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INTERVAL VALUED SECONDARY κ-RANGE SYMMETRIC FUZZY MATRICES

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Abstract

In this paper, we have investigated the characterization of interval valued secondary κ-range symmetric fuzzy matrices. The relationship between interval valued $s-\kappa$ range symmetric, interval valued s-range symmetric, interval valued κ-range symmetric and interval valued range symmetric matrices are discussed. The necessary and sufficient conditions for a matrix to be interval valued $s-\kappa$ range symmetric fuzzy matrices are established.

1. Introduction

All the matrices in this paper are interval valued fuzzy matrices [9]. We have defined a $(IVFM)$ as $X = (x_{ij}) = [x_{ijL}, x_{ijU}]$ where each x_{ij} is the subinterval of interval $[0, 1]$ [5]. Let $\mathcal{F}_{nn}$ be the set of all $n \times n$ fuzzy matrices over the fuzzy algebra with the support $[0, 1]$ under the operation $(+, \cdot)$ as $x, y \in \mathcal{F}, x + y = \max\{x, y\}$ and $x \cdot y = \min\{x, y\}$. Let A^T, A^+, $R(A), C(A), N(A), \rho(A)$ denote the transpose of the matrix A, Moore-pen rose inverse, Row space of A, Column space of A, Null space of A and the rank of A

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respectively. $A\{1\}$ denote the set of all g-inverses of a regular fuzzy matrix A. For a fuzzy matrix, A^{+} if A exist then it is coincide with A^{+} [1]. A fuzzy matrix A is range symmetric if $R(A) = R(A^T)$ and kernels symmetric $N(A) = N(A^T)$ [3]. It is well understood that the idea of range and kernel symmetric are the same for complex matrices. Furthermore, for interval valued fuzzy matrices, this fails. Ann Lee [2] pioneered the research of secondary symmetric matrices, or matrices with symmetric entries around the secondary diagonal. Antoni, Cantoni, and Butler Paul [7] investigated the importance of per symmetric matrices, or matrices that are symmetric about both diagonals, in communication theory. Water and Hill [10] established a theory of s-real and s-Hermitian matrices as a generalization of κ-real and κ-Hermitian matrices [8]. Meenakshi and Jayashree developed the concepts of κ-kernel symmetric fuzzy matrices [4] and of κ-range symmetric fuzzy matrices [11].

We established and developed the notion of interval valued $s-\kappa$ range symmetric matrices for fuzzy matrices as a special example of equivalent to the results on complex matrices, and we enlarged various basic results on $s-\kappa$ Hermitian and interval valued range symmetric matrices. In section 3, an interval valued secondary $s-\kappa$ range symmetric fuzzy matrices can be characterized [6]. In section 4, appropriate criteria are discovered for various g-inverses of an interval valued secondary κ-range symmetric fuzzy matrix to be interval valued secondary κ-range symmetric.

2. Preliminaries

This section contains a few key definitions and outcomes that are required.

Definition 2.1. $A = (a_{ij})_{m\times n}$ is an interval valued fuzzy matrix (IVFM) of order mn where $(a_{ij}) = [a_{ijL}, a_{ijU}]$ the ij^{th} entry of the matrix A is an interval representing the membership values. Every element of a (IVFM) is an interval, and every interval is a subinterval of the interval $[0, 1]$. Any two IVFMs can be represented by C and D. For any two elements $c \in C$ and $d \in D$ where $c = [c_L, c_U]$ and $d = [d_L, d_U]$ are intervals in $[0, 1]$ so that $c_L < c_U$ and $d_L < d_U$.

(i) $c + d = [\max\{c_L, d_L\}, \max\{c_U, d_U\}]$

(ii) $c \cdot d = [\min\{c_L, d_L\}, \min\{c_U, d_U\}]$

In this case, we have used the basic IVFM operation described in [5].

