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CONVOLUTION THEOREM AND THE EXPONENTIAL TRANSFORM OF DERIVATIVES AND INTEGRATIONS OF THE FUNCTION $f(t)$

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Abstract

In the recent paper [1], the concept of exponential transform is introduced. In this paper we have proved the convolution theorem of the exponential transform. The exponential transform of the derivative and integration of a function is obtained. Moreover, some properties of exponential transforms are discussed.

I. Introduction

Integral transforms have been successfully used for almost two centuries in solving many problems in applied mathematics, mathematical physics and engineering science.

The origin of the integral transforms including the Laplace and Fourier transforms can be traced back to back to celebrated work of P. S. Laplace (1729-1827) on probability theory in the 1780s and to monumental Treatise of Joseph Fourier (1768-1830) on 'La Theorie Analytique de la Chaleur' published in 1822, Laplace classic book on 'La Theorie Analytique des Probabilities' includes some basic results of the Laplace transforms which is one of the oldest and most commonly used integral transform available in the

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mathematical literature. This has effectively been used in finding the solution of linear differential equations and integral equations

Many finite transforms like 'finite Laplace Transform', finite Fourier's sine and cosine transforms, finite Hankel transforms have many more applications in applied mathematics, physics and in different branches of engineering also.

Several authors [2-14] discussed the applications of different integral transformations along with its properties. Recently N. S. Ambarkhane, H. A. Dhirbasi, K. L. Bondar [1] introduced an integral transform "Exponential Transform" and proved its existence, some properties like linearity, shifting, second shifting, change of scale. Moreover exponential transform of some basic functions are derived. Main aim of this paper is to prove the further results of the exponential transform for the development and application point of view.

II. Preliminaries

Exponential Transform [1]

Definition 2.1. Let $f(t)$ be function defined for all positive values of t, then

$$\bar{f}(s) = \int_0^\infty a^{-st} f(t) dt,\ a > 1,$$

provided the integral exists is called exponential Transform of $f(t)$. It is denoted as

$$A[f(t)] = \bar{f}(s) = \int_0^\infty a^{-st} f(t) dt,\ a > 1$$

here A is called exponential transformation operator. The parameter s is real or complex number.

In general, the parameter s is taken to be a real positive number.

Theorem 2.2 [Existence of Exponential Transform] [1]**.** *If* $f(t)$ *is a function of class* A, *then exponential transform of* $f(t)$ *exists or suppose* $f(t)$ *is* *piece-wise continuous in every finite interval and is of exponential order k as* $t \to \infty$ *then* $\bar{f}(s)$ *exists for all* $(s\log a) > k$, *that is exponential transform exists.*

Exponential Transform of some functions [1]

(I) $A[1] = \frac{1}{(s\log a)}, a > 1$

(II) $A[t^n] = \frac{n!}{[s\log a]^{n+1}}, a > 1$

(III) $A[e^{kt}] = \frac{1}{(s\log a - k)}, a > 1$

(IV) $A[\cosh kt] = \frac{(s\log a)}{[s\log a^2 - k^2]}, a > 1, (s\log a)^2 > k^2$

(V) $A[\sin kt] = \frac{k}{(s\log a)^2 - k^2}, a > 1, (s\log a)^2 > k^2$

(VI) $A[\sin kt] = \frac{k}{(s\log a)^2 + k^2}$

(VII) $A[\cos kt] = \frac{(s\log a)}{(s\log a)^2 + k^2}$.

