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NEIGHBORHOOD PROPERTIES OF A CERTAIN FAMILY OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

FAROZE AHMAD MALIK[1] and CHITARANJAN SHARMA[2]

Department of Mathematics
Government Holkar (Model
Autonomous) Science College
Indore (M.P.)-452017, India
E-mail: malikferooze@gmail.com
drcsharma3@gmail.com

Abstract

In this paper, we consider a generalized novel family of analytic functions involving complex order and defined by a convolution operator. We study several neighborhood properties of this family. The growth and distortion theorems are also discussed. Some earlier established results are obtained as special cases.

1. Introduction

Let $\mathbb{U} := \{\zeta : \zeta \in \mathbb{C}$ and $|\zeta| < 1\}$ be the open unit disk and let $\mathcal{A}(n)$, $n \in \mathbb{N}$, consist of analytic functions $f : \mathbb{U} \to \mathbb{C}$ of the form $f(\zeta) = \zeta + \sum_{l=n+1}^{\infty} a_j \zeta^j$. Set $\mathcal{A} := \mathcal{A}(1)$. Let $S_n^* \subset \mathcal{A}(n)$ and $\mathcal{C}_n \subset \mathcal{A}(n)$ be, respectively, the well-know classes of starlike and convex functions defined in $\mathbb{U}$. Further, let $\mathcal{T}(n) \subset \mathcal{A}(n)$ be the subclass of $\mathcal{A}(n)$ consisting of all those functions $f(\zeta)$ that are of the form

$$f(\zeta) = \zeta - \sum_{j=n+1}^{\infty} a_j \zeta^j \ (a_j \geq 0, \ j \geq n+1, \ n \in \mathbb{N}). \tag{1.1}$$

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That is $\mathcal{T}(n)$ consists of those analytic functions in $\mathcal{A}(n)$ whose Taylor coefficients are negative. We note that the class $\mathcal{T} := \mathcal{T}(1)$ was first introduced by Silverman [48] and later on studied extensively by a number of authors including the ones in [2, 6, 22, 24, 40, 44, 49, 50]. Also see Srivastava et al. [51, 57]. The importance of the class $\mathcal{T} \subset \mathcal{A}$ in the theory of univalent functions is due to the fact that some conditions which are only sufficient for the members of $\mathcal{A}$ prove to be both necessary and sufficient for the members of $\mathcal{T}$. The coefficient characterization makes several computations in $\mathcal{T}$ manageable which can be very messy and very difficult for the whole of $\mathcal{A}$. For very recent works related to functions with negative coefficients, we refer to [10, 11, 15, 25, 32, 41, 43, 46, 47].

Definition 1.1 (**(n, δ)-Neighborhood**). Let $f \in \mathcal{T}(n)$ be given by (1.1). Then for $\delta \geq 0$, the (n, δ)-neighborhood $\mathcal{N}_{n, \delta}(f)$ of $f(\zeta)$ is defined as

$$\mathcal{N}_{n, \delta}(f) := \left\{ h \in \mathcal{T}(n) : h(\zeta) = \zeta - \sum_{j=n+1}^{\infty} c_j \zeta^j \text{ and } \sum_{j=n+1}^{\infty} j | a_j - c_j | \leq \delta \right\}. \quad (1.2)$$

In particular, if $f(\zeta) = \zeta = e(\zeta)$ is the identity function then

$$\mathcal{N}_{n, \delta}(e) := \left\{ h \in \mathcal{T}(n) : h(\zeta) = \zeta - \sum_{j=n+1}^{\infty} c_j \zeta^j \text{ and } \sum_{j=n+1}^{\infty} j | c_j | \leq \delta \right\}. \quad (1.3)$$

The concept of neighborhood of an analytic function was first introduced by Goodman [21] in 1957. It was later on considered and generalized by Ruscheweyh [42] in 1981. In view of the fact that if $h(\zeta) = \zeta - \sum_{j=n+1}^{\infty} c_j \zeta^j$ satisfies the condition $\sum_{j=n+1}^{\infty} j | c_j | \leq 1$, then $g \in S_n^*$, it is easy to conclude from the definition (1.3) that

$$\mathcal{N}_{n, \infty}(e) \subset S_n^*.$$

Moreover, it has been proved by Ruscheweyh [42] that if $f \in \mathcal{C}_n$, then for $\delta = 2^{-2/n}$, the inclusion

$$\mathcal{N}_{n,\infty}(f) \subset S_n^*$$

holds and the value of δ is best possible. The work on neighborhoods was further continued in [5-8, 13, 18, 26, 29, 30, 37, 45]. For very recent works on neighborhoods, we refer to [1, 3, 4, 14, 16, 17, 19, 36, 38, 39, 58].