For $X = (x_{ij}) = ([x_{ijL}, x_{ijU}])$ and $Y = (y_{ij}) = ([y_{ijL}, y_{ijU}])$ of order $m \times n$ with their sum denoted as $X + Y$

$$X + Y = (x_{ij} + y_{ij}) = ([x_{ijL} + y_{ijL}, x_{ijU} + y_{ijU}])$$

For $X = (x_{ij})_{m \times n}$ and $Y = (y_{ij})_{n \times p}$ with their product denoted as,

$$XY = (z_{ij})_{m \times p} = \left[\sum_{k=1}^{n} c_{ik} d_{kj}\right] \quad i = 1, 2, \ldots, m$$

$$= \left[\sum_{k=1}^{n} (c_{ikL} \cdot d_{kjL}), \sum_{k=1}^{n} (c_{ikU} \cdot d_{kjU}),\right]$$

$X \le Y$ if and only if $x_{ijL} \le y_{ijL}$ and $x_{ijU} \le y_{ijU}$

If $x_{ijL} = x_{ijU}$ and $y_{ijL} = y_{ijU}$, then it is reduces to the standard max. min fuzzy matrix composition.

Definition 2.2. Consider 'V' which has units on its secondary diagonal and K is the fixed product of disjoint transpositions in $S_n = 1$ to n, and K be the related permutation matrix everywhere.

If $a_{ij} = a_{k(j)k(i)}$ for $i, j = 1, 2, \ldots, n$ then a matrix $(a_{ij}) \in IVFM$ is κ-symmetric. For $x = (x_1, \ldots, x_n)^T \in \mathcal{F}_{n \times 1}$. Let us defined the function $\Re(x)$

$\Re(x) = (x_{k(1)}, x_{k(2)}, \ldots, x_{k(n)})^T \in \mathcal{F}_{n \times 1}$ Because K is involuntary, it is possible to verify that the related permutation matrix meets the conditions given in [12].

By the definition of 'V',

$(C \cdot 2 \cdot 2 \cdot 1)$ $KK^T = K^T K = I$, $K = K^T$, $K^2 = I$ and $\Re(x) = Kx$

$(C \cdot 2 \cdot 2 \cdot 2)$ $V = V^T$, $V^T V = VV^T = I$ and $V^2 = I$

$(C \cdot 2 \cdot 2 \cdot 3)$ $(VA)^T = A^T V$, $(AV)^T = VA^T$

$(C \cdot 2 \cdot 2 \cdot 4)$ If A^+ exist, then $(VA)^+ = A^+ V$, $(AV)^+ = VA^+$

$(C \cdot 2 \cdot 2 \cdot 5)$ $R(A) = R(VA)$, $R(A) = R(KA)$.

Theorem 2.3 [3], P.120.

For $A \in \mathcal{F}_n$ *the subsequent statements are equal*

(1) $\rho(A) = r$ *and* A *is a range symmetric and*

(2) $C(A^T) = C(A)$

(3) *For some fuzzy matrices* $AH = KA = A^T H$, K *and* $\rho(A) = r$

(4) PAP^T *is range symmetric matrix of rank of r for few permutation matrix P.*

Lemma. 2.4 [3], P.123

For $A \in \mathcal{F}_n$, *If* A^+ *exists then the subsequent statements are equal*

(1) *A is a range symmetric*

(2) $A^+ A = AA^+$

(3) A^+ *is range symmetric*

(4) A *is normal.*

Lemma 2.5 [3], P.119. *For* $A \in \mathcal{F}_n$ *and a permutation matrix* P, $R(A) = R(B) \Leftrightarrow R(PAP^T) = R(PBP^T)$.

Definition 2.6. For a matrix $A \in \mathcal{F}_{nn}$ is s-symmetric $\Leftrightarrow A = VA^T V$.

Definition 2.7. For a matrix $A \in \mathcal{F}_{nn}$ is s-range symmetric $\Leftrightarrow R(A) = R(VA^T V)$.

Definition 2.8. For a matrix $A \in \mathcal{F}_{nn}$ is $s - \kappa$ range symmetric $\Leftrightarrow R(A) = R(KVA^T VK)$.

Lemma 2.9. *For a matrix* $A \in \mathcal{F}_{nn}$ *is a range symmetric* $\Leftrightarrow AV$ *is range symmetric* $\Leftrightarrow VA$ *is range symmetric.*

Definition 2.10 [12]**.** A matrix $A = [A_L, A_U] \in IVFM_{nn}$ is stated to be interval valued range symmetric if $R(A_L) = R(A_L^T)$ where $R(A_L) = \{x/xA_L = 0 \text{ and } x \in \mathcal{F}_{n\times 1}\}$.

