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ON MAX-IDEAL GRAPH OF THE COMMUTATIVE RING Z_n, n IS NON-PRIME

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Abstract

Max-ideal graph is a graph associated to ring $Z_n, n \neq$ prime , denoted by $M_I(Z_n)$ and is defined as the graph whose vertices are the ideals of the ring Z_n and any two distinct vertices I_1 and I_2 are adjacent in $M_I(Z_n)$ whenever $I_1 \cap I_2 = \{0\}$ or $I_1 \cap I_2 = I_1$ or I_2, which is a maximal ideal of Z_n. In this paper we try to analyze the various structural properties of max-ideal graph of ring $Z_n, n \neq$ prime.

1. Introduction

The study of graph of a commutative ring was first introduced by I. Beck [5] in the year 1988. The graph known as zero-divisor graph of a commutative ring defined by Beck has vertex set to be the ring R and two distinct elements $x, y \in R$ are adjacent if and only if $xy = 0$. Later it was modified by Anderson and Livingston [3]. After that, many authors have discussed about various kind of graphs associated to both commutative and non-commutative rings.

Since many of the algebraic properties of the rings are studied with the help of ideals of rings, so it will be interesting to associate a graph structure to the set of ideals and then study the graph theoretical properties of it. A new approach for constructing a graph for the commutative ring R known as

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a comaximal graph was proposed by Sharma and Bhatwadekar [9]. Also, Ye and Wu [8] defined the comaximal graph on the set of ideals.

In this paper we introduce a new graph structure associated to a commutative ring Z_n called Max-Ideal graph. Max-Ideal graph denoted by $M_I(Z_n)$ is a graph whose vertices are the ideals of the ring Z_n and any two distinct vertices I_1 and I_2 are adjacent in $M_I(Z_n)$ whenever $I_1 \cap I_2 = \{0\}$ or $I_1 \cap I_2 = I_1$ or I_2 which is a maximal ideal of Z_n. In this paper we try to analyse the various structural properties of Max-Ideal graph of ring Z_n, $n \neq$ prime, which comprises girth, diameter, clique number and chromatic number and also investigate whether $M_I(Z_n)$ is bipartite, planar, eulerian, Hamiltonian or not.

2. Some Preliminary Definitions

Definition 2.1. A ring R in which $a \cdot b = b \cdot a$ for every $a, b \in R$ is called a commutative ring.

Definition 2.2. A non-empty subset I of a ring R is called an ideal of R if $a, b \in I \Rightarrow a - b \in I$ and $a \in I$, $r \in R \Rightarrow ar, ra \in I$.

Definition 2.3. Let R be a ring. An ideal $M \neq R$ of R is called a maximal ideal of R if whenever A is an ideal of R such that $M \subseteq A \subseteq R$ then either $A = M$ or $A = R$.

Definition 2.4. A graph G with n vertices is said to be complete graph if every pair of vertices are joined by a line. It is denoted by K_n.

Definition 2.5. A graph G is said to be a bipartite graph, if the vertex set of G can be partitioned into two sets V_1 and V_2 such that every line of G joins a vertex of V_1 with a vertex of V_2.

Definition 2.6. A star graph is a graph with n vertices such that exactly one vertex has degree $n-1$ and the remaining $n-1$ vertices have degree 1.

Definition 2.7. In a graph G the maximal complete subgraph is called a clique. The number of vertices in a clique is called the clique number, denoted by $\omega(G)$.

Definition 2.8. The diameter of $G, diam(G) = \sup\{d(x, y) | x, y \in V(G)\}$.

Definition 2.9. The girth of G, $gr(G)$ is the length of a shortest cycle in G. If G contains no cycle then $gr(G) = \infty$

Definition 2.10. The chromatic number, $\chi(G)$ is defined as the minimum n for which G has an n-coloring (Assignment of colors to its vertices so that no two adjacent vertices have the same color).

Definition 2.11. A graph G is called Eulerian if there exist a walk which traverses each line exactly once and goes through all the vertices and ends at the starting vertex.

Definition 2.12. A graph G is Hamiltonian if it has a spanning cycle.

Definition 2.13. A graph G is planar if it can be embedded in the plane without edges crossing.

3. Main Results

Some obvious consequence of the definition of Max-Ideal graph are:

(I) $M_I(Z_n)$ is connected for all n.

(II) The degree of the vertex corresponding to the ideal $I_1 = \langle n \rangle$ in $M_I(Z_n)$ is $\tau(n) - 1$, where $\tau(n)$ denotes the number of positive divisors of n.

