Advances and Applications in Mathematical Sciences
Volume 21, Issue 5, March 2022, Pages 2511-2525

METRIC DIMENSION OF GRAPHS OBTAINED FROM COMMUTATIVE RING

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Abstract

For a connected graph of order n, the metric basis of a G is a smallest set $S = \{v_1, v_2, \ldots, v_k\}$ of vertices of G such that for vertex $u \in G$, the ordered k-tuples of distances $\{d(u, v_1), d(u, v_2), d(u, v_3), \ldots, d(u, v_k)\}$ are all distinct. The metric dimension of G, denoted as $\dim(G)$, is the cardinality of a metric basis for G. In the present work we investigate metric dimension of intersection graphs and annihilator ideal graphs of commutative ring R.

1. Introduction

Throughout this work, we consider ring R as commutative ring with unity, unless otherwise stated. For the ring R, $Z(R)$ denotes the set of all zero-divisors while $Z^*(G)$ denotes the set of all non-zero zero-divisors. Moreover, we denote the set of all proper ideals of a ring R by $I^*(R)$. For any element $r \in R$, the ideal generated by the element r is denoted by (r) and an ideal I is said to be annihilating ideal of R if there exist an ideal J of R such that $IJ = 0$, where 0 denotes the zero ideal of R. The annihilator of I,

2020 Mathematics Subject Classification: 13A99, 05C12.
Keywords: Commutative ring; intersection graph; annihilator ideal graph; metric dimension.
Received August 13, 2021; Accepted October 29, 2021

denoted as $Ann_R(I)$, is defined as $Ann_R(I) = \{J : IJ = 0\}$. The set of all annihilating ideals of R is denoted as $A(R)$ and, in addition, the set of all non-zero annihilating ideals denoted by $A^*(R)$. For any element x of R, the set of all annihilators of x is, denoted as $ann_R(x)$, defined as $ann_R(x) = \{y : xy = 0\}$. For any ideal I of R, nilpotency $(I) = n$ if $I^n = 0$ for some least integer n. For any rings, R and S the direct product of R and S, denoted as $R \times S$, is defined as $R \times S = \{(r, s) : r \in R, s \in S\}$.

The concept of graph associated with ring was first studied by Beck [7], known as zero-divisor graph, in which all the elements of R were considered as the vertices of graph and two vertices x and y are adjacent if and only if $xy = 0$. Later on, this definition was reformed by Anderson and Livingston [3], in which they considered the set of all non-zero zero-divisors as the set of vertices. Chakra arty et al. [8] have introduced the concept of intersection graph for a commutative ring, denoted as $G(R)$, with the vertex set $I^*(R)$, where two distinct vertices I and J are adjacent if and only if $I \cap J \neq 0$. Salehifar et al. [23] have introduced the concept of annihilator ideal graph for a commutative ring, denoted as $A_I(R)$ with the vertex set $A^*(R)$, where two distinct vertices I and J are adjacent if and only if $Ann_R(IJ) \neq Ann_R(I) \cup Ann_R(J)$. Badawi [5] introduced the concept of annihilator graph for a commutative ring with the set of non-zero zero-divisors considered as the set of vertices and two distinct vertices x and y are adjacent if and only if $ann_R(x) \cup ann_R(x) \neq ann_R(xy)$.

Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. Now recall that a graph is connected if there exist a path between any pair of vertices. The distance between any two vertices x and y is denoted as $d(x, y)$ and defined as the length of the shortest path between them. If such path does not exist then we say $d(x, y) = \infty$. The diameter of a connected graph G, denoted as $diam(G)$, defined as the maximum distance between any pair of vertices of G. A graph in which every pair of vertices is joined by an edge is called complete graph and we denote K_n as the complete graph of n vertices. For the graph theoretic terminology we rely on [6, 25] and for commutative ring theory we refer to [4, 11].

The next section is aimed to provide preliminaries needed for the present work.

2. Preliminaries and Exiting Results

Let G be any connected graph with $n \geq 2$ vertices. For an ordered subset $W = \{v_1, v_2, \ldots, v_k\}$ of vertices of G, we refer to the k-vectors as the metric representation of u with respect to W as

$$r(u \mid W) = \{d(u, v_1), d(u, v_2), d(u, v_3), \ldots, d(u, v_k)\}$$

The set W is said to be resolving set of G if distinct vertices have distinct metric representations. A resolving set containing minimum number of vertices is called a metric basis. The metric dimension, denoted by $\dim(G)$, of G is the cardinality of a metric basis. The metric basis and metric dimension are also known as locating set and locating number. This implies that the metric dimension of G is at most $n - 1$. In fact, for every connected graph G of order $n \geq 2, 1 \leq \dim(G) \leq n - 1$. The metric dimension of graphs arising from rings have been extensively studied in [14, 15, 16, 17, 18, 19].

