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ASSIGNMENT OF SMT AND SMT SECOND LABELINGS ON McGEE GRAPH WITH AN APPLICATION

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Abstract

The McGee graph is proved to admit the Sum Modulo Three labeling (SMT) and the Sum Modulo Three-Second labeling (SMT Second). By numbering the English alphabets in a new way using a PANGRAM, McGee graph and the above two labelings, a new coding technique is developed. Two illustrations are provided.

1. Introduction

Graph Theory, one of the mathematical disciplines, is found to be intimately related to many branches of Mathematics. The concept of labeling of a graph [1] has shown a striking growth with applications in many areas such as coding theory and X-ray Crystallography. In this paper a coding technique with SMT labeling [2], SMT (second) labeling [3], Graph Message Jumble code (GMJ code) [4, 5] and a new way of numbering of alphabets using a PANGRAM on McGee graph is presented.

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2. Pre-Requisites

Definition 2.1. McGee Graph The McGee Graph is a 3-regular graph with 24 vertices and 36 edges. The McGee graph is the unique (3, 7)-cage. (The smallest cubic graph of girth 7) [6].

Definition 2.2. SMT Labeling Let p and q denote the number of vertices and edges of a simple graph $G(p, q)$. Let f be a bijective function from $V(G)$ to $\{1, 2, \ldots p\}$ and f^* be the induced function on $E(G)$ to $\{0, 1, 2\}$ such that $f^*(uv) = s$ where $f(u) + f(v) \equiv s (\bmod 3)$ and $| e_f(i) - e_f(j) | \leq 2$, $i, j \in \{0, 1, 2\}$ where $e_f(i)$ denotes the number of edges assuming the value i. A graph with this labeling is called SMT graph.

Definition 2.3. SMT Second Labeling A SMT Second Labeling on a simple graph G is a bijection $f : V(G) \to \{1, 2, \ldots p\}$ and f^* be the induced function on $E(G)$ to $\{0, 1, 2\}$ such that $f^*(uv) = s$ where $f(u) + f(v) \equiv s (\bmod 3)$ and $| e_f(i) - e_f(j) | \leq 3$, $i, j \in \{0, 1, 2\}$ where $e_f(i)$ denotes the number of edges assuming the value i.

A graph which admits the SMT Second labeling is called SMT Second graph.

Definition 2.4. PANGRAM A PANGRAM in English is a unique sentence in which every letter of the 26 alphabets is used at least once.

3. Main Results

Theorem 3.1. *The McGee graph admits SMT labeling.*

Proof. Let $v_1, v_2, \ldots, v_{24}$ be the vertices of McGee graph taken in the clockwise direction. The vertices are partitioned into four parts as $v_1 \ldots, v_6; v_7 \ldots, v_{12}; v_{13} \ldots v_{18}$ and $v_{19} \ldots v_{24}$ in the following labeling. Define $f : V(G) \to \{1, 2, \ldots, 24\}$ as follows:

$f(v_i) = 1 + 4(i - 1)$ *for* $i = 1$ *to* 6

$f(v_i) = 2 + 4(i - 7)$ *for* $i = 7$ *to* 12

$f(v_i) = 3 + 4(i - 13)$ *for* $i = 13$ *to 18*

$f(v_i) = 4 + 4(i - 19)$ *for* $i = 19$ *to* 24

Define the labeling for the edges by $f^* : E(G) \rightarrow \{0, 1, 2\}$ as follows: Here the edges are classified as circular edges, diametric edges and chordal edges respectively.

