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DI-DOMINATION PAIR OF WHEEL AND ITS RELATED GRAPHS

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Abstract

In this paper the definition of Di-dominating pair set (simply ddp set) is being defined as "Let S_1 and S_2 be two domination sets of G with $|S_1| = |S_2|$, if every vertex $v \in \overline{S}_1 \cap \overline{S}_2$ satisfies $N(v) \cap S_1 \neq N(v) \cap S_2$, then the pair (S_1, S_2) is called Di-dominating pair set (simply ddp set). The minimum of $|S_1|$ (or $|S_2|$) is called Di-domination pair number (ddp number) and is denoted as $\gamma_{ddp}(G)$. Also here we have found a Di-dominating pair set and minimum cardinality of Di-domination pair number of wheel and its related graphs. Further, we have developed some theorems to find Di-dominating pair set and Di-domination number of wheel graph, Helm graph, closed Helm graph, Flower graph, double wheel graph.

1. Introduction

In graph theory the domination number is a recently developing area, this was developed in the years 1950's onwards. But the rate of research on domination significantly increased in the mid-1970's. Oystein Ore [6] introduced the terms "dominating set" and "domination number" in his book on graph theory which was published in 1962. The problems described above were studied in more detail around 1964 by brothers Yaglom and Yaglom [8]. Their studies resulted in solutions to some of these problems for rooks, knights, kings, and bishops.

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A decade later, E. J. Cockayne and S. T. Hedetniemi [7] published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of a graph G. Since this paper was published, domination in graphs has been studied extensively and several additional research papers have been published on this topic. Domination sets are practical interest in several areas such as wireless networks, mobile networks, electrical grids etc.

2. Preliminaries

Definition. Domination set. A subset S of V is said to be a dominating set of G, if every vertex in $V - S$ is adjacent to a vertex in S. The dominating number $\gamma(G)$ of G is the minimum cardinality of a dominating set.

Based on this definition, several dominating sets and corresponding numbers are developed by various persons and are studied in [1] [2] [3] [4] [5]. Motivated by these, we have developed a new type of dominating set and its number will be inducing here.

In a graph $G = (V(G), E(G))$ to be finite, undirected, loopless, and without multiple edges. For every vertex $v \in V(G)$, the open neighbourhood set $N(v)$ is the set of all vertices adjacent to v in G. That is, $N(v) = \{u \in V(G)/uv \in E(G)\}$. The closed neighbourhood set $N[v]$ of v is defined as $N[v] = N(v) \cup \{v\}$. The degree of $v \in V(G)$, denoted by $dG(v)$ and is defined by $dG(v) = |NG(v)|$. The minimum and maximum degrees of G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$ respectively. In a graph, a vertex of degree one is called a pendant vertex and an edge incident with a pendant vertex is called a pendant edge.

Definition. Di-domination Set. Let S_1 and S_2 be two domination sets of G with $|S_1| = |S_2|$, if for every vertex $v \in \overline{S}_1 \cap \overline{S}_2$ satisfies $N(v) \cap S_1 \neq N(v) \cap S_2$, then the pair (S_1, S_2) is called Di-dominating pair set (simply ddp set). The minimum of $|S_1|$ (or $|S_2|$) is called Di-dominating pair number (simply ddp number) and is denoted as $\gamma_{ddp}(G)$, that is $\gamma_{ddp}(G) = \min(|S_1|(\text{or}|S_2|))$.

Theorem 2.1. *If G is a nontrivial connected graph of order n with pendant vertices, then for each pendant vertex must be a Di-dominating pair vertex and this pendant vertices belongs to S_1 (or) S_2 (or) both S_1 and S_2.*

Theorem 2.2. *For the cycle C_n then Di-domination pair number of C_n is*

$$\gamma_{ddp}(C_n) = \begin{cases} \frac{n}{3}; & if\ n \equiv 0 (\text{mod}\, 3) \\ \frac{n+2}{3}; & if\ n \equiv 1 (\text{mod}\, 3) \\ \frac{n+1}{3}; & if\ n \equiv 2 (\text{mod}\, 3) \end{cases}$$

Theorem 2.3. *Let S_1 and S_2 be the dominating sets for Gem graph, then*

$$\gamma_{ddp}(G_n) = \begin{cases} \frac{n}{3}; & if\ n \equiv 0 (\text{mod}\, 3) \\ \frac{n-1}{3}; & if\ n \equiv 1 (\text{mod}\, 3) \\ \frac{n+1}{3}; & if\ n \equiv 2 (\text{mod}\, 3) \end{cases}$$