The following results are some existing results for metric dimension of some graphs.

Lemma 2.1. [20] *A connected graph G of order n has metric dimension* 1 *if and only if* $G \cong P_n$, where P_n *is a path on n vertices*.

Lemma 2.2. [20] *A connected graph G of order $n \geq 2$ has metric dimension $n - 1$ if and only if $G \cong K_n$.*

Lemma 2.3. [20] *For $n \geq 3$, metric dimension of a cycle C_n is* 2.

Lemma 2.4. [20] *For $n \geq 3$, the metric dimension for the bipartite graph $K_{1,\, n-1}$ is $n - 2$ and for $r \geq 2, n \geq 5$, the metric dimension for the bipartite graph $K_{r,\, n-r} = K_{n,\, m}$ with $n = r$ and $m = n - r$ is $n - 2$.*

Lemma 2.5. [20] *Let G be a connected graph with $diam(G) = m < \infty$. If $dim(G) = k < \infty$*, then $\mid V(G) \mid \leq (m + 1)k$.

Lemma 2.6. [20] *Let G be a connected graph with finite diameter. Then $|V(G)|$ is finite if and only if $dim(G)$ is finite.*

Lemma 2.7. [21] *Let R be a commutative ring with unity. Then $dim(\Gamma(R))$ is finite if and only if R is finite.*

Lemma 2.8. [24] *Let $n \neq 2$ be a positive integer and $R = \prod_{i=1}^{n} \mathbb{Z}_2$. Then the following statements hold.*

(1) $dim(AG(\prod_{i=1}^{n} \mathbb{Z}_2)) = n - 1$, *for* $n = 2, 3$.

(2) $dim(AG(\prod_{i=1}^{n} \mathbb{Z}_2)) = n$, *for* $n \geq 4$.

The present work is intended to investigate metric dimension of the graphs obtained from commutative ring.

3. Metric Dimension of Intersection Graph of ideals of Ring

We begin with a commutative ring R with unity such that R is not isomorphic to product of two fields because intersection graph of product of two fields is not a connected graph.

Theorem 3.1. *Let R be any ring. Then*

(i) *$dim(G(R))$ is finite if R is finite.*

(ii) *$dim(G(R))$ is undefined if and only if R is a field.*

Proof. (i) It is clear that R has finitely many ideals as R is finite. Hence by Lemma 2.6, $\dim(G(R))$ is finite.

(ii) It follows from the fact that the metric dimension of $G(R)$ is undefined if and only if the vertex set of $G(R)$ is empty. □

Remark 3.2. The converse of (i) in Theorem 3.1 is not true in general. For example, Let $\mathbb{Z}_8$ be a ring and $\mathbb{Q}$ be a field of rational numbers. Then $\mathbb{Z}_8$ has only 4 ideals namely (0), (2), (4) and $\mathbb{Z}_8$ itself and $\mathbb{Q}$ has only 2 ideals namely (0) and $\mathbb{Q}$. That implies $R = \mathbb{Z}_8 \times \mathbb{Q}$ is infinite ring with 6 proper ideals namely $((0), \mathbb{Q}), ((2), \mathbb{Q}), ((4), \mathbb{Q}), ((2), (0)), ((4), (0)), (\mathbb{Z}_8 (0))$ and 2 trivial ideals namely $((0), (0)), (\mathbb{Z}_8, \mathbb{Q})$. Now it is easy to check that the set $\{((2), \mathbb{Q}), ((4), \mathbb{Q}), ((2), (0)), ((4), (0))\}$ is metric bases of $G(R)$. Hence $\dim(G(R)) = 4$.

Theorem 3.3. *For any ring R*

(i) $dim(G(R)) = 1$ *if and only if* $G(R)$ *is a path.*

(ii) $dim(G(R)) = 2$ *if* $G(R)$ *is a cycle.*

(iii) $dim(G(R)) = |I^*(R)| - 1$ if and only if $G(R)$ is a complete graph.