(III) $M_I(Z_n)$ is always cyclic as it contains K_3.

Theorem 3.1. *Girth of* $M_I(Z_n)$ *is always* 3, $n > 3$.

Proof. In Max-Ideal graph of Z_n, the vertex corresponding to the ideal generated by n, $I_1 = \langle n \rangle = \{0\}$ is adjacent with all the remaining vertices.

Moreover, for any $n \in N$, $n \neq$ prime there exist a prime number p such that p / n. Implying there exist a maximal ideal generated by p, $I_2 = \langle p \rangle$ in Z_n. Now, the vertex corresponding to ideal $I_2 = \langle p \rangle$ is adjacent with the vertex corresponding to the ideal generated by 1, $I_3 = \langle 1 \rangle = Z_n$ in $M_I(Z_n)$.

Hence the vertices I_1, I_2 and I_3 forms a cycle of length 3 in $M_I(Z_n)$. Therefore, girth of $M_I(Z_n)$, $gr[M_I(Z_n)] = 3$.

Theorem 3.2. *Diameter of Max-Ideal graph of* Z_n, *is*

(i) 1, *if* $n = p_1 p_2$.

(ii) 2, *if* $n = p_1^r$ *or* $p_1^r p_2^r$ *or* $p_1 p_2, \ldots, p_r$; *where* $p_1, p_2, \ldots, p_r$ *are distinct primes and* $r > 1$ *is natural number*.

Proof.

(i) The Max-Ideal graph of $Z_{(p_1 p_2)}$ is a complete graph K_4 and so diam $[M_I(Z(p_1 p_2)] = 1$.

(ii) For $n = p_1^r$ or $p_1^r p_2^r$ or $p_1, p_2, \ldots, p_r$; where $p_1, p_2, \ldots, p_r$ are distinct primes and $r > 1$ is natural number, the Max-Ideal graph of Z_n is not complete as the vertex corresponding to the ideal generated by 1, $I_3 = \langle 1 \rangle = Z_n$ is adjacent only with the vertices corresponding to a maximal ideal of Z_n and $I_1 = \langle n \rangle = \{0\}$ in $M_I(Z_n)$ and so there always exist two vertices I_i and I_j such that I_i and I_j are not adjacent in $M_I(Z_n)$.

Since both I_i and I_j are adjacent with the vertex corresponding to ideal generated by n, $I_1 = \langle n \rangle = \{0\}$ in $M_I(Z_n)$, so there exist a path $I_i - I_1 - I_j$ of length 2 joining I_i and I_j in $M_I(Z_n)$. Implying, $d(I_i, I_j) = 2$ in $M_I(Z_n)$. Therefore $diam\,[M_I(Z_n)] = 2$.

Theorem 3.3. *The clique number of the Max-Ideal graph of* Z_n, *is*

(i) 4, *if* $n = p_1 p_2$.

(ii) 3, *if* $n = p_1^r$ *or* $p_1^r p_2$; *where* p_1 *and* p_2 *are distinct primes and* $r > 1$ *is natural number*.

Proof. (i) The result follows from the fact that $M_I(Z_{p_1 p_2})$ is a complete graph of order 4.

(ii) For $n = p_1^r$, $r \geq 2$.

If $r = 2$, then $M_I(Z_{p_1^2}) \cong K_3 \Rightarrow$ Clique number of

$M_I(Z_{p_1^2}), w[M_I(Z_{p_1^2})] = 3.$ Otherwise, if $r > 2$ then the vertices corresponding to the ideals $\langle 1\rangle, \langle p_1\rangle$ and $\langle p_1^r\rangle$ forms a complete graph of order 3 in $M_I(Z_{p_1^r})$. While the remaining vertices $\langle p_1^2\rangle, \langle p_1^3\rangle, \ldots, \langle p_1^{r-1}\rangle$ are adjacent only with the vertex $\langle p_1^r\rangle$. Hence the maximal complete subgraph of $M_I(Z_{p_1^r}), r > 2$ is of order 3. Therefore, $w[M_I(Z_{p_1^r})] = 3, r > 2$.

For $n = p_1^r p_2, r \geq 2$.