$R(A_U) = R(A_U^T)$, *where* $R(A_U) = \{x/xA_U = 0 \text{ and } x \in \mathcal{F}_{n\times 1}\}$.

3. Internal Valued Secondary κ-range Symmetric Fuzzy Matrices

We have classified and acquired some results on interval valued $s - \kappa$ range symmetric fuzzy matrices in this section.

Definition 3.1. A matrix $A = [A_L, A_U] \in IVFM_{nn}$ is an interval valued s-symmetric if and only if

$$A_L = VA_L^TV,\ A_U = VA_U^TV$$

Definition 3.2. A matrix $A = [A_L, A_U] \in IVFM_{nn}$ is an interval valued s-range symmetric if and only if $R(A_L) = R(VA_L^TV)$, $R(A_U) = R(VA_U^TV)$.

Definition 3.3. A matrix $A = [A_L, A_U] \in IVFM_{nn}$ is an interval valued $s - \kappa$ range symmetric $\Leftrightarrow R(A_L) = R(KVA_L^TVK)$, $R(A_U) = R(KVA_U^TVK)$.

Lemma 3.4. *A matrix* $A = [A_L, A_U] \in IVFM_{nn}$ *is an interval valued s-range symmetric* $\Leftrightarrow VA = [VA_L, VA_U]$ *is an interval valued range symmetric* $\Leftrightarrow AV = [A_LV, A_UV]$ *is an interval valued range symmetric.*

Proof. A matrix $A = [A_L, A_U] \in IVFM_{nn}$ is an interval valued s-range symmetric $\Leftrightarrow R(A_L) = R(VA_L^TV)$ [By definition 3.2]

$\Leftrightarrow R(A_LV) = R(A_LV)^T$ [By C 2.2.2]

$\Leftrightarrow A_LV$ is a range symmetric

$\Leftrightarrow R(VA_LVV^T) = R(VVA_L^TV)$

$\Leftrightarrow R(VA_L) = R(VA_L)^T$ [By C 2.2.2]

$\Leftrightarrow VA_L$ is a range symmetric.

Similar manner,

$\Leftrightarrow R(A_U) = R(VA_U^T V)$ [By definition 3.2]

$\Leftrightarrow R(A_U V) = R(A_U V)^T$ [By C 2.2.2]

$\Leftrightarrow A_U V$ is a range symmetric

$\Leftrightarrow R(VA_U VV^T) = R(VVA_U^T V)$

$\Leftrightarrow R(VA_U) = R(VA_U)^T$ [By C 2.2.2]

$\Leftrightarrow VA_U$ is a range symmetric.

$\therefore VA = [VA_L, VA_U]$ is an interval valued symmetric.

Remark 3.5. When $k(i) = i$ for each $i = 1$ to n, the related permutation matrix K becomes the identity matrix, and definition 3.3 becomes $R(A_L) = R(VA_L^T V), R(A_U) = R(VA_U^T V)$, implying that $A = [A_L, A_U]$ are interval valued s-range symmetric matrices.

Remark 3.6. When $k(i) = n - i + 1$, the related permutation matrix K becomes V, and definition 3.3 obtains $R(A_L) = R(A_L^T), R(A_U) = R(A_U^T)$, implying that $A = [A_L, A_U]$ an interval valued range symmetric.

Remark 3.7. We note that, an interval valued $s - \kappa$ symmetric matrix is an interval valued $s - \kappa$ range symmetric for if $A = [A_L, A_U]$ is an interval valued $s - \kappa$ symmetric then $A_L = KVA_L^T VK, A_U = KVA_U^T VK$. Hence $R(A_L) = R(KVA_L^T VK), R(A_U) = R(KVA_U^T VK)$ which implies that A is an interval valued $s - \kappa$ kernel symmetric. The inverse on the other hand does not have to be true. The following studies illustrate this.

Example 3.8. For $K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Let $A = [A_L, A_U] =$

$\begin{bmatrix} [0.1, 0.1] & [0.5, 0.7] \\ [0.5, 0.7] & [0.1, 0.1] \end{bmatrix}$ is an interval valued symmetric, interval valued $s - \kappa$ symmetric and hence therefore interval valued $s - \kappa$ range symmetric.