Following the above cited works, in this chapter, we discuss certain neighborhood properties for the function class $\mathcal{TUM}_\gamma(g, b, k, \alpha)$ given in Definition 1.2 of the following subsection.

1.1. The Function Class $\mathcal{TUM}_\gamma^n(g, b, k, \alpha)$. Let $f \in \mathcal{T}(n)$ be of the form (1.1) and $g \in \mathcal{T}(n)$ be given by

$$g(\zeta) = \zeta + \sum_{j=n+1}^{\infty} b_j \zeta^j \quad (b_j \geq 0,\ j \geq n+1,\ n \in \mathbb{N};\ \zeta \in \mathbb{U}). \tag{1.4}$$

Then the Hadamard product (convolution) of f and g, denoted by $f * g$, is defined as the analytic function

$$(f * g)(\zeta) = \zeta - \sum_{j=n+1}^{\infty} a_j b_j \zeta^j = (g * f)(\zeta).$$

We note that the function $g(\zeta) = \zeta/(1-\zeta)$ acts as an identity under the operation of Hadamard product, i.e.,

$$f(\zeta) * \frac{\zeta}{(1-\zeta)} = f(\zeta) = \frac{\zeta}{(1-\zeta)} * f(\zeta).$$

Definition 1.2. Let $k \geq 0,\ 0 \leq \gamma \leq 1,\ 0 \leq \alpha < 1$ and $b \in C/\{0\}$. For a function $g(\zeta)$ of the form (1.4), we say that the function $f \in \mathcal{T}(n)$ of the form (1.1) belongs to the class $\mathcal{TUM}_\gamma(g, b, k, \alpha)$ if $(f * g)(\zeta) \neq 0$ and

$$\Re\left\{1 - \frac{1}{b}\left(\frac{\zeta \mathcal{G}_\gamma'(\zeta)}{\mathcal{G}_\gamma(\zeta)} - 1\right)\right\} > k\left|\frac{1}{b}\left(\frac{\zeta \mathcal{G}_\gamma'(\zeta)}{\mathcal{G}_\gamma(\zeta)} - 1\right)\right| + \alpha, \quad (\zeta \in \mathbb{U}),$$

where

$$\mathcal{G}_\gamma(\zeta) := (1-\gamma)(f * g)(\zeta) + \gamma\zeta(f * g)'(\zeta)$$

$$= \zeta - \sum_{j=n+1}^{\infty} [1 + \gamma(j-1)] a_j b_j, \quad (n \in \mathbb{N}, \zeta \in \mathbb{U}).$$

We note that for suitable selection of the function $g(\zeta)$ and special choices of the parameters $n, \gamma, b, k, \alpha,$ the class $\mathcal{TUM}_{\gamma}^{n}(g, b, k, \alpha)$ has been considered earlier.

(a) For $g(\zeta) = \zeta/(1-\zeta), k = 0$ and $\gamma = 0$ (or $\gamma = 1$), the class $\mathcal{TUM}_{\gamma}^{n}(g, b, k, \alpha)$ becomes $\mathcal{ST}_{\alpha}^{n}(b)$ (or $\mathcal{C}_{\alpha}^{n}(b)$) comprising of functions that are star like (or convex) of complex order $b(b \in C/\{0\})$ and type $\alpha(1 \leq \alpha < 1)$. Analytically,

$$\mathcal{ST}_{\alpha}^{n}(b) := \left\{ f \in \mathcal{T}(n) : \Re\left[1 + \frac{1}{b}\left(\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right)\right] > \alpha \right\}$$

and

$$\mathcal{C}_{\alpha}^{n}(b) := \left\{ f \in \mathcal{T}(n) : \Re\left[1 + \frac{1}{b} \frac{\zeta f''(\zeta)}{f'(\zeta)} - 1\right] > \alpha \right\}.$$

The classes $\mathcal{ST}_{\alpha}^{1}(b)$ and $\mathcal{C}_{\alpha}^{1}(b)$ were introduced and studied by Frasin [20]. The function classes $\mathcal{ST}_{\alpha}^{1}(b)$ and $\mathcal{C}_{\alpha}^{1}(b)$ stem essentially from the classes $\mathcal{ST}_{0}^{1}(b)$ and $\mathcal{C}_{0}^{1}(b)$, which were considered earlier by Nasr and Aouf [33-35] and Wiatrowski [59].