(i) Labeling for Circular Edges

(a) $f^*(v_i v_{i+1}) = s,\ 2 + 4(2i - 1) \equiv s(\text{mod } 3)$ for $i = 1$ to 5

Here $e_f(0) = 2,\ e_f(1) = 1$ and $e_f(2) = 2$

(b) $f^*(v_i v_{i+1}) = s,\ 4 + 4(2i - 13) \equiv s(\text{mod } 3)$ for $i = 7$ to 11

Here $e_f(0) = 1,\ e_f(1) = 2$ and $e_f(2) = 2$

(c) $f^*(v_i v_{i+1}) = s,\ 6 + 4(2i - 25) \equiv s(\text{mod } 3)$ for $i = 13$ to 17

Here $e_f(0) = 2,\ e_f(1) = 2$ and $e_f(2) = 1$

(d) $f^*(v_i v_{i+1}) = s,\ 8 + 4(2i - 37) \equiv s(\text{mod } 3)$ for $i = 19$ to 23

Here $e_f(0) = 2,\ e_f(1) = 1$ and $e_f(2) = 2$.

Moreover, there are four circular edges $v_6v_7,\ v_{12}v_{13},\ v_{18}v_{19}$ and $v_{24}v_1$ have their end vertices in different parts and their edge values are 2, 1, 0 and 1. From all the above cases, $e_f(0) = 8,\ e_f(1) = 8$ and $e_f(2) = 8, \ldots, (1)$

(ii) Labeling for Diametric Edges

The first two diametric edges v_1v_{13} and v_4v_{16} have their end vertices in I and III part and their edge value is 1. Fourth and fifth diametric edges namely v_7v_{19} and $v_{10}v_{22}$ have their end vertices in II and IV part and their edge value is 0.

Here, $e_f(0) = 2,\ e_f(1) = 2$ and $e_f(2) = 0, \ldots, (2)$

(iii) Labeling for Chordal Edges: The chordal edges v_2v_9 and $v_{15}v_{12}$ have their end vertices in I and II part and their edge values are equal to 0.

Next, the chordal edges v_3v_{20} and v_6v_{23} have their end vertices in I and IV part and their edge value is 2. The chordal edge v_8v_{15} has its end vertices in II and III part and its edge value is 2; the chordal edge $v_{11}v_{18}$ has both its end vertices in III part itself and its edge value is 2. Finally, the chordal edges $v_{14}v_{21}$ and $v_{17}v_{24}$ have their end vertices in III and IV part and their edge values are equal to 1.

$$\text{So, } e_f(0) = 2,\ e_f(1) = 2 \text{ and } e_f(2) = 4, \ldots (3)$$

From (1), (2) and (3)

$$e_f(0) = 12,\ e_f(1) = 12 \text{ and } e_f(2) = 12$$

Thus, $| e_f(0) - e_f(1) | = | e_f(1) - e_f(2) || e_f(2) - e_f(0) | = 0 \leq 3$

Hence, McGee graph admits SMT labeling. Therefore the McGee graph is a SMT graph.

Theorem 3.2. *The McGee graph admits SMT Second labeling.*

Proof. There are 24 vertices $v_1, v_2, \ldots, v_{24}$ taken in the clockwise direction for the McGee Graph G. With respect to the following labeling the vertices are divided into three parts as $v_1, v_2, \ldots, v_8; v_9, \ldots, v_{16}$ and $v_{17}, \ldots, v_{18}$. Define the labelling for the vertices $f : V(G) \to \{1, 2, \ldots, 24\}$ as follows:

$$f(v_i) = 1 + 3(i - 1) \text{ for } i = 1 \text{ to } 8$$

$$f(v_i) = 2 + 3(i - 9) \text{ for } i = 9 \text{ to } 16$$

$$f(v_i) = 3 + 3(i - 17) \text{ for } i = 17 \text{ to } 24.$$

Define the labeling for the edges $f^* : E(G) \to \{0, 1, 2\}$ as follows:

Here the edges are classified as circular edges, diametric edges and chordal edges respectively.

(i) Labeling for Circular Edges:

(a) $f^*(v_iv_{i+1}) = s,\ 2 + 3(2i - 1) \equiv s(\text{mod } 3)$ for $i = 1$ to 7

There are seven circular edges taking the value 2. Therefore $e_f(2) = 7$

(b) $f^*(v_i v_{i+1}) = s,\ 4 + 3(2i - 17) \equiv s(\bmod 3)$ for $i = 9$ to 15

There are seven circular edges taking the value 1. Therefore $e_f(1) = 7$

(c) $f^*(v_i v_{i+1}) = s,\ 6 + 3(2i - 33) \equiv s(\bmod 3)$ for $i = 17$ to 23 There are seven circular edges taking the value 0. Therefore $e_f(0) = 7$

Moreover, there are three circular edges that have their end vertices in different parts. They are $v_8 v_9$, $v_{16} v_{17}$ and $v_{24} v_1$ and their edges values are 0, 2 and 1.

