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ALMOST WEAKLY PSEUDO QUASI-CONFORMALLY SYMMETRIC MANIFOLDS

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Abstract

The motive of the present paper is to study the properties and some example of an Almost weakly pseudo quasi-conformally symmetric manifolds.

1. Introduction

The curvature tensor R of type $(0, 4)$ in an almost weakly pseudo symmetric manifold satisfies the following [1]

$$(\nabla_X R)(Y, Z, U, V) = (\alpha(X) + \beta(Y))R(Y, Z, U, V) + \gamma(Y)R(X, Z, U, V)$$
$$+ \gamma(Z)R(Y, X, U, V) + \lambda(U)R(Y, Z, X, V) + \lambda(V)R(Y, Z, U, X), \tag{1}$$

where ∇ is covariant derivatives operator with respect to the metric tensor g and $\alpha, \beta, \gamma, \lambda$ are non-zero 1-form related with the vector field $\sigma, \rho, \delta, \nu$ respectively defined as $\alpha(X) = g(X, \sigma), \beta(X) = g(X, \rho), \gamma(X) = g(X, \delta), \lambda(X) = g(X, v)$ for all X. An almost weakly pseudo symmetric manifold of dimension n can be written as $A(WPS)_n$.

The quasi-conformal curvature tensor is defined as [4]

$$C(X, Y, Z, U) = aR(X, Y, Z, U) + b[S(Y, Z)g(X, U) - S(X, Z)g(Y, U)$$

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$$+ S(X, U)g(Y, Z) - S(Y, U)g(X, Z)] - \frac{r}{n(n-1)}(a + 2(n-1)b)$$

$$[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)], \tag{2}$$

If $a = 1$ and $b = -\frac{1}{n-2}$, then above equation takes the form

$$C(X, Y, Z, U) = R(X, Y, Z, U) - \frac{1}{n-2}[S(Y, Z)g(X, U)$$

$$-S(X, Z)g(Y, U) + S(X, U)g(Y, Z) - S(Y, U)g(X, Z)]$$

$$+ \frac{r}{(n-1)(n-2)}[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]$$

$$= \mathcal{C}(X, Y, Z, U), \tag{3}$$

$$\mathcal{W}(X, Y, Z, U) = R(X, Y, Z, U) - \frac{r}{n(n-1)}[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]. \tag{4}$$

where $\mathcal{C}$ represent the conformal curvature tensor, r represent scalar curvature of curvature tensor R and S is the Ricci tensor [6].

By the combination of equation (3) and (4), we have

$$C(X, Y, Z, U) = -(n-2)b\mathcal{C}(X, Y, Z, U) + [a + (n-2)b]\mathcal{W}(X, Y, Z, V). \tag{5}$$

In the Einstein manifold of Riemannian manifold $(M^n, g), (n > 2)$, we have [1]

$$S(X, Y) = \frac{r}{n} g(X, Y). \tag{6}$$

If in a non-quasi-conformally flat Riemannian manifold (M^n, g), quasi conformal curvature tensor C satisfies the condition

$$(\nabla_X C)(Y, Z, U, V) = (\alpha(X) + \beta(Y))C(Y, Z, U, V) + \gamma(Y)C(X, Z, U, V)$$

$$+ \gamma(Z)C(Y, X, U, V) + \lambda(U)C(Y, Z, X, V) + \lambda(V)C(Y, Z, U, X), \tag{7}$$

then the manifold is said to be almost weakly pseudo quasi-conformally symmetric manifold and it is denoted by $A(WPCS)_n$ where $\alpha, \beta, \gamma, \lambda$ has already defined in (1).