(b) If we set $g(\zeta) = \zeta/(1-\zeta)$ and $b = 1$ in Definition 1.2, then for $\gamma = 0$ we obtain the class

$$\mathcal{US}^{n}(\alpha, k) := \left\{ f \in \mathcal{T}(n) : \Re\left[\frac{\zeta f'(\zeta)}{f(\zeta)} - \alpha\right] > k\left|\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right| \right\}$$

consisting of uniformly k-starlike functions of order α and for $\gamma = 1$ we obtain the class

$$\mathcal{UC}^{n}(\alpha, k) := \left\{ f \in \mathcal{T}(n) : \Re\left[1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} - \alpha\right] > k\left|\frac{\zeta f''(\zeta)}{f'(\zeta)}\right| \right\}$$

comprising of uniformly k-convex functions of order α. Describing geometrically, $f \in \mathcal{US}^n(\alpha, k)$ (or $\mathcal{UC}^n(\alpha, k)$) if and only if all the values taken by the expression $zf'(z)/f(z)$ (or $1 + zf''(z)/f(z)$) lie in the conic domain $\Delta_{\alpha, k}$ given by

$$\Delta_{\alpha, k} := \{u + iv \in \mathbb{C} : (u - \alpha)^2 > k^2((u - 1)^2 + v^2)\}.$$

The conic domain $\Delta_{\alpha, k}$ represents:

(i) the right-half plane $\Re(w) > \alpha$ for $k = 0$,

(ii) a hyperbolic domain for $0 < k < 1$,

(iii) a parabolic domain for $k = 1$, and

(iv) an elliptic domain for $k > 1$.

For $\alpha = 1/2$ and $k \in \{0, 0.5\ 1, 1.5\}$, a pictorial representation of the conic domain $\Delta_{\alpha, k}$ is presented in Figure 1. Observe that the conic regions are symmetric about the u-axis. For further details, see Kanas and Wisniowska [27, 28].

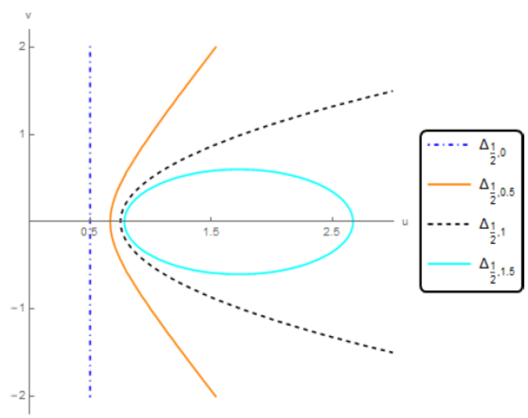

Figure 1. The conic domain $\Delta_{\alpha, k}$ with $\alpha = 1/2$ and $k \in \{0, 0.5\ 1, 1.5\}$.

Referring to other special cases of the class $\mathcal{TUM}_\gamma^n(g, b, k, \alpha)$, we note that.

(1) For $\alpha = 0$, this class was considered by Bukhari et al. [12].

(2) Aouf et al. [9] considered it for $b = 0$.

(3) For $k = 0, 0 \leq \alpha < 1$ and $g(\zeta)$ to be the operator D^{Ω} defined by

$$D^{\Omega} f(\zeta) = \zeta - \sum_{j=n+1}^{\infty} j^{\Omega} a_j \zeta^j (f \in \mathcal{T}(n);\ \Omega \in \mathbb{N}_0),$$

the class $\mathcal{TUM}_{\gamma}^{n}(g, b, k, \alpha)$ was discussed by Orhan et al. [37].