Here $[A_L] = \begin{bmatrix} 0.1 & 0.5 \\ 0.5 & 0.1 \end{bmatrix}$, $[A_U] = \begin{bmatrix} 0.1 & 0.7 \\ 0.7 & 0.1 \end{bmatrix}$

Consider $KVA_L^T VK = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0.5 \\ 0.5 & 0.1 \end{bmatrix}$$

$$= A_L$$

$KVA_L^T VK = A_L$. Similar we can get $KVA_U^T VK = A_U$

$\Rightarrow A = [A_L, A_U]$ is an interval valued $s - \kappa$ symmetric.

Example 3.9. For $\kappa = (1, 2)(3)$ $K = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Here $K \neq I$, $K \neq V$ and $V \neq VK$

Now $A = [A_L, A_U] = \begin{bmatrix} [0, 0] & [0, 0] & [1, 1] \\ [0.3, 0.4] & [1, 1] & [0, 0] \\ [0.3, 0.6] & [0.1, 0.1] & [0, 0] \end{bmatrix}$ is an interval valued $s - \kappa$ range symmetric but not an interval valued $s - \kappa$ symmetric.

Consider $KVA_L^T VK = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0.3 & 0.3 \\ 0 & 1 & 0.1 \\ 1 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0.3 & 0.3 \\ 0 & 1 & 0.1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.1 & 0 \\ 0 & 0 & 1 \\ 0.3 & 0.3 & 0 \end{bmatrix} \neq A_L$$

Similarly, we can prove that

$$KVA_U^T VK \neq A_U$$

Hence $A = [A_L, A_U]$ is not an interval valued $s - \kappa$ symmetric

But $R(A_L) = N(KVA_L^T VK) = \{0\}$

$$R(A_U) = N(KVA_U^T VK) = \{0\}$$

$\therefore A = [A_L, A_U]$ an interval valued $s - \kappa$ range symmetric

Theorem 3.10 (Characterization Theorem)**.** *The following statements are identical for* $A \in IVFM_{n \times n}$

(1) $A = [A_L, A_U]$ *is an interval valued* $s - \kappa$ *range symmetric*

(2) $KVA = [KVA_L, KVA_U]$ *is an interval valued range symmetric*

(3) $AKV = [A_L KV, A_U KV]$ *is an interval valued range symmetric*

(4) $VA = [VA_L, VA_U]$ *is an interval valued* κ*-range symmetric*

(5) $AK = [A_L K, A_U K]$ *is an interval valued s-range symmetric*

(6) A^T *is an interval valued* $s - \kappa$ *range symmetric*

(7) $R(A_L) = R(A_L^T VK), R(A_U) = R(A_U^T VK)$

(8) $R(A_L^T) = R(A_L VK), R(A_U^T) = R(A_U VK)$

(9) $C(KVA_L) = C((KVA_L)^T)\ C(KVA_U) = C((KVA_U)^T)$

(10) $A_L = VKA_L^T VKH_1, \; A_U = VKA_U^T VKH_1 \; for \; H_1 \in IVFM$

(11) $A_L = H_1 KVA_L^T KV, \; A_U = H_1 KA_U^T VK \; for \; H_1 \in IVFM$

(12) $A_L^T = KVA_L VKH_1, \; A_U^T = KVA_U VKH_1 \; for \; H_1 \in IVFM$

(13) $A_L^T = H_1 KVA_L KV, \; A_U^T = H_1 KVA_U VK \; for \; H_1 \in IVFM.$

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) Consider $A = [A_L, A_U]$ is an interval valued $s - \kappa$ range symmetric

Let us consider A_L is a $s - \kappa$ range symmetric

$$\Leftrightarrow R(A_L) = R(KVA_L^T VK), \; R(A_U) = R(KVA_U^T VK)$$

[By Definition 3.3]

$$\Leftrightarrow R(KVA_L) = R((KVA_L)^T) \, R(KVA_U) = R((KVA_U)^T)$$

[By C 2.2.5]

$\Leftrightarrow KVA = [KVA_L, KVA_U]$ is an interval valued range symmetric

$\Leftrightarrow VA = [VA_L, VA_U]$ is an interval valued κ-range symmetric

As a result (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) are true

(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5)

$A = [A_L, A_U]$ is an interval valued $s - \kappa$ range symmetric

$$\Leftrightarrow R(A_L) = R(KVA_L^T VK), \; R(A_U) = R(KVA_U^T VK)$$

[By Definition 3.3]

$$\Leftrightarrow R(KVA_L) = R((KVA_L)^T) R(KVA_U) = R((KVA_U)^T)$$

[By C 2.2.5]