(iv) $dim(G(R)) = |I^*(R)| - 2$ *if and only if* $G(R)$ *is a star graph* (other *than* $K_{1, 1}$) *or a bipartite graph.*

Proof. The proof is immediate from Lemma 2.1 to Lemma 2.4. □

Theorem 3.4. *If R is a finite local principal ideal ring with nil potency* $(R) = n,\ n \geq 2,$ *then* $dim(G(R)) = n - 2$.

Proof. Since R is finite local principal ideal ring with nil potency $(R) = n$ there exist a maximal ideal m of R such that $m^n = (0)$. So R has only $n - 1$ proper ideals, namely $m, m^2, m^3, \ldots, m^{n-1}$ such that $m^{n-1} \subset m^{n-2} \subset \ldots \subset m^2 \subset m$. Moreover $m^i \cap m^j \neq (0)$ for any $i, j \in \{1, 2, \ldots, n-1\}$. Hence $G(R)$ is complete graph with $n - 1$, vertices and so by Lemma 2.2 $\dim(G(R)) = n - 2$. □

Theorem 3.5. *Let R be any ring with n proper ideals and* $\mathbb{F}$ *be any field.*

If $dim(G(R)) = n - 1$ then $dim(G(R \times \mathbb{F})) = 2n$.

Proof. Since $\dim(G(R)) = n - 1$ which implies $G(R) = K_n$. Let $I_1, I_2, \ldots, I_n$ be proper ideals of R, then $I_i \cap I_j \neq (0), \forall i \neq j$. Then $R \times \mathbb{F}$ has total $2n + 4$ ideals in which $2n + 2$ proper ideals which are $(R, (0))$,

$((0), \mathbb{F})$, $(I_i, (0))$ and $(I_i, \mathbb{F})$, where $i = 1, 2, 3, \ldots, n$ also $(I_i, (0)) \cap (I_j, \mathbb{F}) \neq (0)$, $\forall i, j$. Since $I_i \cap I_j \neq (0)$ and also we have $(I_i, (0)) \cap (R, (0)) \neq (0)$, $(I_i, \mathbb{F}) \cap (R, (0)) \neq (0)$, $(I_i, \mathbb{F}) \cap ((0), \mathbb{F}) \neq (0)$. Now let $A = \{(R, (0)), (I_i(0)), (I_i, \mathbb{F}), \forall i = 1, 2, 3, \ldots, n$ be the set of vertices such that they are mutually adjacent and the vertex $((0), \mathbb{F})$ is adjacent only with $(I_i, \mathbb{F})$, $\forall i = 1, 2, 3, \ldots, n$. So, set of vertices A forms a complete sub graph of $G(R \times \mathbb{F})$ with $2n + 1$ vertices say it G_1, then $\dim(G_1) = 2n$. Hence $\dim(G(R \times \mathbb{F})) \geq 2n$. Now suppose that $\dim(G(R \times \mathbb{F})) \geq 2n + 1$, but $(G(R \times \mathbb{F}))$ has $2n + 2$ vertices that implies $G(R \times \mathbb{F}) = K_{2n+2}$ which is not true. Therefore $\dim(G(R \times \mathbb{F})) \geq 2n$. □

Corollary 3.6. *If* $G(\mathbb{Z}_n)$ *is connected and non Hamiltonian, then* $\dim(G(\mathbb{Z}_n)) \in \{0, 1, 2\}$.

Proof. As proved by Chakrabarty et al. [8] $G(\mathbb{Z}_n)$ is non Hamiltonian if and only if $n = P^2, pq, p^2q, p^3$ and $G(\mathbb{Z}_n)$ is disconnected for $n = pq$. If $n = p^2$, then $G(\mathbb{Z}_n)$ is a graph with a single vertex. Hence $\dim(G(\mathbb{Z}_n)) = 0$. If $n = p^3$, then $G(\mathbb{Z}_n)$ is path of order 2. Hence $\dim(G(\mathbb{Z}_n)) = 1$. If $n = p^2q$, then $G(\mathbb{Z}_n)$ is shown in Figure 1 which has dimension 2. Hence proved.

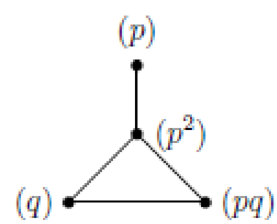

Figure 1. $G(\mathbb{Z}_{p^2q})$.