In $M_I(Z_{p_1^r p_2})$, the vertices corresponding to the ideals $\langle p_1^r p_2\rangle, \langle 1\rangle, \langle P_2\rangle$ and $\langle p_1^r\rangle$ have degree greater than or equal to 3 while all the remaining vertices have degree 1 or 2. But the vertex corresponding to the ideal $\langle 1\rangle$ is not adjacent with the vertex corresponding to ideal $\langle p_1^r\rangle$ and so the vertices corresponding to the ideals $\langle p_1^r p_2\rangle, \langle 1\rangle, \langle P_2\rangle$ and $\langle p_1^r\rangle$ will not form a complete subgraph, hence the maximal complete subgraph of $M_I(Z_{p_1^r p_2})$ is of order 3 which is formed by the vertices corresponding to the ideals $\langle p_1^r p_2\rangle, \langle 1\rangle$ and $\langle P_2\rangle$. Hence , $w[M_I(Z_{p_1^r p_2})] = 3$.

Theorem 3.4. *The clique number of the graph* $M_I(Z_{p_1, p_2, \ldots, p_n})$ *is* $n + 1, n \geq 3$.

Proof. For $n = 3, M_I(Z_{p_1 p_2 p_3})$ has 8 vertices and the vertices corresponding to the ideals $\langle p_1 p_2 p_3\rangle, \langle p_1 p_2\rangle, \langle p_2 p_3\rangle$ and $\langle p_1 p_3\rangle$ forms a complete graph K_4.

Moreover, no other vertex in $M_I(Z_{p_1 p_2 p_3})$ have degree greater than or equal to 4 except the vertex corresponding to the ideal $\langle 1\rangle$ but the vertex corresponding to ideal $\langle 1\rangle$ is not adjacent with all four of the above mentioned vertices and so the maximal complete subgraph in $M_I(Z_{p_1 p_2 p_3})$ is of order 4.

For $n = 4$, $M_I(Z_{p_1p_2p_3p_4})$ has 16 vertices and the vertices corresponding to the ideals $\langle p_1p_2p_3p_4\rangle, \langle p_1p_2p_3\rangle, \langle p_2p_3p_4\rangle, \langle p_3p_4p_1\rangle$ and $\langle p_4p_1p_2\rangle$ forms a complete graph K_5.

Moreover, no other vertex in $M_I(Z_{p_1p_2p_3p_4})$ have degree greater than or equal to 5 except the vertex corresponding to the ideal $\langle 1\rangle$ but the vertex corresponding to ideal $\langle 1\rangle$ is not adjacent with all five of the above mentioned vertices and so the maximal complete subgraph in $M_I(Z_{p_1p_2p_3p_4})$ is of order 5.

In this way, for $n = r$, $M_I(Z_{p_1p_2p_3, \ldots, p_r})$ has 2^r vertices and the vertices corresponding to the ideals $\langle p_1p_2, \ldots, p_r\rangle, \langle p_1p_2p_3, \ldots, p_{r-1}\rangle$, $\langle p_2p_3, \ldots, p_r\rangle, \langle p_3p_4, \ldots, p_rp_1\rangle, \ldots, \langle p_rp_1p_2, \ldots, p_{r-2}\rangle$ forms a complete graph K_{r+1}, which is maximal complete subgraph in $M_I(Z_{p_1p_2p_3, \ldots, p_r})$.

Hence, $\omega[M_I(Z_{p_1p_2p_3, \ldots, p_r})] = r + 1,\ r \geq 3$. □

Theorem 3.5. *The Max-Ideal graph of* Z_n *is neither a bipartite nor a star*.

Proof. By Theorem 3.1, there always exist a cycle of length 3 in $M_I(Z_n)$ and since a graph is bipartite if and only if all its cycle are of even length, so $M_I(Z_n)$ is not bipartite.

Next, a star S_k is the complete bipartite graph $K_{1,k}$ and since $M_I(Z_n)$ will never be a bipartite graph. So $M_I(Z_n)$ is not a star as well. □

Theorem 3.6. *The Max-Ideal graph of* Z_n *is planar whenever* $n = p_1p_2$ *or* $n = p_1^r$.

Proof. For $n = p_1p_2$, the result follows from the fact that $M_I(Z_{p_1p_2})$ is a complete graph of order 4. Next, for $n = p_1^r$.

In the ring $Z_{p_1^r}$, the ideal generated by $p_1, \langle p_1\rangle$ is the only maximal ideal. So the vertices corresponding to the ideals $\langle 1\rangle$ and $\langle p_1\rangle$ are adjacent in

$M_I(Z_{p_1^r})$. Apart from that all the remaining vertices $\langle p_1^2 \rangle, \langle p_1^3 \rangle, \ldots, \langle p_1^{r-1} \rangle$ are adjacent with the vertex $\langle p_1^r \rangle$ only.