2. Almost Weakly Pseudo Quasi-Conformally Symmetric Manifolds

Let orthonormal basis of tangent space is $\{e_i, i = 1, 2, \ldots, n\}$. Summing over $i(1 \leq i \leq n)$ by replacing $X = Y = e_i$ in (6) and (7), we obtained

$$\widetilde{\mathcal{C}}(Y, Z) = 0, \tag{8}$$

and

$$\widetilde{\mathcal{W}}(Y, Z) = S(Y, Z) - \frac{r}{n} g(Y, Z) \tag{9}$$

where $\widetilde{\mathcal{W}}(Y, Z) = \sum_{i=1}^{n} \mathcal{W}(e_i, Y, Z, e_i)$ and $\widetilde{\mathcal{C}}(Y, Z) = \sum_{i=1}^{n} \mathcal{C}(e_i, Y, Z, e_i)$

By virtue of $r = \sum_{i=1}^{n} S(e_i, e_i)$ we obtain from (9), that

$$\sum_{i=1}^{n} \widetilde{\mathcal{W}}(e_i, e_i) = 0. \tag{10}$$

By replacing X and Y with e_i in (5) and using (8) and (9), we get

$$\sum_{i=1}^{n} C(e_i, Y, Z, e_i) = [a + (n-2)b]\widetilde{\mathcal{W}}(Y, Z). \tag{11}$$

By writing the equation (7) in local co-ordinate system it takes the following form

$$\nabla_i C_{jklm} = [\alpha_i + \beta_i] C_{jklm} + \gamma_j C_{iklm} + \gamma_k C_{jilm} + \lambda_l C_{jkim} + \lambda_m C_{jkli}, \tag{12}$$

and also the equation (11) reduces to the form

$$C_{kl} = [a + (n-2)2]\widetilde{\mathcal{W}}_{kl} = [a + (n-2)b]\left(S_{kl} - \frac{r}{n} g_{kl}\right). \tag{13}$$

Transvecting (12) with g^{jm}, we obtain

$$\nabla_i C_{kl} = [\alpha_i + \beta_i] C_{kl} + \gamma_j C^j_{ikl} + \gamma_k C_{il} + \lambda_l C_{ki} + \lambda_m C^m_{kli}. \tag{14}$$

Again transvecting (14) with g^{kl} and using (10) and (13), we get

$$(\gamma_j C^k_{ikl} + \lambda_j C^j_{kli}) g^{kl} = 0. \tag{15}$$

From (13), the above equation yields

$$(\gamma_j + \lambda_j)[a + (n-2)b](S^j_i - \frac{r}{n}\delta^j_i) = 0, \tag{16}$$

which reduce in the form

$$[a + (n-2)b](\widetilde{\mathcal{W}}(X, \delta) + \widetilde{\mathcal{W}}(X, v)) = 0, \tag{17}$$

which gives

$$\widetilde{\mathcal{W}}(X, \delta) + \widetilde{\mathcal{W}}(X, v) = 0 \text{ provided } [a + (n-2)b] \neq 0. \tag{18}$$

In view (9), the relation (18) yield

$$S(X, \delta + v) = \frac{r}{n} g(X, \delta + v). \tag{19}$$

This lead to the following:

Theorem 1. *In an* $A(WPCS)_n$. *The eigenvector* $\delta + v$ *will correspond to the eigenvalue* $\frac{r}{n}$. *Now, let us consider the scalar curvature r is zero in an* $A(WPCS)_n$, *then* (19) *gives* $S(X, \delta + v) = 0$.

On putting $X = Y$ in above equation, we get

$$S(X, \delta + v) = 0,$$

which gives

$$S(Y, \delta) = -S(Y, v), \tag{20}$$

then from (3), (4) and (20), equation (5) yield

$$C(X, Y, \delta, v) = aR(X, Y, \delta, v) + b[S(Y, \delta)\lambda(X) - S(X, \delta)\lambda(Y) + S(X, v)\gamma(Y) - S(Y, v)\gamma(X)]. \tag{21}$$

Hence, we have:

Theorem 2. *In an* $A(WPCS)_n$, *the quasi-conformal curvature tensor C with zero scalar curvature will be as in* (21).