Theorem 3.7. *If* R_1 *and* R_2 *are any finite local principle ideal rings with nilpotency* $(R_1) = n_1$ *and nilpotency* $(R_2) = n_2$ *then* $dim(G(R_1 \times R_2)) = n_1n_2 + n_1 + n_2 - 3$.

Proof. Let R_1 and R_2 be any finite local principle ideal rings with nilpotency n_1 and n_2 respectively then $G(R_1) = K_{n_1-1}$ and $G(R_2) = K_{n_2-1}$. Moreover R_1 and R_2 has total n_1+1 and n_2+1 ideals in which n_1-1 and n_2-1 are proper ideals. Let $I_1, I_2, \ldots, I_{n_1-1}$ and $J_1, J_2, \ldots, J_{n_1-1}$ be proper ideals of R_1 and R_2 respectively, then $R_1 \times R_2$ will have $(n_1+1)(n_2+1)-2$ proper ideals namely $(R_1, 0), (0, R_2), (I_i, 0), (0, j_j)$ $(I_i, J_j), (I_i, R_2), (R_1, J_j)$ for $i = 1, 2, \ldots, n_1 - 1$ and $j = 1, 2, \ldots, n_2 - 1$. Now consider all the proper ideals of $R_1 \times R_2$ as vertices of $G(R_1 \times R_2)$. Since ideal I_i is adjacent with all other ideals of R_1, so the vertices $(I_i, 0), (I_i, J_j)$ and (I_i, R_2) are mutually adjacent and also ideal J_j is adjacent with all other ideals of R_2 then the vertices $(0, J_j), (I_i, J_j)$ and (R_1, J_j) are mutually adjacent. Moreover the vertices $(R_1, 0), (R_1, J_j), (I_i, J_j)$ are mutually adjacent and vertices $(0, R_2), (I_i, R_2), (I_i, J_j)$ are also mutually adjacent. Now consider three disjoint subsets of vertices $V_1 = \{(R_1, J_j), (I_i, R_2), (I_i, j_j)\}$, $V_2 = \{(R_1, 0), (I_i, 0)\}$ and $V_3 = \{(0, R_2), (0, J_j)\}$. Then sub graph induced from V_1, V_2, and V_3 will form complete graphs $K_{n_1 n_2 - 1}, K_{n_1}, K_{n_2}$ respectively. Hence the result follows. □

Corollary 3.8. *Let* $R_1, R_2, \ldots, R_n$ *be rings and* $R \cong R_1 \times \ldots \times R_n$, *then* $dim(G(R)) < \infty$ *if and only of* $dim(G(R_i)) < \infty$, *for every* $i, 1 \le i \le n$.

Proof. Let $R_1, R_2, \ldots, R_n$ be rings and $R \cong R_1 \times R_2 \times \ldots \times R_n$. Now suppose that $\dim(G(R_i)) < \infty$ for every $i, 1 \le i \le n$, then it is clear that R_i has finitely many ideals, that means $|I^*(R_i)|$ is finite. Since R is direct product of R_i, R has finitely many ideals which implies $|V(G(R))|$ is finite. Hence $\dim(G(R)) < \infty$. Conversely suppose that $\dim(G(R)) < \infty$. Now suppose, if possible, that there is a ring R_i such that $\dim(G(R)) < \infty$, then $|V(G(R_i))| = \infty$, which implies that $|V(G(R))| = \infty$. This contradicts to our supposition that $\dim(G(R))$ is finite. Hence $\dim(G(R)) = \infty$, for every $i, 1 \le i \le n$. □

4. Metric Dimension of Annihilator Ideal Graph of a Ring

A very obvious but important result has been proved in following theorem.

Theorem 4.1. *Let R be any commutative ring, then*

(i) *If* $dim(\Gamma(R))$ *is finite then* $dim(\mathbb{A}_{\mathbb{I}}(R))$ *is finite.*

(ii) $dim(\mathbb{A}_{\mathbb{I}}(R))$ *is undefined if and only if R is an integral domain.*

Proof. (i) Suppose $\dim(\Gamma(R))$ is finite then, by Lemma 2.7 R is finite.

(ii) It follows from the fact that an integral domain has no non-zero annihilating ideals. □

It can be seen from Illustration 4.2 that converse of Theorem 4.1 (i) need not be true.