(4) This class was considered by Murugusundaramoorthy [31] for $k = \gamma = 0, 0 \leq \alpha < 1$ and $g(\zeta)$ to be the Ruscheweyh derivative operator $D^{\lambda} (\lambda > -1)$ defined by

$$D^{\lambda} f(\zeta) := \frac{\zeta}{(1-\zeta)^{1+\lambda}} * f(\zeta) \quad (f \in \mathcal{T}(n))$$

or, equivalently, by

$$D^{\lambda} f(\zeta) := \zeta - \sum_{j=n+1}^{\infty} \binom{\lambda + j - 1}{j - 1} a_j \zeta^j \quad (f \in \mathcal{T}(n)),$$

where

$$\binom{\eta}{m} = \frac{\eta(\eta - 1)\ldots(\eta - m + 1)}{m!} \quad (\eta \in \mathbb{C};\ m \in \mathbb{N}_0).$$

(5) For $k = 0, 0 \leq \alpha < 1$ and

$$g(\zeta) = \frac{\zeta}{1 - \zeta}$$

see Altintlas et al. [6].

2. Neighborhood Properties of $\mathcal{TUM}_{\gamma}^{n}(g, b, k, \alpha)$ involving $\mathcal{N}_{n,\delta}(e)$

To prove the main results we need the following Lemma.

Lemma 2.1. *Let* $n \in \mathbb{N}, k \geq 0, 0 \leq \gamma \leq 1, 0 \leq \alpha < 1$ *and* $b \in \mathbb{C}/\{0\}$ *A necessary and sufficient condition for* $f(\zeta)$ *of the form* (1.1) *to be in the class* $\mathcal{TUM}_{\gamma}^{n}(g, b, k, \alpha)$ *is that*

$$\sum_{j=n+1}^{\infty}[(k+1)(j-1)+(1-\alpha)|b|][1+\gamma(j-1)]a_j b_j \leq (1-\alpha)|b|. \tag{2.1}$$

The result is sharp for the function

$$f_i(\zeta) = \zeta - \frac{(1-\alpha)|b|}{[(k+1)(j-1)+(1-\alpha)|b|][1+\gamma(j-1)]b_j}\zeta^j, \tag{2.2}$$

for $j \geq n+1,\ n = 1, 2, 3, \ldots.$

Proof. Let $f \in \mathcal{TUM}_{\gamma}^{n}(g, b, k, \alpha)$. Then from Definition 1.2, we have

$$\Re\left\{1+\frac{1}{b}\left(\frac{\zeta(f*g)'(\zeta)+\gamma\zeta^2(f*g)''(\zeta)}{(1-\gamma)(f*g)(\zeta)+\gamma\zeta(f*g)'(\zeta)}-1\right)\right\}$$

$$> k\left|\frac{1}{b}\left(\frac{\zeta(f*g)'(\zeta)+\gamma\zeta^2(f*g)''(\zeta)}{(1-\gamma)(f*g)(\zeta)+\gamma\zeta(f*g)'(\zeta)}-1\right)\right|+\alpha.$$

On using the representation (1.1) of $f(\zeta)$, we get

$$\Re\left\{1+\frac{1}{b}\left(\frac{\sum_{j=n+1}^{\infty}(j-1)[1+\gamma(j-1)]a_j b_j \zeta^{j-1}}{1-\sum_{j=n+1}^{\infty}[1+\gamma(j-1)]a_j b_j \zeta^{j-1}}\right)\right\}$$

$$> k\left|\frac{1}{b}\left(\frac{\sum_{j=n+1}^{\infty}(j-1)[1+\gamma(j-1)]a_j b_j \zeta^{j-1}}{1-\sum_{j=n+1}^{\infty}[1+\gamma(j-1)]a_j b_j \zeta^{j-1}}\right)\right|+\alpha.$$

Letting $\zeta \to 1^-$ along the real axis and then simplifying we obtain

$$1-\frac{1}{|b|}\left(\frac{\sum_{j=n+1}^{\infty}(j-1)[1+\gamma(j-1)]a_j b_j}{1-\sum_{j=n+1}^{\infty}[1+\gamma(j-1)]a_j b_j}\right)$$

$$> k\left[\frac{1}{|b|}\left(\frac{\sum_{j=n+1}^{\infty}(j-1)[1+\gamma(j-1)]a_j b_j}{1-\sum_{j=n+1}^{\infty}[1+\gamma(j-1)]a_j b_j}\right)\right]+\alpha,$$

which on simplification yields the inequality (2.1). Conversely, suppose the inequality (2.1) holds true. We need to show $f \in \mathcal{TUM}_{\gamma}^{n}(g, b, k, \alpha)$. In view of Definition 1.2, it is sufficient to show that

$$k\left|\left(\frac{\zeta\mathcal{G}_{\gamma}(\zeta)}{\mathcal{G}_{\gamma}(\zeta)}-1\right)\right|-\Re\left\{\frac{\bar{b}}{|b|}\left(\frac{\zeta\mathcal{G}_{\gamma}(\zeta)}{\mathcal{G}_{\gamma}(\zeta)}-1\right)\right\} \leq (1-\alpha)|b|.$$