Adversely, if C satisfy the condition (21) then equation (2) yield

$$\frac{r}{n(n-1)}(a + 2(n-1)b)\ (\gamma(Y)\lambda(X) - \gamma(X)\lambda(Y)) = 0. \tag{22}$$

Since γ and λ are nowhere vanishing 1-form, so from (22), we get $r = 0$, provided that $(a + 2(n-1)b) \neq 0.$ Thus it lead following theorem:

Theorem 3. *An* $A(WPCS)_n$ *will have zero scalar curvature if it satisfy* (22), *provided that* $a + 2(n-1)b \neq 0.$ *Let us consider the vanishing Ricci tensor S of an* $A(WPCS)_n$ *then from* (2), *we have*

$$C(X, Y, Z, U) = aR(X, Y, Z, U), \tag{23}$$

where $a \neq 0$, *so by use of* (7), *equation* (23) *gives the sufficient condition for an* $A(WPCS)_n$ *to be an* $A(WPS)_n$. *This lead the following*:

Theorem 4. *For vanishing curvature tensor, an* $A(WPCS)_n$ *will be an* $A(WPS)_n$, *provided that* $a \neq 0.$

3. Einstein $A(WPCS)_n$

Ricci tensor S in an Einstein manifold of $A(WPCS)_n$ defined by (7) will give the following condition

$$dr(Y) = 0 \text{ and } (\nabla_X S)(Y, Z) = 0. \tag{24}$$

In view of (6) and (24), the relation (2) yields

$$C(X, Y, Z, U) = aR(X, Y, Z, U) - \frac{ar}{n(n-1)}[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]. \tag{25}$$

Differentiating the equation (25) covariantly with respect to $\mathcal{L}$, we get

$$(\nabla_{\mathcal{L}}C)(X, Y, Z, U) = a(\nabla_{\mathcal{L}}R)(X, Y, Z, U). \tag{26}$$

Differentiating (5) covariantly, we get

$$(\nabla_{\mathcal{L}}C)(X, Y, Z, U) = -(n-2)b(\nabla_{\mathcal{L}}\mathcal{C})(X, Y, Z, U) - (n-2)b(\nabla_{\mathcal{L}}\mathcal{C})(X, Y, Z, U)$$
$$+[a + (n-2)2]((\nabla_{\mathcal{L}}\mathcal{W})(X, Y, Z, U)). \tag{27}$$

Using (3), (4), (7) and (26) in (27)

$$a(\nabla_{\mathcal{L}}R)(X, Y, Z, U) = -(n-2)b[\{\alpha(\mathcal{L}) + \beta(\mathcal{L})\}\mathcal{C}(X, Y, Z, U) + \gamma(X)\mathcal{C}(\mathcal{L}, Y, Z, U)$$
$$+\gamma(Y)\mathcal{C}(X, \mathcal{L}, Z, U) + \lambda(Z)\mathcal{C}(X, Y, \mathcal{L}, U) + \lambda(U)\mathcal{C}(X, Y, Z, \mathcal{L})]$$
$$+a[a + (n-2)b][\{\alpha(\mathcal{L}) + \beta(\mathcal{L})\}\mathcal{C}(X, Y, Z, U) + \gamma(X)\mathcal{C}(\mathcal{L}, Y, Z, U)$$
$$+\gamma(Y)W(X, \mathcal{L}, Z, U) + \lambda(Z)W(X, Y, \mathcal{L}, U) + \lambda(U)W(X, Y, Z, \mathcal{L})]$$
$$+a[a + (n-2)b][\{\alpha(\mathcal{L}) + \beta(\mathcal{L})\}\mathcal{C}(X, Y, Z, U) + \gamma(X)\mathcal{C}(\mathcal{L}, Y, Z, U)$$
$$+\gamma(Y)W(X, \mathcal{L}, Z, U) + \lambda(Z)W(X, Y, \mathcal{L}, U) + \lambda(U)W(X, Y, Z, \mathcal{L})]. \tag{28}$$

In an Einstein manifold of $A(WPCS)_n$ the conformal curvature tensor (3) follows that

$$\mathcal{C}(X, Y, Z, U) = R(X, Y, Z, U) - \frac{r}{n(n-1)}[g(Y, Z)g(X, U) - g(X, z)g(Y, U)]. \tag{29}$$