$\Leftrightarrow R(VK(KVA_L)) = R((VK)A_L^T VK(VK)^T),$

$R(VK(KVA_U)) = R((VK)A_U^T VK(VK)^T)$ [By Lemma. 2.2]

$\Leftrightarrow R(A_L KV) = R((A_L KV)^T), R(A_U KV) = R((A_U KV)^T)$

$\Leftrightarrow AKV = [A_L KV, A_U KV]$ is an interval valued range symmetric

$\Leftrightarrow AK = [A_L K, A_U K]$ is an interval s-range symmetric

As a result (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) are true

(2) $\Leftrightarrow$ (9)

$KVA = [KVA_L, KVA_U]$ is interval valued range symmetric

$$\Leftrightarrow R(KVA_L) = R((KVA_L)^T), R(KVA_U) = R((KVA_U)^T)$$

$$\Leftrightarrow C((KVA_L)^T) = C(KVA_L) C((KVA_U)^T) = C(KVA_U)$$

As a result, (2) $\Leftrightarrow$ (9) hold

(2) $\Leftrightarrow$ (7)

$KVA = [KVA_L, KVA_U]$ is an interval valued range symmetric

$\Leftrightarrow R(KVA_L) = R((KVA_L)^T), R(KVA_U) = R((KVA_U)^T)$

$\Leftrightarrow R(A_L) = R((KVA_L)^T), R(A_U) = R((KVA_U)^T)$ [By C 2.2.5]

$\Leftrightarrow R(A_L) = R(A_L^T VK), R(A_L) = R(A_L^T VK)$

As a conclusion (2) $\Leftrightarrow$ (7) is true

(3) $\Leftrightarrow$ (8)

$AVK = [A_L VK, A_U VK]$ is an interval valued range symmetric

$$\Leftrightarrow R(A_L VK) = R((A_L VK)^T), R(A_U VK) = R((A_U VK)^T)$$

$$\Leftrightarrow R(A_L VK) = R((A_L VK)^T), R(A_U VK) = R((A_U VK)^T)$$

$$\Leftrightarrow R(A_L VK) = R((A_L)^T), R(A_U VK) = R((A_U)^T)$$

As a conclusion (3) $\Leftrightarrow$ (8) hold

(1) $\Leftrightarrow$ (6)

$A = [A_L, A_U]$ is an interval valued $s - \kappa$ range symmetric

$\Leftrightarrow R(A_L) = R(KVA_L^T VK), R(A_U) = R(KVA_U^T VK)$ [By Definition 3.3]

$\Leftrightarrow R(KVA_L) = R((KVA_L)^T), R(KVA_U) = R((KVA_U)^T)$

$\Leftrightarrow (KVA)^T = (KVA_L, KVA_U)^T$ is an interval valued range symmetric

$\Leftrightarrow A^T VK = [A_L VK, A_U VK]$ is an interval valued range symmetric

$\Leftrightarrow A^T = [A_L^T, A_U^T]$ is an interval valued $s - \kappa$ range symmetric

As a conclusion (1) $\Leftrightarrow$ (6) is true

(1) $\Leftrightarrow$ (12) $\Leftrightarrow$ (11)

$A = [A_L, A_U]$ is an interval valued $s - \kappa$ range symmetric

$\Leftrightarrow R(A_L) = R(KVA_L^T VK), R(A_U) = R(KVA_U^T VK)$

$\Leftrightarrow C(A_L^T) = C(KVA_L VK), C(A_U^T) = C(KVA_U VK)$

$\Leftrightarrow A_L^T = KVA_L VK, A_U^T = KVA_U VK$ [By Theorem 2.3]

$\Leftrightarrow A_L = H_1 KVA_L^T VK, A_U = H_1 KVA_U^T VK$ for $H_1 \in IVFM$

As a result (1) $\Leftrightarrow$ (12) $\Leftrightarrow$ (11) hold

(2) $\Leftrightarrow$ (13) $\Leftrightarrow$ (10)