Now, for $\zeta \in \partial\mathbb{U} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$, we have

$$k\left|\left(\frac{\zeta\mathcal{G}_{\gamma}(\zeta)}{\mathcal{G}_{\gamma}(\zeta)}-1\right)\right|-\Re\left\{\frac{\bar{b}}{|b|}\left(\frac{\zeta\mathcal{G}_{\gamma}(\zeta)}{\mathcal{G}_{\gamma}(\zeta)}-1\right)\right\}$$

$$\leq (k+1)\left|\left(\frac{\zeta\mathcal{G}_{\gamma}(\zeta)}{\mathcal{G}_{\gamma}(\zeta)}-1\right)\right|$$

$$\leq (k+1)\,\frac{\sum_{j=n+1}^{\infty}(j-1)[1+\gamma(j-1)]a_j b_j}{1-\sum_{j=n+1}^{\infty}[1+\gamma(j-1)]a_j b_j}.$$

This last expression is bounded above by $(1-\alpha)|b|$ if

$$\sum_{j=n+1}^{\infty}[(k+1)(j-1)+(1-\alpha)|b|][1+\gamma(j-1)]a_j b_j \leq (1-\alpha)|b|.$$

The result now follows by an appeal to maximum modulus theorem. □

Remark 1. For $g(\zeta) = \zeta/(1-\zeta)$ and $k = 0$, a special case of Lemma 2.1 was proved by Altintlas et al. [6]

Theorem 2.1. *Let* $b_j \geq b_{n+1}$, $j \geq n+1$, *and* δ *be given by*

$$\delta = \frac{(n+1)(1-\alpha)|b|}{(1+n\gamma)[n(k+1)+(1-\alpha)|b|]b_{n+1}}.$$

Then

$$\mathcal{TUM}_{\gamma}^{n}(g, b, k, \alpha) \subset \mathcal{N}_{n,\delta}(e).$$

Proof. Let $f \in \mathcal{TUM}_{\gamma}^{n}(g, b, k, \alpha)$. Then from Lemma 2.1, we have

$$\sum_{j=n+1}^{\infty}[(k+1)(j-1)+(1-\alpha)|b|][1+\gamma(j-1)]a_j b_j \leq (1-\alpha)|b|. \qquad (2.3)$$

Given $b_j \geq b_{n+1}$, it is easy to observe that for each $j \geq n+1$, the following inequality holds

$$[n(k+1)+(1-\alpha)|b|][1+n\gamma]b_{n+1}$$

$$\leq [(k+1)(j-1)+(1-\alpha)|b|][1+\gamma(j-1)]b_j$$

Therefore from (2.3) it follows that

$$[n(k+1)+(1-\alpha)|b|][1+n\gamma]b_{n+1}\sum_{j=n+1}^{\infty}a_j$$

$$= \sum_{j=n+1}^{\infty}[n(k+1)+(1-\alpha)|b|][1+n\gamma]a_j b_{n+1}$$

$$\leq \sum_{j=n+1}^{\infty}[(k+1)(j-1)+(1-\alpha)|b|][1+\gamma(j-1)]a_j b_j$$

$$\leq (1-\alpha)|b|.$$

This further implies that

$$\sum_{j=n+1}^{\infty}a_j \leq \frac{(1-\alpha)|b|}{[n(k+1)+(1-\alpha)|b|][1+n\gamma]b_{n+1}}, \ (n \in \mathbb{N}). \qquad (2.4)$$