Illustration 4.2. Let $R = \mathbb{Z}_6 \times \mathbb{Q}$ then $\dim(\Gamma(R))$ is infinite as R is infinite. Also R has 8 ideals and $V(\mathbb{A}_{\mathbb{I}}(R)) = \{((3), 0), ((3), \mathbb{Q}), (0, \mathbb{Q}), ((1), \mathbb{Q}), ((2), \mathbb{Q}), ((2), 0)\}$. Then $\mathbb{A}_{\mathbb{I}}(R)$ is as shown in Figure 2 and $\dim(\mathbb{A}_{\mathbb{I}}(R)) = 2$.

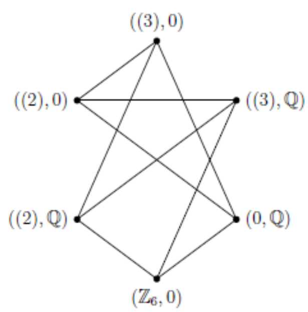

Figure 2. Graph $\mathbb{A}_{\mathbb{I}}(R)$.

Theorem 4.3. *Let R be a ring. Then* $dim(\mathbb{A}_{\mathbb{I}}(R))$ is *finite if and only if every vertex of* $\mathbb{A}_{\mathbb{I}}(R)$ *has finite degree.*

Proof. Suppose that $\dim(\mathbb{A}_{\mathbb{I}}(R))$ is finite, say k. Hence the metric representation of any vertex v has k-tuple. As proved by Salehifar et al. [23]

$\text{diam}(\mathbb{A}_{\mathbb{I}}(R)) \leq 2,$ it follows that each co-ordinate is either of 0, 1 or 2. Hence the number of vertices in $\mathbb{A}_{\mathbb{I}}(R)$ is at most 3^k. Therefore every vertex has finite degree. Conversely suppose that every vertex of $\mathbb{A}_{\mathbb{I}}(R)$ has finite degree. Then $\mathbb{A}_{\mathbb{I}}(R)$ is finite as $\text{diam}(\mathbb{A}_{\mathbb{I}}(R)) \leq 2.$ □

Theorem 4.4. *Let R be a reduced commutative ring such that $|\mathbb{A}_{\mathbb{I}}(R)| \geq 3.$ If $gr(\mathbb{A}_{\mathbb{I}}(R)) = \infty$ then $dim(\mathbb{A}_{\mathbb{I}}(R)) = |\mathbb{A}^*(R)| - 2.$*

Proof. If R is a reduced commutative ring with $|\mathbb{A}_{\mathbb{I}}(R)| \geq 3$ then as proved by Salehifar et al. [23] $(\mathbb{A}_{\mathbb{I}}(R)) = K_{1, |\mathbb{A}^*(R)|-1}$ *and by Lemma* 2.4

$dim(\mathbb{A}_{\mathbb{I}}(R)) = |\mathbb{A}^*(R)| - 2.$ □

Theorem 4.5. *Let R be any ring then*

(i) $diam(\mathbb{A}_{\mathbb{I}}(R)) = 0$ *if and only if* $dim(\mathbb{A}_{\mathbb{I}}(R)) = 0.$

(ii) $diam(\mathbb{A}_{\mathbb{I}}(R)) = 1$ *if and only if* $dim(\mathbb{A}_{\mathbb{I}}(R)) = |\mathbb{A}^*(R)| - 1.$

Proof. (i) $\text{diam}(\mathbb{A}_{\mathbb{I}}(R)) = 0$ if and only if it has a single vertex.

Hence $\dim(\mathbb{A}_{\mathbb{I}}(R)) = 0.$ □

(ii) $\text{diam}(\mathbb{A}_{\mathbb{I}}(R)) = 1$ if and only if it is complete graph. Hence result follows. □

Theorem 4.6. *If R is any local principle ideal ring with nil potency $(R) = n,$ then $dim(\mathbb{A}_{\mathbb{I}}(R)) = n - 2.$*

Proof. Let R be any finite local principal ideal ring with nilpotency $(R) = n,$ this implies that there exist a maximal ideal m of R such that $m^n = (0).$ Then ring R has only $n-1$ proper ideals, namely $m, m^2, m^3, \ldots, m^{n-1}$ and $Ann(m) = \{m^{n-1}\}, Ann(m^2) = \{m^{n-1}, m^{n-2}\},$ $Ann(m^3) = \{m^{n-1}, m^{n-2}, m^{n-3}\}, \ldots, Ann(m^{n-1}) = \{m^{n-1}, m^{n-2}, \ldots, m\},$ in general we can write $Ann(m^i) = \{m^{n-j} : 1 \leq j \leq i\}.$ Now by the definition of annihilator ideal graph, the ideals m^i and m^j are adjacent if $i + j \geq n$ as