3. Di-domination Number of Wheel and its Related Graph

Theorem 3.1. *For any wheel graph W_n, if $n \geq 4$. Then Di-domination pair number is*

$$\gamma_{ddp}(W_n) = \begin{cases} \frac{n}{3}; & if\ n \equiv 0 (\text{mod}\, 3) \\ \frac{n-1}{3}; & if\ n \equiv 1 (\text{mod}\, 3). \\ \frac{n+1}{3}; & if\ n \equiv 2 (\text{mod}\, 3) \end{cases}$$

Proof. Consider any wheel graph W_n with n vertices formed by sum of complete graph with one vertex v_1 and cycle graph with $n-1$ vertices are $v_2, \ldots, v_{n-1}, v_n$, that is the wheel W_n can be defined as the graph $K_1 + C_{n-1}$. Here v_1 has degree $n-1$ so it is internal vertex to all other vertices and $d_G(v_2) = d_G(v_3) = \ldots d_G(v_n) = 3$. Now collecting minimum Di-domination pair number as following wheel reducing possible cases of other graphs we will be analysis minimum.

(a) Suppose we eliminate the internal vertex v_1, then new graph forms cycle with $n-1$ vertices and $n-1$ edges. Hence by Theorem 2.2

$$\gamma_{ddp}(C_{n-1}) = \gamma_{ddp}(W_n) = \begin{cases} \dfrac{n-1}{3}; & \text{if } n-1 \equiv 0(\bmod 3) \\ \dfrac{n-1+2}{3}; & \text{if } n-1 \equiv 1(\bmod 3) \\ \dfrac{n-1+1}{3}; & \text{if } n-1 \equiv 2(\bmod 3) \end{cases}$$

$$= \begin{cases} \dfrac{n-1}{3}; & \text{if } n-1 \equiv 0(\bmod 3) \\ \dfrac{n-1}{3}; & \text{if } n-1 \equiv 1(\bmod 3) \\ \dfrac{n}{3}; & \text{if } n-1 \equiv 2(\bmod 3) \end{cases}$$

$$= \begin{cases} \dfrac{n-1}{3}; & \text{if } n-1 \equiv 0(\bmod 3) \\ \dfrac{n+1}{3}; & \text{if } n-1 \equiv 2(\bmod 3) \\ \dfrac{n}{3}; & \text{if } n-1 \equiv 0(\bmod 3) \end{cases}$$

Therefore wheel graph reduce cycle with $n-1$ vertices of Di-dominating pair sets dominates all n vertices and it is minimum.

(b) Suppose we eliminate any one edge from rim vertices of wheel, then new graph forms gem graph with n vertices and $2n-3$ edges. So by theorem 2.3, we get

$$\gamma_{ddp}(G_n) = \gamma_{ddp}(W_n) = \begin{cases} \dfrac{n}{3}; & \text{if } n \equiv 0(\bmod 3) \\ \dfrac{n-1}{3}; & \text{if } n \equiv 1(\bmod 3) \\ \dfrac{n-1}{3}; & \text{if } n \equiv 2(\bmod 3) \end{cases}$$

This shows that minimum Di-domination pair number of wheel and gem graph are equal in any other way.

Theorem 3.2. *For any Helm graph* $H_n(n > 3)$, *then* $\gamma_{ddp}(H_n) = n-1$.

Proof. The Helm H_n is obtained from wheel W_n by attaching a pendant edge of vertex to each of its $n-1$ rim vertex. So it contains wheel W_n and $n-1$ pendant vertices, it has $2n-1$ vertices and $3(n-1)$ edges. Now collect minimum Di-domination pair number, in order to dominate the pendant vertices of H_n using by theorem 2.1 "every pendant vertex must be a Di-dominating pair vertex". Our choice to select either the pendant vertex or rim vertex that will be the dominating vertex and the dominating sets S_1 and S_2 formed at least one rim vertex and remaining vertices at most $n-2$ pendant vertices.

Therefore it is possible to take two set of $n-1$ vertices that dominate the all vertices of H_n, which implies that $\gamma_{ddp}(H_n) = n-1$.