In this case, $e_f(0) = 8,\ e_f(1) = 8$ and $e_f(2) = 8, \ldots (1)$

(ii) Labeling for Diametric Edges

The first two diametric edges $v_1 v_{13}$ and $v_4 v_{16}$ have their end vertices in I and II part and their edge values are equal to zero.

The third diametric edge $v_7 v_{19}$ has its end vertices in I and III part and its edge value is 1. Finally, the fourth diametric edge $v_{10} v_{22}$ has its end vertices in II and III part and its edge value is 2. In this case,

$$e_f(0) = 2,\ e_f(1) = 1 \text{ and } e_f(2) = 1, \ldots, \tag{2}$$

(iii) Labeling for Chordal Edges

The chordal edges $v_2 v_9$, $v_5 v_{12}$ and $v_8 v_{15}$ have their end vertices in I and II part and their edge values are equal to zero.

The chordal edges $v_3 v_{20}$ and $v_6 v_{23}$ have their end vertices in I and III part and their edge values are 1.

The chordal edges $v_{11} v_{18}$ and $v_{14} v_{21}$ have their end vertices in II and III part and their edge values are 2.

Finally, the chordal edge $v_{17} v_{24}$ has both its end vertices in III part and its edge value is 0.

In this case, $e_f(0) = 4,\ e_f(1) = 2$ and $e_f(2) = 2, \ldots, (3)$

From (1), (2) and (3)

$e_f(0) = 14,\ e_f(1) = 11$ and $e_f(2) = 11$

Thus, $| e_f(0) - e_f(0) | \leq 3$

$| e_f(1) - e_f(2) | \leq 3$

$| e_f(2) - e_f(0) | \leq 3$

Hence, McGee graph admits SMT Second labeling. Therefore McGee graph is a SMT Second graph.

4. Application

A coding technique is developed using a PANGRAM, GMJ code on McGee graph with SMT and SMT Second labelings. Two illustrations are given, one for each.

4.1 Numbering of alphabets

The numbers 1 to 26 are assigned to the letters of the PANGRAM in order, avoiding repetitions.

The PANGRAM used for coding is "THE QUICK BROWN FOX JUMPS OVER A LAZY DOG."

A	B	C	D	E	F	G	H	I	J	K	L	M
21	9	7	25	3	14	26	2	6	16	8	22	17
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
13	11	18	4	10	19	1	5	20	12	15	24	23

The graph taken for discussion consists of only 24 vertices, so the numbers 25 and 26 can be assigned to any vertex of the graph. By writing 25 in any of the following form $24 + 1,\ 23 + 2, \ldots 13 + 12$ and using the coding suggested, the letter corresponding to 25 is expressed. Here 25 correspond to *D*. Similarly it is done for 26 which correspond to *G*.

4.2 Coding a message: A coding technique to communicate any message is developed. For an example a quotable quote of Albert Einstein is taken. It is:

"In the middle of every difficulty lies opportunity".

Clue for the graph : A named graph unique with respect to regularity and girth.

Clue for SMT Labeling : While you add and divide remember the number for wisdom and harmony.

Clue for SMT Second Labeling : Remember the number for wisdom and harmony and its special neighbour while you are busy with the basic operations.

Clue for PANGRAM : Is it not nice to see all at a time?

4.3 Coding of a letter with respect to SMT Labeling:

The allotment of numbers assigned to the vertices should be known.