It shows that the $\mathcal{C}$ and $\mathcal{W}$ are equivalent. By (28) and (29), we have

$$(\nabla_{\mathcal{L}}R)(X, Y, Z, U) = \{\alpha(\mathcal{L}) + \beta(\mathcal{L})\}$$
$$\left[R(X, Y, Z, U) - \frac{r}{n(n-1)}\{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}\right]$$
$$+\gamma(X)\left[R(\mathcal{L}, Y, Z, U) - \frac{r}{n(n-1)}\{g(Y, Z)g(\mathcal{L}, U) - g(\mathcal{L}, Z)g(\mathcal{L}, U)\}\right]$$
$$+\gamma(Y)\left[R(X, \mathcal{L}, Z, U) - \frac{r}{n(n-1)}\{g(\mathcal{L}, Z)g(X, U) - g(X, Z)g(\mathcal{L}, U)\}\right]$$
$$+\lambda(Z)\left[R(X, Y, \mathcal{L}, U) - \frac{r}{n(n-1)}\{g(Y, \mathcal{L})g(X, U) - g(X, \mathcal{L})g(Y, U)\}\right]$$

$$+\gamma(U)\left[R(X, Y, Z, \mathcal{L})-\frac{r}{n(n-1)}\{g(Y, \mathcal{L})g(X, Z)-g(X, Z)g(Y, \mathcal{L})\}\right] \tag{30}$$

Let us consider that an Einstein $A(WPCS)_n$ is an $A(WPS)_n$. Then by virtue of (30), we get

$$\frac{r}{n(n-1)}[\{\alpha(\mathcal{L})+\beta(\mathcal{L})\}\{g(Y, Z)g(X, U)-g(X, Z)g(Y, U)\}$$
$$+\gamma(X)\{g(Y, Z)g(\mathcal{L}, U)-g(\mathcal{L}, Z)g(Y, U)\}$$
$$+\gamma(Y)\{g(\mathcal{L}, Z)g(X, U)-g(X, Z)g(\mathcal{L}, U)\}$$
$$+\gamma(Z)\{g(Y, \mathcal{L})g(X, U)-g(X, \mathcal{L})g(Y, U)\}$$
$$+\lambda(U)V\{g(Y, \mathcal{L})g(X, Z)-g(X, Z)g(Y, \mathcal{L})\}=0. \tag{31}$$

Taking summation over $i(1 \leq i \leq n)$ by changing X and Y in e_i in above equation, we get

$$r[(n-1)\{\alpha(\mathcal{L})+\beta(\mathcal{L})+\gamma(\mathcal{L})+\lambda(\mathcal{L})\}g(Y, Z)+(n-2)\gamma(Y)g(\mathcal{L}, Z)$$
$$+(n-2)\lambda(Z)g(Y, \mathcal{L})]=0. \tag{32}$$

Substituting $Y=Z=e_i$ in (32), we get

$$r[n(n-1)\{\alpha(\mathcal{L})+\beta(\mathcal{L})\}+2(n-1)\{\gamma(\mathcal{L})+\lambda(\mathcal{L})\}]=0. \tag{33}$$

Again substituting $\mathcal{L}=Z=e_i$ in (32), we obtained

$$r(n-1)[\alpha(Z)+\beta(Z)+\gamma(Z)+(n-1)\lambda(Z)]=0.$$

which reduce into following form by replacing Z with $\mathcal{L}$

$$r(n-1)[\alpha(\mathcal{L})+\beta(\mathcal{L})+\gamma(\mathcal{L})+(n-1)\lambda(\mathcal{L})]=0. \tag{34}$$

Similarly, putting $\mathcal{L}=Z=e_i$ in (32), we get

$$r(n-1)[\alpha(Y)+\beta(Y)+\gamma(Y)+(n-1)+\lambda(Y)]=0,$$

which gives the following form by replacing Z with $\mathcal{L}$

$$r(n-1)[\alpha(\mathcal{L})+\beta(\mathcal{L})+\gamma(\mathcal{L})+(n-1)+\lambda(\mathcal{L})]=0. \tag{35}$$

Adding (33), (34) and (35), we get

$$r(n^2 + n - 2)[\alpha(\mathcal{L}) + \beta(\mathcal{L}) + \gamma(\mathcal{L}) + (n - 1) + \lambda(\mathcal{L})] = 0. \tag{36}$$

Therefore, $r = 0$ provided that $\alpha(\mathcal{L}) + \beta(\mathcal{L}) + \gamma(\mathcal{L}) + (n - 1)\lambda(\mathcal{L}) \neq 0.$ Thus we have the following:

Theorem 5. *The scalar curvature of the manifold will vanish if an Einstein $A(WPCS)_n$ is an $A(WPS)_n$, provided that $\alpha + \beta + \gamma + \lambda \neq 0$.*