Theorem 3.3. *For any closed Helm graph* $CH_n (n > 3)$, *then*

$$\gamma_{ddp}(H_n) = \left\{\begin{array}{ll} \frac{n}{2}-1, & if\ n = 4k\ for\ k = 1, 2, \ldots \\ \frac{n-1}{2}, & if\ n = 4k+1\ for\ k = 1, 2, \ldots \\ \frac{n}{2}, & if\ n = 4k+2\ for\ k = 1, 2, \ldots \\ \frac{n-1}{2}+1 & if\ n = 4k+3\ for\ k = 1, 2, \ldots \end{array}\right\}.$$

Proof. The closed Helm CH_n is a helm by joining each pendant vertex to form a cycle. Let $v_1, v_2, \ldots, v_{n-1}$ be the inner cycle, v be a common vertex to inner cycle and also $u_1, u_2, \ldots, u_{n-1}$ be the outer cycle of CH_n. It has $2n-1$ vertices and $4(n-1)$ edges. Now collect the Di-dominating pair set, suppose we eliminate v from CH_n. Thus we get two cycle of length $2n-2$ and each vertex has degree three. For any one vertex will be dominating three vertices and it self also. Our choice to select set S_1 one form inner cycle and next one form outer cycle at a distant three alternatively to end of all vertices be dominate as left to right (i.e., v_1 to u_{n-1}) and also its neighbour of first one of inner cycle will get in S_2 and having same processes at v_2 to u_1 as shown in figure in the following cases.

Case (i). If $n = 4k$, in Figure 1 we get vertex v_1 and u_{n-1} of S_1 are dominating both u_1, v_{n-1}. Also vertex v_2 and u_1 of S_2 are dominating both

u_2, v_1 and all other vertex of S_1 and S_2 are being dominate by different vertices. So that $\gamma_{ddp}(CH_n) = \frac{(2n-2)-2}{4} = \frac{n}{2} - 1$.

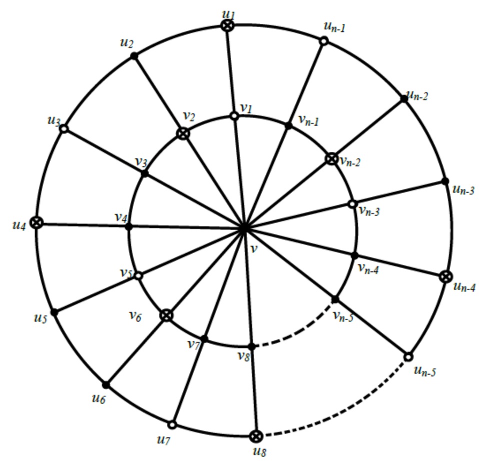

Figure 1. $CH_{n=4k}$.

Case (ii). If $n = 4k + 1$, see Figure 2 the dominating set S_1 of vertices are dominating different undominated vertices and thus also S_1.

Therefore $\gamma_{ddp}(CH_n) = \frac{2n-2}{4} = \frac{n-1}{2}$.

Case (iii). If $n = 4k + 2$ we have the dominating set $S_1 = \{v_1, u_3, v_5, \ldots, u_{n-3}, v_{n-1}\}$ and vertex v_1 is dominating v_{n-1}, also v_{n-1}, is dominating v_1. From the set $S_2 = \{v_2, u_4, v_6, \ldots, u_{n-2}, v_1\}$ for a vertex $u_1 \in V - (S_1, S_2)$ we have $N(u_1) \cap S_1 = v_1 = N(u_1) \cap S_2$ which contradicts the hypotheses, so we eliminate v_1 in S_2 and our choices to add u_{n-1} is in S_2.

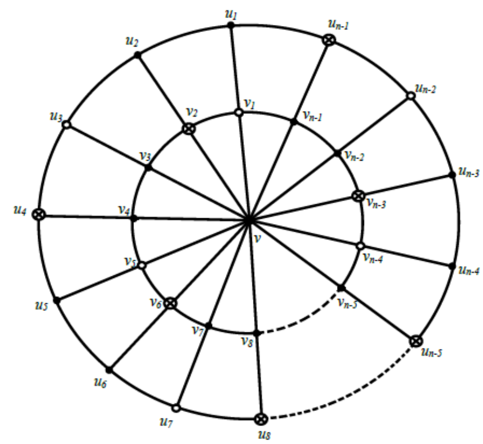

Figure 2. $CH_{n=4k+1}$.