Using Lemma 2.1 again in conjunction with (2.4), we have

$$(k+1)(1+n\gamma)b_{n+1}\sum_{j=n+1}^{\infty}ja_j$$

$$\leq (1-\alpha)|b| + (1+n\gamma)[(k+1)-(1-\alpha)|b|]b_{n+1}\sum_{j=n+1}^{\infty}a_j$$

$$\leq (1-\alpha)|b| + (1+n\gamma)[(k+1)-(1-\alpha)|b|]b_{n+1}$$

$$\times \frac{(1-\alpha)|b|}{[n(k+1)+(1-\alpha)|b|][1+n\gamma]b_{n+1}}$$

$$= \frac{(n+1)(k+1)(1-\alpha)|b|}{n(k+1)+(1-\alpha)|b|}.$$

The above inequality further yields that

$$\sum_{j=n+1}^{\infty} ja_j \leq \frac{(1-\alpha)|b|}{(1+n\gamma)[n(k+1)+(1-\alpha)|b|][1+n\gamma]b_{n+1}} = \delta. \tag{2.5}$$

Thus, the theorem follows in light of the definition of $\mathcal{N}_{n,\delta}(e)$.

3. Neighborhood Properties for the Family $\mathcal{TUM}_{\gamma}^{n(\varpi)}(g, b, k, \alpha)$

The aim of this section is to find the neighborhood for the class $\mathcal{TUM}_{\gamma}^{n(\varpi)}(g, b, k, \alpha)$ defined as follows.

Definition 3.1. Let $0 \leq \varpi < 1$ and $h \in \mathcal{TUM}_{\gamma}^{n}(g, b, k, \alpha)$. A function $f \in \mathcal{T}(n)$ is said to belong to the class $\mathcal{TUM}_{\gamma}^{n(\varpi)}(g, b, k, \alpha)$ if it satisfies

$$\left|\frac{f(\zeta)}{h(\zeta)} - 1\right| < 1 - \varpi \quad (\zeta \in \mathbb{U}) \tag{3.1}$$

Theorem 3.1. *Let* $h(\zeta)$ *be in the class* $\mathcal{TUM}_{\gamma}^{n}(g, b, k, \alpha)$ *and*

$$\varpi = 1 - \frac{\delta[n(k+1)+(1-\alpha)|b|][1+n\gamma]b_{n+1}}{(n+1)\{[n(k+1)+(1-\alpha)|b|][1+n\gamma]b_{n+1} - (1-\alpha)|b|\}}. \tag{3.2}$$

Then

$$\mathcal{N}_{n,\delta}(h) \subset \mathcal{TUM}_{\gamma}^{n(\varpi)}(g, b, k, \alpha).$$

Proof. Let us take

$$h(\zeta) = \zeta - \sum_{j=n+1}^{\infty} c_j \zeta^j.$$

Suppose that $f(\zeta) = \zeta - \sum_{j=n+1}^{\infty} a_j \zeta^j \in \mathcal{N}_{n,\,\delta}(h)$. Then from Definition 1.2, it follows that

$$\sum_{j=n+1}^{\infty} j | a_j - c_j | \leq \delta,$$

which readily implies the inequality

$$\sum_{j=n+1}^{\infty} | a_j - c_j | \leq \frac{\delta}{n+1} \, (n \in \mathbb{N}).$$

Next, since $h \in \mathcal{TUM}_{\gamma}^{n}(g, b, k, \alpha)$ we have from (2.4) that

$$\sum_{j=n+1}^{\infty} c_j \leq \frac{(1-\alpha)|b|}{[n(k+1) + (1-\alpha)|b|][1 + n\gamma] b_{n+1}} \, (n \in \mathbb{N}).$$

Now utilizing the above inequality and the representations of f and h, we have

$$\left| \frac{f(\zeta)}{h(\zeta)} - 1 \right| = \left| \frac{\zeta - \sum_{j=n+1}^{\infty} a_j \zeta^j}{\zeta - \sum_{j=n+1}^{\infty} c_j \zeta^j} - 1 \right|$$

$$= \left| \frac{\sum_{j=n+1}^{\infty} (a_j - c_j) \zeta^j}{\zeta - \sum_{j=n+1}^{\infty} c_j \zeta^j} \right|$$

$$= \left| \frac{\sum_{j=n+1}^{\infty} (a_j - c_j) \zeta^{j-1}}{\zeta - \sum_{j=n+1}^{\infty} c_j \zeta^{j-1}} \right|$$

$$\leq \frac{\sum_{j=n+1}^{\infty} | a_j - c_j |}{1 - \sum_{j=n+1}^{\infty} c_j}$$

$$\leq \frac{\delta/(n+1)}{1 - \frac{(1-\alpha)|b|}{[n(k+1)+(1-\alpha)|b|][1+n\gamma]b_{n+1}}}$$