$m^i m^j = (0)$. Now what happens when $i + j < n$. In other words, we have to check the adjacency of ideals whenever $m^i m^j = m^k \neq (0)$ and also in this case, it is clear that $k > i, j$. In this case $Ann(m) \subset Ann(m^2) \subset Ann(m^3) \subset \ldots \subset Ann(m^{n-1})$ as mentioned earlier $Ann(m) = Ann\{m^{n-1}\}$, $Ann(m^2) = \{m^{n-1}, m^{n-2}\}$, $Ann(m^3) = \{m^{n-1}, m^{n-2}, m^{n-3}\}, \ldots, Ann(m^{n-1}) = \{m^{n-1}, m^{n-2}, \ldots, m\}$. Therefore $Ann(m^i, m^j) \supset Ann(m^i) \cup Ann(m^j)$ which implies that m^i and m^j are also adjacent whenever $i + j < n$. Hence all ideals of the ring R are mutually adjacent in $\mathbb{A}_{\mathbb{I}}(R)$, that means $\mathbb{A}_{\mathbb{I}}(R) = K_{n-1}$. Hence the result follows. □

Theorem 4.7. *If R is a finite principle ideal ring with nil potency $(R) = n$ and $\mathbb{F}$ is any field, then $dim(\mathbb{A}_{\mathbb{I}}(R \times \mathbb{F})) = 2n - 3$.*

Proof. As nilpotency $(R) = n$ and $\mathbb{F}$ is a field, $R \times \mathbb{F}$ has total $2n + 2$ ideals, in which $2n$ ideals are proper. Since R is a finite local principal ideal ring, there exist maximal ideal m of R such that $m^n = (0)$. Therefore proper ideals of R and $R \times \mathbb{F}$ are $\{m, m^2, m^3, \ldots, m^{n-1}\}$ and $\{((0), \mathbb{F}), (R, (0)), (m^i, (0)), (m^i, \mathbb{F}), \forall i = 1, 2, 3, \ldots, n-1\}$ respectively. Now, we will check the adjacency of ideals of $R \times \mathbb{F}$. Thus, we have $Ann_R((0), \mathbb{F}) = \{(m^i, (0)) : 1 \leq i \leq n-1\}$, $Ann_R(R, (0)) = \{((0), \mathbb{F})\}$, $Ann_R(m^i, (0)) = \{((0), \mathbb{F}), (m^{n-j}, (0)), (m^{n-j}, \mathbb{F}) : 1 \leq j \leq i\}$ and $Ann_R(m^i, \mathbb{F}) = \{(m^{n-j}, (0)) : 1 \leq j \leq i\}$. As we know that if $IJ = (0)$ then I and J are adjacent in annihilator ideal graph, the ideal $((0), \mathbb{F})$ is adjacent to all elements of the set $\{(m^i, (0)), \forall i = 1, 2, \ldots, n-1\}$ and similarly, $(R, (0))$ is adjacent to $\{((0), \mathbb{F})\}$, $(m^i, (0))$ is adjacent to all the elements of the set $\{((0), \mathbb{F})\}, (m^{n-j}, (0)), (m^{n-j}, \mathbb{F})_{j=1}^{i}$, and $(m^i, \mathbb{F})$ is adjacent to all elements of the set $\{(m^{n-j}, (0))\}_{j=1}^{i}$. Now, the mutual product of remaining combinations of ideals are $(m^i, (0))(m^j, \mathbb{F}) = ((m^{i+j}, (0))), (m^i, \mathbb{F}), (m^j, \mathbb{F}) = (m^{i+j}, \mathbb{F})$ whenever $i + j < n$, $(m^i, (0))(R, (0)) = ((m^i, (0)), (m^i, \mathbb{F})((0), \mathbb{F}) = ((0), \mathbb{F})$, and $(m^i, \mathbb{F})(R, (0)) = (m^i, (0))$. The pairs of vertices which are adjacent in annihilator ideal graphs are $((m^i, (0)), (m^j, \mathbb{F})), ((m^i, (0)), ((m^j, (0))$ $((m^i, \mathbb{F}), (m^j, \mathbb{F}))$ and $((m^j, \mathbb{F}), (R, (0))$, while $((m^j, (0)), (R, (0)))$ and $((m^i, \mathbb{F}), ((0), \mathbb{F}))$ are also non-adjacent. Onsider disjoint partitions of vertex set $V(\mathbb{A}_{\mathbb{I}}(\mathbb{R} \times \mathbb{F}))$ such that $A = \{(m^i, (0)) : i = 1, 2, 3, \ldots, n-1\}, B = \{(m^i, \mathbb{F}) : i = 1, 2, 3, \ldots, n-1\}$, $C = \{((0), \mathbb{F})\}$ and $D = \{(R, (0))\}$, then sub graphs induced from A as well as B are complete. Moreover, the element of C is adjacent with all the elements of A and D, the element of D is adjacent with all the elements of B and all the elements of A are adjacent with all the elements of B. Therefore the annihilator ideal graph $\mathbb{A}_{\mathbb{I}}(\mathbb{R} \times \mathbb{F})$ has complete sub graph of order $2n - 2$. Hence result follows. □