Properties of Exponential Transform [1]

(I) Linear Property

$$A[k_1 f_1(t) + k_2 f_2(t)] = k_1 A[f_1(t)] + k_2 A[f_2(t)]$$

(II) Shifting Property

If $A[f(t)] = \bar{f}(s)$, then $A[e^{kt} f(t)] = \bar{f}\left(s - \frac{k}{\log a}\right)$

(III) Change of Scale Property

If $A[f(t)] = \bar{f}(s)$ then $A[f(kt)] = \frac{1}{k}\bar{f}\left(\frac{s}{k}\right)$

(IV) Second Shifting Theorem

If $A[f(t)] = \bar{f}(s)$ and $G(t) = \begin{cases} F(t-k), \; t > k \\ \\ 0, \; t < k \end{cases}$ then $A[G(t)] = a^{-ks}\bar{f}(s).$

III. Main Results

Exponential Transform of the derivative of a function

Theorem 3.1. *If* $A[f(t)] = \bar{f}(s),$ *then* $A[f'(t)] = (s\log a).A[f(t)] - f(0).$

Proof. We have

$$A[f'(t)] = \int_0^\infty a^{-st}(t)dt$$

$$= \int_0^\infty e^{-st\log a} f'(t)dt.$$

Integrating by parts, we get

$$A[f'(t)] = [e^{-st\log a} f(t)]_0^\infty - \int_0^\infty (-s\log a).e^{-st\log a} f(t)dt$$

$$= [0 - f(0)] + (s\log a)\int_0^\infty e^{-st\log a} f(t)dt$$

$$= (s\log a)\int_0^\infty a^{-st} f(t)dt - f(0)$$

$\therefore A[f'(t)] = (s\log a)\cdot A[f(t)] - f(0).$

Theorem 3.2. *If* $A[f(t)] = \bar{f}(s)$ *then* $A[f''(t)] = (s\log a)^2 A[f(t)]$ $-(s\log a).f(0) - f'(0).$

Proof. We have

$$A[f''(t)] = \int_0^\infty a^{-st} f''(t)dt$$

$$= \int_0^\infty e^{-st\log a} f''(t)dt.$$

Integrating by parts, we get

$$A[f''(t)] = [e^{-st\log a} f'(t)]_0^\infty - \int_0^\infty -s\log a \,.\, e^{-st\log a} f'(t)dt$$

$$= [0 - f'(0)] + (s\log a)\int_0^\infty e^{-st\log a} f'(t)dt$$

$$= (s\log a)\int_0^\infty a^{-st} f'(t)dt - f'(0)$$

$$= (s\log a) \,.\, A[f'(t)] - f'(0)$$

$$= (s\log a) \,.\, A[\langle s\log a\rangle \,.\, A[f(t)] - f(0)] - f'(0)$$

$\because\ A[f'(t)] = (s\log a) \,.\, A[f(t)] - f(0)$

$\therefore\ A[f''(t)] = (s\log a)^2 A[f(t)] - (s\log a) \,.\, f(0) - f'(0).$

Theorem 3.3. *If* $A[f(t)] = \bar{f}(s)$ *then* $A[f'''(t)] = (s\log a)^3 A[f(t)] - (s\log a)^2$ $f(0) - (s\log a)f'(0) - f''(0).$

Proof. We have

$$A[f'''(t)] = \int_0^\infty a^{-st} f'''(t)dt$$

$$= \int_0^\infty e^{-st\log a} f'''(t)dt.$$

Integrating by parts, we get

$$A[f'''(t)] = [e^{-st\log a} f''(t)]_0^\infty - \int_0^\infty -s\log a \,.\, e^{-st\log a} f''(t)dt$$

$$= [0 - f''(0)] + (s\log a)\int_0^\infty e^{-st\log a} f''(t)dt$$

$$= (s\log a)\int_0^\infty a^{-st} f''(t)dt - f''(0)$$

$$= (s\log a).A[f''(t)] - f''(0)$$

$$= (s\log a)(s\log a^2 A[f(t)] - (s\log a).f(0) - f'(0)) - f''(0)$$

$$\because A[f''(t)] = (s\log a)^2 A[f(t)] - (s\log a).f(0) - f'(0)$$

$$A[f'''(t)] = (s\log a)^3 A[f(t)] - (s\log a)^2 f(0) - (s\log a)f'(0) - f''(0).$$

Exponential Transform of Derivative of a function

Theorem 3.4. *If* $A[f(t)] = \bar{f}(s)$, *then* $A[f^n(t)] = (s\log a)^n A[f(t)] - (s\log a)^{n-1} f(0) - (s\log a)^{n-2} f'(0) - (s\log a)^{n-3} f''(0) - \ldots - f^{n-1}(0)$.

