\mu \in [0, 1]$ (the case $\mu = 0, \lambda \in [0, 1]$ investigated. Similarly), we write

$$F(0, \mu) = T_1 + T_2\mu + (T_3 + T_4)\mu^2 = G(\mu)$$

It is clear that $T_3 + T_4 \leq 0$ and $T_3 + T_4 \geq 0$ for some values of $h \in (0, 2)$.

In the case $T_3 + T_4 \leq 0$, the function $G(\mu)$ cannot have a local maximum in the interval $(0, 1)$, but $G(0) = T_1$ and $G(1) = T_1 + T_2 + T_3 + T_4$ in the extremes of the interval $[0, 1]$.

Let $T_3 + T_4 \leq 0$ for some values of $h \in (0, 2)$. Then, the function $G(\mu)$ is an increasing function and the maximum occurs at $\mu = 1$.

Therefore,

$$\max\{G(\mu) : \mu \in [0, 1]\} = G(1) = T_1 + T_2 + T_3 + T_4.$$

For $\lambda = 1$ and $\mu \in [0, 1]$ (the case $\mu = 1$ and $\lambda \in [0, 1]$ investigated Similarly), we write

$$F(1, \mu) = (T_3 + T_4)\mu^2 + (T_2 + 2T_4)\mu + (T_1 + T_2 + T_3 + T_4) = H(\mu).$$

Similar to the above, we write

$$\max\{F(1, \mu) : \mu \in [0, 1]\} = H(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Thus, $G(1) \leq H(1)$, the maximum of the function $F(\lambda, \mu)$ occurs at the point (1, 1) and

$$\max\{F(\lambda, \mu) : \lambda, \mu \in S\} = F(1, 1) = H(1)$$

on the boundary of the square S.

Define the function $\phi : (0, 2) \to \mathbb{R}$ as follows:

$$\phi(h) = T_1 + 2T_2 + 2T_3 + 4T_4 = F(1, 1).$$

Substituting the values of T_1, T_2, T_3 and T_4 in the expression of ϕ, we obtain

$$\phi(h) = \left[\frac{(1+2\alpha)}{3(1+\alpha)^3(1+3\alpha)} - \frac{1}{8(1+\alpha)^2(1+2\alpha)} - \frac{1}{3(1+\alpha)(1+3\alpha)} + \frac{1}{(1+\alpha)^4} + \frac{1}{16(1+2\alpha)^2}\right]h^4$$

$$+ \left[\frac{1}{2(1+\alpha)^2(1+2\alpha)} + \frac{7}{3(1+\alpha)(1+3\alpha)} - \frac{1}{2(1+2\alpha)^2}\right]h^2 + \frac{1}{(1+2\alpha)^4}$$

$$= Pt^2 + Qt + R, \text{ where } t = h^2.$$

Thus we have

$$\max \phi(h) = \begin{cases} R, & \left(Q \le 0,\ P \le -\frac{Q}{4}\right) \\ 16P + 4Q + R, & \left(Q \ge 0,\ P \ge -\frac{Q}{8}\right) (\text{or}) \left(Q \le 0,\ P \ge -\frac{Q}{4}\right) \\ \frac{4PR - Q^2}{4P}, & \left(Q > 0,\ P \le -\frac{Q}{8}\right). \end{cases}$$

$$\text{i.e. } |\, a_2 a_4 - a_3^2 \,| \le \begin{cases} R, & \left(Q \le 0,\ P \le -\frac{Q}{4}\right) \\ 16P + 4Q + R, & \left(Q \ge 0,\ P \ge -\frac{Q}{8}\right) (\text{or}) \left(Q \le 0,\ P \ge -\frac{Q}{4}\right) \\ \frac{4PR - Q^2}{4P}, & \left(Q > 0,\ P \le -\frac{Q}{8}\right). \end{cases}$$

which completes the proof.

Acknowledgement

The authors are thankful to the referee for their insightful suggestions. The work of the first author was supported by the grant given under minor research project 21-22, Tamilnadu state council for higher education.

References

[1] Serap Bulut, Coefficient estimates for a class of analytic and bi-univalent functions, Novi Sad. J. Math. 43(2) (2013), 59-65.

[2] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, in Mathematical Analysis and Its Applications (Kuwait; 1985), KFAS Proceedings Series, Vol. 3, Pergamon, Oxford, (1988), 53-60, see also Studia Univ. Babes-Bolyai Math. 31(2) (1986), 70-77.

[3] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. 18 (1967), 63-68.

[4] D. A. Brannan and J. G. Clunie (Eds.), Aspects of contemporary complex analysis (Proceedings of the NATO Advanced Study Institute held at the University of Durham, July 20, 1979, Academic Press, New York and London, (1980).

[5] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. 23 (2010), 1188-1192.

[6] Q. H. Xu, Y. C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and biunivalent functions, Appl. Math. Lett. 25 (2012), 990-994.

[7] Murat Caglar, Erhan Deniz and Hari Mohan Srivastava, Second Hankel determinant for certain subclasses of bi-univalent functions, Turkish Journal of Mathematics 41 (2017), 694-706.

[8] J. W. Noonan and D. K. Thomas, On the second Hankel determinant of a neatly mean p-valent functions, Trans. Am. Math. Soc. 223 (1976), 337-346.

[9] Halit Orhan, Nanjundan Mahesh and Jagadeesan Yamini, Bounds for the second Hankel dterminant of certain bi-univalent functions, Turkish Journal of Mathematics 40 (2016), 679-687.

[10] Sahsene Altinkaya and Sibel Yalsin, Coefficient Estimates for two new subclasses of bi-univalent functions with respect to symmetric points, Journal of Function Spaces, 2015.

[11] R. M. Ali, S. K. Lee, V. Ravichandran et al., Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, Appl. Math. Lett. 25 (2012), 344-251.

[12] V. Kumar, S. Kumar and V. Ravichandran, Third Hankel determinant for certain classes of analytic functions, International Conference on Recent Advances in Pure and Applied Mathematics (2018), 223-231.

[13] Mulutin Obradovic and Nikola Tuneski, On the fifth coefficients for the class $u(\lambda)$, Advances in Mathematics: Scientific Journal 10(1) (2021), 1-7.

[14] Liangpend Xiong and Xiasli Liu, Some extension of coefficient problems for Bi-univalent Ma-Minda starlike and convex functions, Filomat 29(7) (2015), 1645-1650.

[15] V. Ravichandran and Shelly Verma, Bound for the fifth coefficient of certain starlike functions, C.R. Acad. Sci. Paris. Ser. I 353 (2015), 505-510.

[16] C. Pommerenke, Univalent functions, Vandenhoech and Rupercht, Gottingen, 1975.

[17] U. Grenander and G. Szego, Toeplitz forms and their applications, California, Monographs in Mathematical Sciences, Berkeley, CA, USA, University of California Press, 1958.