Theorem 4.8. *Let* $R_1 = \mathbb{Z}_n$ *for* $n = p_1 p_2 \ldots p_k$, *where* p_i*'s are distinct primes and* $R_2 = F_1 \times F_2 \times \ldots \times F_k$, *where each* $F_i, 1 \le i \le k$, is *a field and* $R_3 = \prod_{i=1}^{k} \mathbb{Z}_2$. *Then* $\mathbb{A}_{\mathbb{I}}(R_1) = \mathbb{A}_{\mathbb{I}}(R_2) = AG(R_3)$.

Proof. First of all note that $V(AG(R_3)) = \{(x_1, x_2, \ldots, x_k) : x_i \in \{0, 1\}\}$, but not all x_i's are 0 or 1 together. Also, the vertices of $\mathbb{A}_{\mathbb{I}}(R_1)$ are ideals generated by the divisors of n which are 2^k because $n = p_1 p_2 \ldots p_k$ has total 2^k divisors.

Let $\mathcal{D}$ be the set of all divisors of n. for $d \in \mathcal{D}$ and $(x_1, x_2, \ldots, x_k) \in R_3$, define a map $\varrho : \mathcal{D} \to R_3$ by

$$\varrho(d) = (x_1, x_2, \ldots, x_k), \text{ where } x_i = \begin{cases} 0, & \text{if } p_i \mid d \\ 1, & \text{otherwise} \end{cases}$$

For $d_1, d_2 \in \mathcal{D}$, if $d_1 = d_2$, then $p_i \mid d_1$ if and only if $p_i \mid d_2$. Therefore, ϱ is bijection. Let d_1 and d_2 be two divisors of n such that d_1 and d_2 are adjacent in $\mathbb{A}_{\mathbb{I}}(R_1)$, then $Ann_{R_1}(d_1) \cup Ann_{R_1}(d_2) \neq Ann_{R_1}(d_1 d_2)$. Let

$\varrho(d_1) = (x_1, x_2, \ldots, x_k)$, where $x_i = 0$ if $p_i \mid d_1$ and $\varrho(d_2) = (y_1, y_2, \ldots, y_k)$, where $y_i = 0$ if $p_i \mid d_2$ then $(\varrho(d_1))(\varrho(d_2)) = (z_1, z_2, \ldots, z_k)$, where $z_i = 0$ if $p_i \mid d_1$ or $p_i \mid d_2$. Also $ann_{R_3}(\varrho(d_1)) = \{(x'_1, x'_2, \ldots, x'_k) : x'_i = 0 \text{ if } p_i \nmid d_1\}$ and $ann_{R_3}(\varrho(d_2)) = \{(y'_1, y'_2, \ldots, y'_k) : y'_i = 0 \text{ if } p_i \nmid d_2\}$ implies that $ann_{R_3}(\varrho(d_1))\, ann_{R_3}(\varrho(d_2)) = \{(w_1, w_2, \ldots, w_k) : w_i = 0 \text{ if } p_i \nmid d_1 \text{ or } p_i \nmid d_2\}$ and $ann_{R_3}(\varrho(d_1))(\varrho(d_2)) = \{(z'_1, z'_2, \ldots, z'_k) : z'_i = 0 \text{ if } p_i \nmid d_1 \text{ and } p_i \nmid d_2\}$. By using our assumption that d_1 and d_2 are adjacent, that means $Ann_{R_1}(d_1) \cup Ann_{R_1}(d_2) \neq Ann_{R_1}(d_1 d_2)$, then it is clear that $ann_{R_3}(\varrho(d_1)) \cup ann_{R_3}(\varrho(d_2)) \neq ann_{R_3}(\varrho(d_1))(\varrho(d_2))$. Therefore $\varrho(d_1)$ and $\varrho(d_2)$ are adjacent in $AG(R_3)$. Thus ϱ preserves adjacency. Similarly we can prove that ϱ preserves non-adjacency and hence $\mathbb{A}_{\mathbb{I}}(R_1) \cong AG(R_3)$. Since each field F_i has two ideals, the numbers of ideals of R_2 is 2^k. The vertices of $\mathbb{A}_{\mathbb{I}}(R_2)$ are of the form $I_1 \times I_2 \times \ldots \times I_k$, where at least one $I_i = (0)$ and at least one $I_i = F_i$. Therefore, $|V(\mathbb{A}_{\mathbb{I}}(R_2))| = 2^k - 2$. Also, we have $|V(AG(R_3))| = 2^k - 2$. Now, define a map $\varrho : V(\mathbb{A}_{\mathbb{I}}(R_2)) \to V(AG(R_3))$ by $\varrho(I_1 \times I_2 \times \ldots \times I_k) = (a_1, a_2, \ldots, a_k)$, where $a_i = 0$ if and only if $I_i = (0)$. Then, ϱ clearly a bijection and also preserves the adjacencies and non-adjacencies in $\mathbb{A}_{\mathbb{I}}(R_2)$ and $AG(R_3)$ and therefore, $\mathbb{A}_{\mathbb{I}}(R_1) \cong AG(R_3)$. □