$$\leq \frac{\delta[n(k+1)+(1-\alpha)|b|][1+n\gamma]b_{n+1}}{(n+1)\{[n(k+1)+(1-\alpha)|b|][1+n\gamma]b_{n+1} - (1-\alpha)|b|\}}$$

$$= 1 - \varpi,$$

provided that ϖ is given precisely by (3.2). This implies that $f \in \overset{(\varpi)}{\mathcal{TUM}}{}_{\gamma}^{n}(g, b, k, \alpha)$. Hence proves the inclusion $\mathcal{N}_{n,\delta}(h) \subset \overset{(\varpi)}{\mathcal{TUM}}{}_{\gamma}^{n}(g, b, k, \alpha)$ and completes the proof of Theorem 3.1. □

Taking $g(\zeta) = \zeta/(1-\zeta)$, $b = 1$ and $\gamma = 0$, Theorem 3.1 yields the following.

Corollary 3.1. *Let $h(\zeta)$ be in the class $\mathcal{US}^n(\alpha, k)$ and*

$$\varpi = 1 - \frac{\delta[1 + n(k+1) - \alpha]}{n(n+1)(k+1)}.$$

Then

$$\mathcal{N}_{n,\delta}(h) \subset \overset{(\varpi)}{\mathcal{US}}{}^{n}(\alpha, k),$$

where $\overset{(\varpi)}{\mathcal{US}}{}^{n}(\alpha, k)$ is defined in accordance to the Definition 3.1.

Remark 2. For $k = 0$, the Corollary 3.1 was obtained by Altintas et al. [5, Theorem 4.2].

Again for $g(\zeta) = \zeta/(1-\zeta)$ and $b = 1$, Theorem 3.1 yields the following result upon taking $\gamma = 1$.

Corollary 3.2. *Let* $h(\zeta)$ *be in the class* $\mathcal{UC}^n(\alpha, k)$ *and*

$$\varpi = 1 - \frac{\delta[1 + n(k+1) - \alpha]}{n[2 + k - a]}.$$

Then

$$\mathcal{N}_{n,\,\delta}(h) \subset \overset{(\varpi)}{\mathcal{UC}^n}(\alpha, k),$$

where $\overset{(\varpi)}{\mathcal{UC}^n}(\alpha, k)$ *is defined in accordance with the Definition* 3.1.

4. Growth and Distortion Estimates

Before going to the main result, we combine the inequalities (2.4) and (2) proved in Theorem 2.1 in the following lemma.

Lemma 4.1. *Let* $f(\zeta)$ *given by* (1.1) *be a member of* $\mathcal{TUM}_\gamma^n(g, b, k, \alpha)$ *and let* $b_j \geq b_{n+1}$. *Then*

$$\sum_{j=n+1}^{\infty} a_j \leq \frac{(1-\alpha)|\,b\,|}{[n(k+1) + (1-\alpha)|\,b\,|][1 + n\gamma]b_{n+1}}, \tag{4.1}$$

and

$$\sum_{j=n+1}^{\infty} j a_j \leq \frac{(n+1)(1-\alpha)|\,b\,|}{(1+n\gamma)[n(k+1) + (1-\alpha)|\,b\,|]b_{n+1}}. \tag{4.2}$$

Theorem 4.1. *Let* $f(\zeta) = \zeta - \sum_{j=n+1}^{\infty} a_j \zeta^j \in \mathcal{TUM}_\gamma^n(g, b, k, \alpha)$, *where* $a_j \geq 0$ *and* $\zeta \in \mathbb{U}$. *Then we have the following.*

(*i*) *The growth estimates on* $f(\zeta)$ *are given by*

$$|\,f(\zeta)\,| \leq |\,\zeta\,| + \frac{(1-\alpha)|\,b\,|}{[n(k+1) + (1-\alpha)|\,b\,|][1 + n\gamma]b_{n+1}}|\,\zeta\,|^{n+1}$$

and

$$|f(\zeta)| \geq |\zeta| - \frac{(1-\alpha)|b|}{[n(k+1)+(1-\alpha)|b|][1+n\gamma]b_{n+1}}|\zeta|^{n+1}.$$

(*ii*) *The distortion estimates on* $f(\zeta)$ *are given by*

$$|f'(\zeta)| \leq 1 + \frac{(n+1)(1-\alpha)|b|}{(1-n\gamma)[n(k+1)+(1-\alpha)|b|]b_{n+1}}|\zeta|^{n},$$

and

$$|f'(\zeta)| \geq 1 + \frac{(n+1)(1-\alpha)|b|}{(1-n\gamma)[n(k+1)+(1-\alpha)|b|]b_{n+1}}|\zeta|^{n}.$$

The result is sharp for the function given by (2.2).