Hence however large the value of r may be, the Max-Ideal graph of $Z_{p_1^r}$ can always be embedded in a plane without edge crossing. Hence $M_I(Z_{p_1^r})$ is planar. □

Theorem 3.7. *The Max-Ideal graph of* $Z_{p_1 p_2}$ *is Hamiltonian but not Eulerian.*

Proof. The Max-Ideal graph of $Z_{p_1 p_2}$, $M_I(Z_{p_1 p_2})$ is a complete graph K_4. So it contains a spanning cycle and moreover degree of each of its vertices is odd.

Hence, $M_I(Z_{p_1 p_2})$ is Hamiltonian but not Eulerian.

Theorem 3.8. *The Max-Ideal graph of* $Z_{p_1 p_2, \ldots, p_n}$ *is not Eulerian.*

Proof. In the Max-Ideal graph of $Z_{p_1 p_2, \ldots, p_n}$, the degree of the vertex corresponding to the ideal $I_1 = \langle p_1 p_2, \ldots, p_n \rangle = \{0\}$ is $2^n - 1$, which is odd for all n. Hence, $M_I(Z_{p_1 p_2, \ldots, p_n})$ is not Eulerian.

Theorem 3.9. *The Max-Ideal graph of* Z_{p^n} *is*

(i) *Both Eulerian and Hamiltonian if* $n = 2$.

(ii) *Neither Eulerian nor Hamiltonian if* $n > 2$.

Proof. (i) The Max-Ideal graph of Z_{p^2} is complete graph K_3 and so is both Eulerian and Hamiltonian.

(ii) For $n > 2$, the ideals of Z_{p^n} are $I_1 = \langle p^n \rangle = \{0\}$, $I_2 = \langle 1 \rangle = Z_{p^n}$, $I_3 = \langle p \rangle$, $I_4 = \langle p^2 \rangle, \ldots, I_{n+1} = \langle p^{n-1} \rangle$.

Out of all the ideals of Z_{p^n}, I_3 is the only maximal ideal and so the vertex corresponding to I_3 is adjacent with the vertex corresponding to I_2 in

$M_I(Z_{p^n})$. While all the remaining vertices corresponding to the ideals $I_4, I_5, \ldots, I_{n+1}$ are of degree 1. Which implies that $M_I(Z_{p^n})$ is neither Eulerian nor Hamiltonian. □

Theorem 3.10. *The Max-Ideal graph of Z_n is neither Eulerian nor Hamiltonian whenever $n = p_1^r p_2$ or $p_1^r p_2^r$ where $r \geq 3$.*

Proof. For $n = p_1^r p_2$,

The ideals of the ring $Z_{p_1^r p_2}$ are $I_1 = \langle p_1^r p_2 \rangle = \{0\}$, $I_2 = \langle 1 \rangle = Z_{p_1^r p_2}$, $I_3 = \langle p_1 \rangle$, $I_4 = \langle p_2 \rangle$, $I_5 = \langle p_1^2 \rangle$, $I_6 = \langle p_1^3 \rangle, \ldots, I_{r+2} = \langle p_1^{r-1} \rangle$, $I_{r+3} = \langle p_1^r \rangle$, $I_{r+4} = \langle p_1 p_2 \rangle$, $I_{r+5} = \langle p_1^2 p_2 \rangle, \ldots, I_{r+(r-1+3)} = I_{2(r+1)} = \langle p_1^{r-1} p_2 \rangle$.

Now, in Max-Ideal graph of $Z_{p_1^r p_2}$, the vertices corresponding to the ideals $I_5, I_6, \ldots, I_{r+2}$ are adjacent with the vertex corresponding to the ideal I_1 only. which implies that $M_I(Z_{p_1^r p_2})$ is neither Eulerian nor Hamiltonian.

For $n = p_1^r p_2^r$.

As above the vertices corresponding to the ideals $\langle p_1^2 \rangle, \langle p_1^3 \rangle, \ldots, \langle p_1^{n-1} \rangle, \langle p_2^2 \rangle, \langle p_2^3 \rangle, \ldots, \langle p_2^{n-1} \rangle$ are adjacent only with the vertex corresponding to the ideal $\langle p_1^r p_2^r \rangle$ in $M_I(Z_{p_1^r p_2^r})$.