Theorem 4.9. *If* $R = F_1 \times F_2 \times \ldots \times F_n$ *or* $Z_{p_1 p_2 \ldots p_n}$, *then*

(i) $dim(\mathbb{A}_{\mathbb{I}}(R)) = n - 1$ *for* $n = 2, 3$.

(ii) $dim(\mathbb{A}_{\mathbb{I}}(R)) = n$ *for* $n \geq 4$.

Proof. The result follows from Lemma 2.8 and Theorem 4.8. □

Corollary 4.10. *Let* $R_1, R_2, \ldots, R_n$ *be rings and* $R \cong R_1 \times R_2 \times \ldots \times R_n$.

Then $dim(\mathbb{A}_{\mathbb{I}}(R)) < \infty$ *if and only if* $dim(\mathbb{A}_{\mathbb{I}}(R)) < \infty$, *for every* $i, 1 \leq i \leq n$.

Proof. Let $R_1, R_2, \ldots, R_n$ be rings and $R \cong R_1 \times R_2 \times \ldots \times R_n$. Now suppose that $\dim(\mathbb{A}_{\mathbb{I}}(R_i)) < \infty$ for every $i, 1 \leq i \leq n,$ then it is clear that it has finitely many ideals. That means $|I^*(R_i)|$ is finite. Since R is direct product of R_i, R has finitely many ideals which implies $|V(\mathbb{A}_{\mathbb{I}}(R))|$ is finite. Hence $\dim(\mathbb{A}_{\mathbb{I}}(R)) < \infty.$ Conversely suppose that $\dim(\mathbb{A}_{\mathbb{I}}(R)) < \infty,$ Now suppose that there is a ring R_i, such that $\dim(\mathbb{A}_{\mathbb{I}}(R_i)) = \infty,$ then $|V(\mathbb{A}_{\mathbb{I}}(R_i))| = \infty.$ Which implies that $|V(\mathbb{A}_{\mathbb{I}}(R))| = \infty.$ This is contradicts to our supposition that $\dim(\mathbb{A}_{\mathbb{I}}(R))$ is finite. Therefore $\dim(\mathbb{A}_{\mathbb{I}}(R_i)) = \infty,$ for every $i, 1 \leq i \leq n.$

5. Conclusion

There are ample number of graphs arising from commutative rings available in literature and study of properties of these graphs is also interesting. The metric dimension has been studied in zero-divisor graphs, annihilator graphs and total graphs of commutative ring. To investigate such parameter for graphs from commutative ring is an open area of research.

Acknowledgement

Our thanks are due to the anonymous refrees' for careful reading and constructive suggestions for the improvement in the first draft of this paper while M. R. Jadeja is thankful to Atmiya University, Rajkot, Gujarat (India) for the partial support to the present work under the seed Money Financial Assistance Programme (No. SL/SMFAP/Rhase 1/2021/008 Dated 27/10/2021).

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