Thus we get $S_2 = \{v_2, u_4, v_6, \ldots, u_{n-2}, v_1\}$ by the vertex u_{n-1} is dominating u_{n-2}, also u_{n-2} is dominating u_{n-1} and all other vertex of S_1 and S_2 are dominating different vertices as shown in figure 3. Then $\gamma_{ddp}(CH_n) = \dfrac{(2n-2)+2}{4} = \dfrac{n}{2}$.

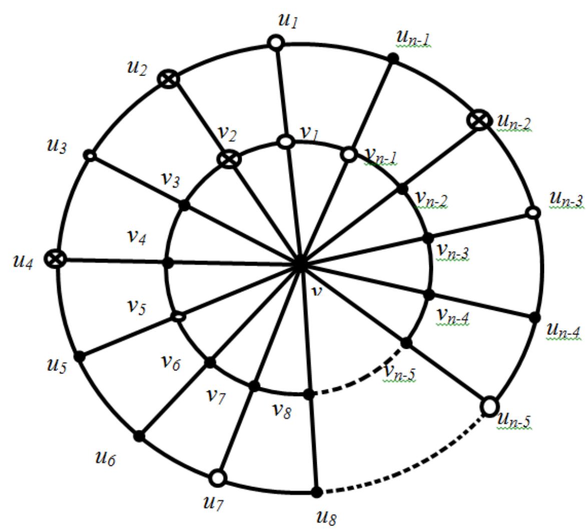

Figure 3. $CH_{n=4k+2}$.

Case (iv). If $n = 4k+3$, in Figure 4 now the set

$\{v_1, u_3, v_5, \ldots, u_{n-4}, v_{n-2}\}$ will taking from inner to outer cycle at distant three. This set is not dominating set, because u_{n-1} is not dominated. So we take any one vertex that will dominate u_{n-1} and we include this in $S_1 = \{v_1, u_3, v_5, \ldots, u_{n-4}, v_{n-2}\}$. Also the set $S_2 = \{v_2, u_4, v_6, \ldots, u_{n-2}, v_{n-1}\}$ is not dominating u_1 we include any one of the neighbour of u_1 in S_2. Thus we get Di-dominating pair number as

$$\gamma_{ddp}(CH_n) = \frac{2n-2}{4} + 1 = \frac{n-1}{2} + 1.$$

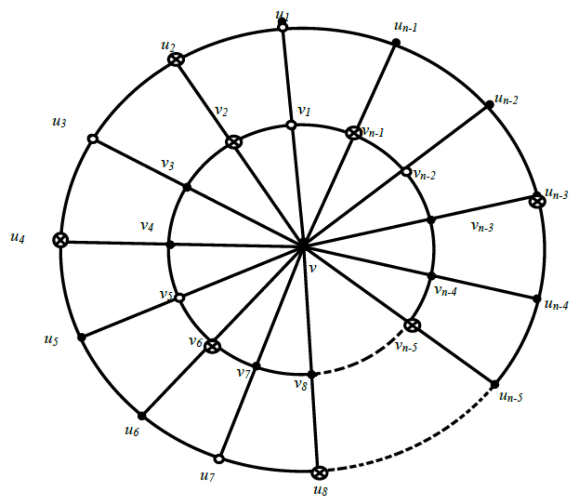

Figure 4. $CH_{n=4k+3}$.

Theorem 3.4. *If* $G = Fl_n$ *be a flower graph. Then* $\gamma_{ddp}(Fl_n)$ $= \begin{cases} 1 + \frac{n-1}{2}; \text{ if } n \text{ is odd} \\ 1 + \frac{n}{2}; \text{ if } n \text{ is even} \end{cases}$.

Proof. The flower graph Fl_n is obtained from a helm by joining each pendant vertex to the central vertex of the Helm. It has $2n-1$ vertices and $4(n-1)$ edges, here $v_1, v_2, \ldots, v_{n-1}$ vertices are of degree four, $u_1, u_2, \ldots, u_{n-1}$ vertices are of degree two and v be a central vertex that has degree $2(n-1)$. The minimum Di-domination pair set will be constructed as follows. The central vertex be dominated all other vertices, so assume it belongs to both S_1 and S_2, since by hypothesis $N(v) \cap S_1 \neq N(v) \cap S_2$ and $|S_1| = |S_2|$. Our choice is to select some additional vertices to include in the sets of flower graph.