Proof. (i) Since $f(\zeta) = \zeta - \sum_{j=n+1}^{\infty} a_j \zeta^j \in \mathcal{TUM}_{\gamma}^{n}(g, b, k, \alpha)$, we have from (4.1),

$$|f(\zeta)| = \left|\zeta - \sum_{j=n+1}^{\infty} a_j \zeta^j\right|$$

$$\leq |\zeta| + \sum_{j=n+1}^{\infty} a_j |\zeta^j|$$

$$\leq |\zeta| + |\zeta|^{n+1} \sum_{j=n+1}^{\infty} a_j.$$

$$\geq |\zeta| + |\zeta|^{n+1} \times \frac{(1-\alpha)|b|}{[n(k+1)+(1-\alpha)|b|][1+n\gamma]b_{n+1}}.$$

Again from (4.1), we have

$$|f(\zeta)| = \left|\zeta - \sum_{j=n+1}^{\infty} a_j \zeta^j\right|$$

$$\leq |\zeta| + \sum_{j=n+1}^{\infty} a_j |\zeta^j|$$

$$\leq |\zeta| + |\zeta|^{n+1} \sum_{j=n+1}^{\infty} a_j$$

$$\geq |\zeta| + |\zeta|^{n+1} \times \frac{(1-\alpha)|b|}{[n(k+1)+(1-\alpha)|b|][1+n\gamma]b_{n+1}}.$$

This establishes the first part of the theorem.

(ii) Since $f'(\zeta) = 1 - \sum_{j=n+1}^{\infty} ja_j\zeta^{j-1}$, we have

$$|f'(\zeta)| = \left|1 - \sum_{j=n+1}^{\infty} ja_j\zeta^{j-1}\right|$$

$$\leq |\zeta| + \sum_{j=n+1}^{\infty} ja_j|\zeta^{j-1}|$$

$$\leq |\zeta| + |\zeta|^n \sum_{j=n+1}^{\infty} ja_j.$$

Now applying the inequality (4.2), we have

$$|f'(\zeta)| \leq 1 + \frac{(n+1)(1-\alpha)|b|}{(1+n\gamma)[n(k+1)+(1-\alpha)|b|]b_{n+1}}|\zeta|^n.$$

The other inequality can easily be proved in a similar fashion.

This establishes the growth and distortion theorems for the function class $\mathcal{TUM}_{\gamma}^{n}(g, b, k, \alpha)$. □

5. Concluding Remarks

(1) In literature, there exist a number of unsolved problems related to neighborhoods of special families of univalent functions. For example, Ruscheweyh asked for a hypothetical geometric characterization of the class $\mathcal{N}_{n,1/4}(\mathcal{C}) \subset S_n^*$. There is also an important but arduous question of Sheil-Small and Silvia [45] concerning the inclusion of $\mathcal{N}_{n,1/2}(\mathcal{C}_n(1/2))$ in S_n^*,

where $\mathcal{C}_n(1/2)$ denotes the class of convex functions of order 1/2, i.e.,

$$f \in \mathcal{C}_n(1/2) \Leftrightarrow \Re\left(1 + \frac{\zeta f'(\zeta)}{f(\zeta)}\right) > \frac{1}{2}, \quad \zeta \in \mathbb{U}.$$

In fact, the paper [45] contains many open problems and conjectures of this type connected with the Duality Theory developed by Ruscheweyh.

(2) In [42], Ruscheweyh proved that for $f \in \mathcal{T}(n)$ and $\delta > 0$, if

$$\frac{f(\zeta) + \epsilon\zeta}{1 + \epsilon} \in \mathcal{S}_n^*, \ \forall \ \epsilon \text{ with } |\epsilon| < \delta,$$

then $\mathcal{N}_{n,\,1/4}(f) \subset S_n^*$. Ruscheweyh [42] asked whether or not the above result is valid if we replace the class S_n^* by the class of close-to-convex univalent functions denoted by $\mathcal{K}_n$. To the best of our knowledge, this problem is still unsolved up to now.

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