Adversely, let us consider the scalar curvature $r = 0$ in an Einstein $A(WPCS)_n$. Then equation (30) yield that an Einstein $A(WPCS)_n$ is an $A(WPS)_n$, provided that $a \neq 0.$ Hence, it lead the following theorem:

Theorem 6. *If the scalar curvature vanishes in an Einstein $A(WPCS)_n$, then the manifold is an $A(WPS)_n$.*

Next, let us consider 1-form γ and λ associated with the vector field δ and v are parallel in an Einstein $A(WPCS)_n$.

Then we get

$$\nabla_X \delta = 0 \text{ and } \nabla_X v = 0, \text{ for all } X. \tag{37}$$

Also we have

$$R(X, Y, v) = \nabla_X \nabla_Y \delta - \nabla_Y \nabla_X \delta - \nabla_{[X, Y]} \delta = 0, \tag{38}$$

and

$$R(X, Y, \nu) = 0. \tag{39}$$

Contracting (38) and (39), we have

$$S(Y, \delta) = 0 \text{ and } S(Y, \nu) = 0. \tag{40}$$

In view if (40), the relation (19) yields

$$rg(Y, \delta + \nu) = 0.$$

If $\| \delta + v \|^2 \neq 0,$ then the above equation will give $r = 0.$

Conversely, if $r = 0,$ then from (19) it is clear that the manifold is an

$A(WPCS)_n$ then it is an $A(WPS)_n$ for $a \neq 0$. Hence we can state the theorem:

Theorem 7. *If the* 1*-form* γ *and* λ *associated with the parallel vector field* δ *and* v *in an Einstein* $A(WPCS)_n$ *then the manifold is an* $A(WPS)_n$, *provided that* $a \neq 0$ *and* $\| \delta + v \|^2 \neq 0$.

4. Example of 5-dimentional $A(WPCS)_5$

Let $M^5 = \{(x^1, x^2, x^3, x^4, x^5) \in \mathbb{R}^5 : x^1 \neq -1\}$ be a five dimensional manifold $\mathbb{R}^5$ and g be a Riemannian metric defined on M^5 as [5]

$$ds^2 = (x^1 + 1)(x^3)^2(dx^1) + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2 + (dx^5)^2. \tag{41}$$

Then the component of contra variant and covariant of the metric are as follows:

$$g_{11} = (x^1 + 1)(x^3)^2,\ g_{12} = g_{21} = 1,\ g_{22} = 0,\ g_{33} = g_{44} = g_{55} = 1,$$

$$g^{11} = 0,\ g^{22} = -(x^1 + 1)(x^3)^2,\ g^{12} = g^{21} = g^{33} = g^{44} = g^{55} = 1. \tag{42}$$

By using (41) and (42) we get the only non zero terms of the curvature tensor and the Christoffel symbols are

$$\Gamma_{11}^{2} = \frac{1}{2}(x^3)^2,\ \Gamma_{11}^{3} = -(x^1 + 1)(x^3),\ \Gamma_{11}^{2} = (x^1 + 1)(x^3),\ R_{1331} = x^1 + 1. \tag{43}$$

Due to (42) and (43) the non-zero term of Ricci tensor are

$$S_{11} = x^1 + 1. \tag{44}$$

The non-zero term of scalar curvature are

$$r = g^{ij}S_{ij} = g^{11}S_{11} + g^{22}S_{22} + g^{33}S_{33} + g^{44}S_{44} + g^{55}S_{55}.$$

By the use of (42) and (44) in above equation, we have

$$r = 0. \tag{45}$$

Therefore, the scalar curvature of (M^5, g) is zero.

Now, by using (3) and (4) in quasi-conformal curvature tensor, the non vanishing term of $\mathcal{W}$ and $\mathcal{C}$ are

$$\mathcal{W}_{1551} = x^1 + 1 \text{ and } \mathcal{C}_{1331} = \frac{2}{3}(x^1 + 1), \mathcal{C}_{1441} = \mathcal{C}_{1551} = -\frac{1}{3}(x^1 + 1)$$

respectively. Using above equation in (5), we get

$$C_{1331} = (a + b)(x^1 + 1) \neq 0, C_{1441} = C_{1551} = b(x^1 + 1) \neq 0. \quad (46)$$

Hence (M^5, g) is a non zero quasi-conformally manifold.