Proof. We have

$$A[f'(t)] = (s\log a).\,A[f(t)] - f(0). \qquad (3.4.1)$$

Replacing $f(t)$ by $f'(t)$ and $f'(t)$ by $f''(t)$ in equation (3.4.1), we get

$$A[f''(t)] = (s\log a).A[f'(t)] - f'(0). \qquad (3.4.2)$$

From equation (3.4.1) and (3.4.2) we get

$$A[f''(t)] = (s\log a).[(s\log a).A[f(t)] - f(0)] - f'(0)$$

$$A[f''(t)] = (s\log a)^2 A[f(t)] - (s\log a).f(0) - f'(0).$$

Similarly we get

$$A[f'''(t)] = (s\log a)^3 A[f(t)] - (s\log a)^2 f(0) - (s\log a)f'(0) - f''(0)$$

$$A[f^{iv}(t)] = (s\log a)^4 A[f(t)] - (s\log a)^3 f(0) - (s\log a)^2 f'(0) - (s\log a)f''(0) - f'''(0)$$

$$\therefore\ A[f^n(t)] = (s\log a)^n A[f(t)] - (s\log a)^{n-1} f(0) - (s\log a)^{n-2} f'(0)$$

$$-(s\log a)^{n-3} f''(0) - \ldots - f^{n-1}(0).$$

Exponential Transform of Integral of function

Theorem 3.5. *If* $A[f(t)] = \bar{f}(s)$, *then*

$$A\left[\int_0^t f(t)dt\right] = \frac{1}{(s\log a)}\bar{f}(s).$$

Proof. Let $\phi(t) = \int_0^t f(t)dt$ and $\phi(0) = 0$. Then $\phi'(t) = f(t)$.

By using Exponential Transform of $\phi'(t)$

$$A[\phi'(t)] = (s\log a).A[\phi(t)] - \phi(0)$$

$$A[\phi'(t)] = (s\log a).A[\phi(t)] \qquad \because \phi(0) = 0$$

$$A[\phi(t)] = \frac{1}{(s\log a)}A[\phi'(t)].$$

Putting values of $\phi(t)$ and $\phi'(t)$ in above equation we get,

$$A\left[\int_0^t f(t)dt\right] = \frac{1}{(s\log a)}A[f(t)]$$

$$\text{i.e. } A\left[\int_0^t f(t)dt\right] = \frac{1}{(s\log a)}\bar{f}(s).$$

Theorem 3.6. *If* $A[f(t)] = \bar{f}(s)$ *then* $A[t^n . f(t)] = \left[\frac{(-1)^n}{(\log a)^n}\right]\frac{d^n}{ds^n}[\bar{f}(s)].$

Proof. We have

$$A[f(t)] = \bar{f}(s) = \int_0^\infty a^{-st}f(t)dt. \tag{3.6.1}$$

Differentiating equation (3.6.1) with respective s, we get

$$\frac{d}{ds}[\bar{f}(s)] = \frac{d}{ds}\left[\int_0^\infty a^{-st}f(t)dt\right]$$

$$= \int_0^\infty \frac{\partial}{\partial s}(e^{-st\log a})f(t)dt$$

$$= \int_0^\infty -t\log a \,.\, e^{-st\log a} f(t)dt$$

$$= -(\log a)\int_0^\infty e^{-st\log a} f(t)\,.\,tdt$$

$$= -(\log a)\int_0^\infty a^{-st}(t\,.\,f(t))dt$$

$$= (-1)(\log a)\,.\,A[t\,.\,f(t)]$$

$$\therefore\ A[t\,.\,f(t)] = \left[\frac{(-1)^1}{(\log a)^1}\right]\frac{d}{ds}[\bar{f}(s)].$$