Case (i). n is odd

The collection of additional vertices of S_1 is either v_i or u_i $i = 1, 2, \ldots, n-1$, only similar vertex with same suffix has $n-1$ even number of vertices) and also S_2 is different from S_1 with respect to other vertices to either v_i or u_i (i.e., these vertices are not in both set). Thus we get i elements in S_1 and S_2 will be different. This implies that $\frac{n-1}{2}$ vertices are in S_1, also in S_2. Therefore $\gamma_{ddp}(Fl_n) = 1 + \frac{n-1}{2}$ for n is odd.

Case (ii). n is even

Now let us remove one vertex from v_i $i = 1, 2, \ldots, n-1$, and its nearer vertex u_i $i = 1, 2, \ldots, n-1$. In Fl_n graph is reduced to new graph Fl_{n-1} if $n-1$ is odd. Using case (i) in the sets with $1 + \frac{n-1}{2}$ vertices, the hypothesis of (S_1, S_2) is violated, because the removed vertex has $N(v) \cap S_1 = N(v) \cap S_2$, if $v \in V - (S_1, S_2)$. So we include one removed vertex in S_1 and S_2, that is removal of v_i and u_i is either $v_i \in S_1$, $u_i \in S_2$ or $u_i \in$ both S_1 and S_2. It follows that minimum Di-domination pair is $\gamma_{ddp}(Fl_n) = 1 + \frac{(n-1)-1}{4} + 1 = 1 + \frac{n}{2}$.

Remark. For every graph G with central vertex, if we construct the sets S_1 and S_2 within central vertex, then this sets has some vertex include to S_1 and S_2 since by hypothesis $N(v) \cap S \neq N(v) \cap S_2$, and $|S_1| = |S_2|$. Also without central vertex we get $\gamma_{ddp}(G)$ is greater than one.

Theorem 3.5. *For any double wheel graph*, $\gamma_{ddp}(W_{n-1, n-1}) = 1 + \gamma(C_{n-1})$.

Proof. Let us consider the double wheel is composed by sum of complete graph with one vertex v and two cycles with $n-1$ vertices as $v_1, v_2, \ldots, v_{n-1}$

and $u_1, u_2, \ldots, u_{n-1}$ (i.e., double wheel consists as the graph $K_1 + 2C_{n-1}$, where the vertices of two cycles be connected to common vertex v). So it has $2n-1$ vertices and $4(n-1)$ edges. Now let us collect the Di-dominating pair set. Suppose v belongs to both in S_1 and S_2.

Our choices to assume cycle with $n-1$ vertices of ordinary minimum domination set of $\gamma(C_{n-1})$. If double wheel has two cycles with $n-1$ vertices, then S_1 of dominating vertices has one common vertex and any one cycle of minimum Di-dominating set. Also, S_2 has same common vertex and another one cycle with $\gamma(C_{n-1})$. The selection in any other way of (S_1, S_2) will increase the Di-domination number. Thus we have the minimum Di-domination pair number as $\gamma_{ddp}(W_{n-1, n-1}) = 1 + \gamma(C_{n-1})$.

Theorem 3.6. *For any Gear graph G_n if $(n > 3)$, then Di-domination pair number is $\gamma_{ddp}(G_n) = \gamma_{ddp}(C_{2n-2})$.*

Proof. The gear graph contains from wheel W_n by attaching an extra vertex between each pair of adjacent vertices on its $n-1$ rim vertex. It has $2n-1$ vertices of which one internal vertex v of degree $n-1$, $n-1$ vertices of degree three, other $n-1$ vertices of degree two. We eliminate the internal vertex v. Now the gear graph is reducing to new graph forms cycle with $2n-2$ vertices. By using the result of theorem 2.5 with $2n-2$ vertices of cycle, the minimum Di-domination pair number is also minimum number for gear graph G_n.

Hence $\gamma_{ddp}(G_n) = \gamma_{ddp}(C_{2n-2})$.

Corollary 1. *Let G be any double Gear graph, then $\gamma_{ddp}(G) = 2\gamma_{ddp}(C_{2n-2})$. In general let G be any n leaves gear graph, then $\gamma_{ddp}(G) = n\gamma_{ddp}(C_{2n-2})$.*

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