- With respect to SMT labeling, the 24 vertices are divided into four parts v_1 to $v_6; v_7$ to $v_{12}; v_{13}$ to v_{18} and v_{19} to v_{24}.
- The vertices v_1, v_7, v_{13} and v_{19} take the value 1, 2, 3 and 4 respectively.
- The remaining vertices are obtained by the rule $a + (i - 1)d$, $a = 1$ and $d = 4$ for $i = 1$ to 6, $a = 2$ and $d = 4$ for $i = 7$ to $12; a = 3$ and $d = 4$ for $i = 13$ to 18 and $a = 4$, $d = 4$ for $i = 19$ to 24.

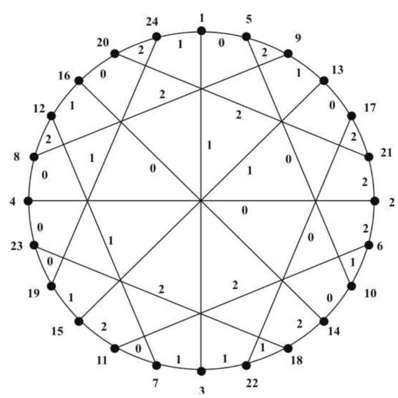

Figure. McGee Graph with SMT Labeling.

• From numbering of alphabets, the letter I takes the value 6. From the graph, 6 is assigned to v_8 that is the second vertex of the second part. So I is denoted by II_2.

• The letter N takes the value 13. From the graph, 13 is assigned to v_4 that is the fourth vertex of the first part. So N is denoted by I_4. Similarly all the other words are coded.

The message in the cipher text:

$$II_2 I_4 I_1 II_1 III_1 III_3 II_4 II_6 II_2 III_1 III_5 I_5 II_2 (IV_6 I_1)(I_4 IV_3) II_6 III_1 III_3 (II_5)_2 III_3$$
$$II_3 I_1 I_2 I_4 II_2 I_1 IV_6 III_1 IV_5 III_1 II_3 IV_6 IV_6 I_1 II_2 (II_4)_2 II_2 III_2 I_2 II_6 I_1 IV_6, \ldots, (A)$$

The words are jumbled.

4.4. Coding of a letter with respect to SMT Second Labelling. The vertices are divided into three parts v_1 to $v_8; v_9$ to v_{16}, and v_{17} to v_{24}. The procedure is the same as for SMT labeling. The message in the cipher text:

$$II_4 II_5 I_8 III_2 III_1 I_7 III_2 I_5 II_6 III_2 (III_8 I_1)(I_5 III_4) I_8 III_1 III_1 II_7 III_1 I_4 III_8 I_1 II_1 III_1$$
$$(III_8 I_1) III_2 (II_5)_2 III_2 I_3 II_2 I_8 I_1 III_8 II_4 (III_6)_2 II_4 I_4 I_1 II_2 I_5 III_2 I_1 III_8, \ldots, (B)$$

A and B denote the cipher texts of the same message with respect to two different labelings on the same graph.

5. Conclusion

The usage of PANGRAM in coding a message is very new and is expected to increase the level of secrecy and intricacy in communicating any kind of message at personal, official or governmental levels.

References

[1] J. A. Gallian, A dynamic survey of graph labelling, The Electronic Journal of Combinatorics 19 (2009), 1-219.

[2] V. Shendra Shainy, G. Margaret Joan Jebarani and V. Balaji, A Labeling Sum Modulo Three with Verification and an Application, Journal of Physics (Communicated).

[3] V. Shendra Shainy, G. Margaret Joan Jebarani and V. Balaji, SMT Labeling on Generalized prism Graphs GP(m, 1), South East Asian Journal, Accepted.

[4] G. UmaMaheswari, G. Margaret Joan Jebarani and V. Balaji, Coding through a two star and Super Mean Labeling, Applied Mathematics and Scientific Computing 2 (2019), 469-

478.

[5] D. Sathiya, G. Margaret Joan Jebarani and A. Merceline Anita, Coding Messages through Prime Cordial Labeling on McGee graph, Advances in Mathematics: Scientific Journal 9(8) (2020), 6169-6175.

[6] F. Harary, Graph Theory, Addison-Wesley, Reading, Mass, 1972.