Hence the graph $M_I(Z_{p_1^r p_2^r})$ is neither Eulerian nor Hamiltonian. □

Theorem 3.11. *The chromatic number of the Max-Ideal graph of Z_n is*

(i) 4, *if* $n = p_1 p_2$ *or* $p_1 p_2 p_3$

(ii) 3, *if* $n = p_1^r, r > 1$ *or* $n = p_1^r p_2, r > 2$ *where* p_1, p_2, p_3 *are distinct primes and r is natural number.*

Proof. (i) For $n = p_1 p_2$, $M_I(Z_{p_1 p_2}) \cong K_4 \Rightarrow \chi[M_I(Z_{p_1 p_2})] = 4$.

For $n = p_1 p_2 p_3$,

The vertices corresponding to the ideals $\langle p_1 p_2 p_3 \rangle, \langle p_1 p_2 \rangle, \langle p_2 p_3 \rangle$ and $\langle p_1 p_3 \rangle$ forms a complete graph K_4 So we can assign color-1 to color-4 to the subgraph K_4. No other vertex is adjacent with all four of them and so we can assign the above mentioned colors to the remaining vertices of $M_I(Z_{p_1 p_2 p_3})$ in such a way that no two adjacent vertices have same color. Hence, $\chi[M_I(Z_{p_1 p_2})] = 4$.

(ii) For $n = p_1^r, r \geq 2$,

If $r = 2$ then $M_I(Z_{p_1^2}) \cong K_3 \Rightarrow \chi[M_I(Z_{p_1^2})] = 3$. Otherwise, if $r > 2$ then the vertices corresponding to the ideals $\langle 1 \rangle, \langle p_1 \rangle$ and $\langle p_1^r \rangle$ forms a complete graph K_3 in $M_I(Z_{p_1^r})$. So we can assign color-1 to color-3 to the subgraph K_3.

Moreover the remaining vertices $\langle p_1^2 \rangle, \langle p_1^3 \rangle, \ldots, \langle p_1^{r-1} \rangle$ are adjacent only with the vertex $\langle p_1^r \rangle$. So we can assign either of the two color which is not assign to the vertex $\langle p_1^r \rangle$ to the remaining vertices of $M_I(Z_{p_1^r})$.

Hence, $\chi[M_I(Z_{p_1^r})] = 3$.

For $n = p_1^r p_2, r \geq 2$. The ideals of the ring $Z_{p_1^r p_2}$ are $I_1 = \langle p_1^r p_2 \rangle = \{0\}, I_2 = \langle 1 \rangle = Z_{p_1^r p_2}, I_3 = \langle p_1 \rangle, I_4 = \langle p_2 \rangle, I_5 = \langle p_1^2 \rangle, I_6 = \langle p_1^3 \rangle, \ldots,$ $I_{r+2} = \langle p_1^{r-1} \rangle, I_{r+3} = \langle p_1^r \rangle, I_{r+4} = \langle p_1 p_2 \rangle, I_{r+5} = \langle p_1^2 p_2 \rangle, \ldots, I_{r+(r-1+3)}$ $= I_{2(r+1)} = \langle p_1^{r-1} p_2 \rangle$.

We can assign color-1 to vertex I_1.

In $M_I(Z_{p_1^r p_2})$, the vertex corresponding to the ideal I_1 is adjacent with all the remaining vertices and so we can not assign color-1 to the remaining vertices.

Next, we can assign color-2 to vertex I_2.

The vertices corresponding to the ideals $I_5, I_6, \ldots, I_{r+2}$ are adjacent only with the vertex I_1, so we can assign color-2 to each of the vertices $I_5, I_6, \ldots, I_{r+2}$.

Also, the vertex corresponding to the ideal I_{r+3} is not adjacent with the vertices corresponding to the ideals $I_2, I_5, I_6, \ldots, I_{r+2}$ and so we can assign color-2 to the vertex I_{r+3}.

Next, the vertices corresponding to the maximal ideal I_3 and I_4 are adjacent with both the vertices I_1 and I_2 and so we assign color-3 to each of them.

Moreover the vertices corresponding to the ideals $I_{r+4}, I_{r+5}, \ldots, I_{2r+2}$ are adjacent only with vertices I_1 and I_{r+3} and hence we can assign color-3 to all of them.

Hence, $\chi[M_I(Z_{p_1^r p_2})] = 3, r > 2$.

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