Similarly, we get

$$A[t^2 f(t)] = \left[\frac{(-1)^2}{(\log a)^2}\right]\frac{d^2}{ds^2}[\bar{f}(s)]$$

$$A[t^3 f(t)] = \left[\frac{(-1)^3}{(\log a)^3}\right]\frac{d^3}{ds^3}[\bar{f}(s)]$$

$$A[t^n f(t)] = \left[\frac{(-1)^n}{(\log a)^n}\right]\frac{d^n}{ds^n}[\bar{f}(s)].$$

Theorem 3.7. *If* $A[f(t)] = \bar{f}(s)$ *then* $A\left[\frac{1}{t}f(t)\right] = (\log a)\int_s^\infty \bar{f}(s)ds.$

Proof. We have

$$A[f(t)] = \bar{f}(s) = \int_0^\infty a^{-st} f(t)dt. \tag{3.7.1}$$

Integrating equation (3.7.1) with respective s, we get

$$\int_s^\infty \bar{f}(s)ds = \int_s^\infty\left[\int_0^\infty a^{-st} f(t)dt\right]ds$$

$$= \int_0^\infty\left[\int_s^\infty e^{-st\log a} f(t)ds\right]dt$$

$$= \int_0^\infty \left[\frac{e^{-st\log a} f(t)}{-t\log a}\right]_s^\infty dt$$

$$= \int_0^\infty \frac{f(t)}{-t\log a} [e^{-st\log a}]_s^\infty dt$$

$$= \int_0^\infty \frac{f(t)}{-t\log a} [0 - e^{-st\log a}] dt$$

$$= \frac{1}{\log a} \int_0^\infty \left[e^{-st\log a}\left(\frac{f(t)}{t}\right)\right] dt$$

$$= \frac{1}{(\log a)} \int_0^\infty a^{-st}\left(\frac{f(t)}{t}\right) dt$$

$$\int_s^\infty \bar{f}(s) ds = \frac{1}{(\log a)} A\left[\frac{f(t)}{t}\right]$$

$$\therefore\ A\left[\frac{1}{t} f(t)\right] = (\log a) \int_s^\infty \bar{f}(s) ds.$$

Theorem 3.8 (Convolution Theorem). *If* $A[f_1(t)] = \bar{f}_1(s)$ *and* $A[f_2(t)] = \bar{f}_2(s)$ *then*

$$A\left\{\int_0^t f_1(x) f_2(t - x) dx\right\} = \bar{f}_1(s) \, . \, \bar{f}_2(s).$$

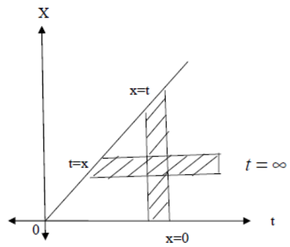
Proof. We have

$$A\left\{\int_0^t f_1(x) f_2(t - x) dx\right\} = \int_0^\infty a^{-st} \int_0^t f_1(x) f_2(t - x) dx dt$$

$$= \int_0^\infty \int_0^t a^{-st} f_1(x) f_2(t - x) dx dt.$$

Where the double integral is taken over the infinite region in the first quadrant lying between the lines $x = 0$ and $x = t$.

On changing the order of integration. The above integral becomes



$$A\left\{\int_0^t f_1(x)f_2(t-x)dx\right\} = \int_0^\infty \int_0^\infty a^{-st} f_1(x)f_2(t-x)dxdt$$

$$= \int_0^\infty a^{-sx} f_1(x)dx \int_0^\infty a^{-s(t-x)} f_2(t-x)dt$$

$$= \int_0^\infty a^{-sx} f_1(x)dx \int_0^\infty a^{-sz} f_2(z)dz$$

($\therefore$ on putting $t - x = z$

$$= \left[\int_0^\infty a^{-sx} f_1(x)dx\right] . \bar{f}_2(s)$$

$$\therefore A\left\{\int_0^t f_1(x)f_2(t-x)dx\right\} = \bar{f}_1(s) . \bar{f}_2(s).$$

IV. Conclusions

In this work we obtained exponential transform of derivative and integration of a function. Some properties like exponential transform of a function multiplied and divide by an independent variable. Moreover convolution theorem is verified.

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