The non-vanishing terms of covariant derivatives of (46) are

$$\nabla_1 C_{1331} = a + b \neq 0, \nabla_1 C_{1441} = \nabla_1 C_{1551} = b \neq 0, \quad (47)$$

and the other components vanishes identically which shows that the manifold M^5 is neither quasi-conformally flat nor quasi-conformally symmetric.

Let us suppose that the member of the 1-form $\alpha, \beta, \gamma, \lambda$ as

$$\alpha_i = \begin{cases} \frac{1}{6(x^1 + 1)}, & \text{if } i = 1 \\ 0, & \text{if } i \neq 1 \end{cases}, \quad \beta_i = \begin{cases} \frac{1}{2(x^1 + 1)}, & \text{if } i = 1 \\ 0, & \text{if } i \neq 1 \end{cases}$$

$$\gamma_i = \begin{cases} \frac{1}{3(x^1 + 1)}, & \text{if } i = 1 \\ 0, & \text{if } i \neq 1 \end{cases}, \quad \lambda_i = \begin{cases} -\frac{1}{(x^1 + 1)}, & \text{if } i = 1 \\ 0, & \text{if } i \neq 1 \end{cases} \quad (48)$$

In the considered manifold (M^5, g) the 1-form (7) reduces to the following equations:

(1) $\nabla_1 C_{1331} = (\alpha_1 + \beta_1) C_{1331} + \gamma_1 C_{1331} + \gamma_3 C_{1131} + \lambda_3 C_{1311} + \lambda_1 C_{1331}$,

(2) $\nabla_3 C_{1131} = (\alpha_3 + \beta_3) C_{1131} + \gamma_1 C_{3131} + \gamma_1 C_{1331} + \lambda_3 C_{1131} + \lambda_1 C_{1133}$,

(3) $\nabla_3 C_{1311} = (\alpha_3 + \beta_3) C_{1311} + \gamma_1 C_{3311} + \gamma_3 C_{1311} + \lambda_1 C_{1331} + \lambda_1 C_{1313}$,

(4) $\nabla_1 C_{1441} = (\alpha_1 + \beta_1) C_{1441} + \gamma_1 C_{1441} + \gamma_4 C_{1141} + \lambda_4 C_{1411} + \lambda_1 C_{1441}$,

(5) $\nabla_4 C_{1141} = (\alpha_4 + \beta_4) C_{1141} + \gamma_1 C_{4141} + \gamma_1 C_{1441} + \lambda_4 C_{1141} + \lambda_1 C_{1144},$

(6) $\nabla_4 C_{1411} = (\alpha_4 + \beta_4) C_{1411} + \gamma_1 C_{4411} + \gamma_4 C_{1411} + \lambda_4 C_{1441} + \lambda_1 C_{1414},$

(7) $\nabla_1 C_{1551} = (\alpha_1 + \beta_1) C_{1551} + \gamma_1 C_{1551} + \gamma_5 C_{1151} + \lambda_5 C_{1511} + \lambda_1 C_{1551},$

(8) $\nabla_5 C_{1151} = (\alpha_5 + \beta_5) C_{1151} + \gamma_1 C_{5151} + \gamma_1 C_{1551} + \lambda_5 C_{1151} + \lambda_1 C_{1155},$

(9) $\nabla_5 C_{1511} = (\alpha_5 + \beta_5) C_{1511} + \gamma_1 C_{5511} + \gamma_5 C_{1511} + \lambda_5 C_{1551} + \lambda_1 C_{1515}.$

It is clear that the equation (1)-(9) hold for every member of (48). Thus the manifold M^5 is an $A(WPCS)_5$ with vanishing scalar curvature. Hence it lead the following:

Theorem 8. *The Riemannian manifold* (M^5, g) *with the metric*

$$ds^2 = (x^1 + 1)(x^3)^2(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2 + (dx^5)^2$$

is an $A(WPCS)_5$ *with zero scalar curvature which is neither quasi-conformally flat nor quasi-conformally symmetric.*

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