$KVA = [KVA_L, KVA_U]$ is an interval valued range symmetric

$\Leftrightarrow VA = [VA_L, VA_U]$ is an interval valued κ-range symmetric

$\Leftrightarrow R(VA_L) = R(K(VA_L)^T K), R(VA_U) = R(K(VA_U)^T K)$

$\Leftrightarrow R(A_L) = R(A_L^T VK), R(A_U) = R(A_U^T VK)$ [By C2.2.5]

$\Leftrightarrow C(A_L^T) = C(KVA_L), C(A_U^T) = C(KVA_U)$

$\Leftrightarrow A_L^T = HKVA_L, A_U^T = HKVA_U$ for $H \in IVFM$

$\Leftrightarrow A_L^T = H_1 KVA_L KV,\ A_U^T = H_1 KVA_U KV$

$\Leftrightarrow A_L = VKA_L^T VKH_1,\ A_U = VKA_U^T VKH_1$ for $H_1 \in IVFM$

As a conclusion (2) $\Leftrightarrow$ (13) $\Leftrightarrow$ (10) are true. As a result, the theorem is valid.

The preceding theorem simplifies to the analogous constraint for a matrix to be an interval valued s-range symmetric for $K = I$ in particular.

Corollary 3.11. *The following propositions are equal for* $A \in IVFM_{n\times n}$

1. $A = [A_L, A_U]$ *is an interval valued s-range symmetric*

2. $VA = [VA_L, VA_U]$ *is an interval valued range symmetric*

3. $AV = [A_L V, A_U V]$ *is an interval valued range symmetric*

4. $A^T = [A_L^T, A_U^T]$ *is an interval valued s-range symmetric*

5. $R(A_L) = R(A_L^T V),\ R(A_U) = R(A_U^T V)$

6. $R(A_L^T) = R(A_L V),\ R(A_U^T) = R(A_U V)$

7. $C(KVA_L) = C((VA_L)^T),\ C(KVA_U) = C((VA_L)^T)$

8. $A_L = VA_L^T VH_1,\ A_U = VA_U^T VH_1$ *for* $H_1 \in IVFM$

9. $A_L = H_1 VA_L^T V,\ A_U = H_1 VA_U^T V$ *for* $H_1 \in IVFM$

10. $A_L^T = VA_L VH_1,\ A_U^T = VA_U VH_1$ *for* $H_1 \in IVFM$

11. $A_L^T = H_1 VA_L V,\ A_U^T = H_1 VA_U VH_1$ *for* $H_1 \in IVFM$

Lemma 3.12. *For* $A = [A_L, A_U] \in IVFM_{nn},\ A = [A_L^+, A_U^+]$ *exist* $\Leftrightarrow (KA_L)^+, (KA_U)^+$ *exits* $\Leftrightarrow (VKA_L)^+, (VKA_U)^+$ *exits*

Proof. For $A = [A_L, A_U] \in IVFM_{nn}$ if A_L^+ exists then $A_L^+ = A_L^T$ which A_L^T is a generalized inverse of A_L.

Consider A_L^+, A_U^+ exists $\Leftrightarrow (KA_L)^+, (KA_U)^+$ exist [lemma 3.4 in [11]]

$\Leftrightarrow (KA_L)(KA_L)^T(KA_L), (KA_U)(KA_U)^T(KA_U)$

$\Leftrightarrow (KA_L)^+, (KA_U)^+$ exits

$\Leftrightarrow (VKA_L)(VKA_L)^T(VKA_L), (VKA_U)(VKA_U)^T(VKA_U)$

$\Leftrightarrow (VKA_L)^T \in (VKA_L)\{1\}, (VKA_L)^T \in (VKA_U)\{1\}$

$\Leftrightarrow (VKA_L)^+, (VKA_U)^+$ exists

Hence the result

Remark 3.13. For $A = [A_L, A_U] \in IVFM_{nn}, A_L^+, A_U^+$ exits $\Leftrightarrow (KVA_L)^+$, $(KVA_U)^+$ exits.

Corollary 3.14. *The subsequent statements are equivalent for* $A = [A_L, A_U] \in IVFM_{nn}, A = [A_L^+, A_U^+]$ *exits*

(1) $A = [A_L, A_U]$ *is an interval valued* $s - \kappa$ *range symmetric*

(2) $(VKA_L)(VKA_L)^+ = (VKA_L)^+(VKA_L), (VKA_U)(VKA_U)^+ = (VKA_U)^+(VKA_U)$

(3) $A^+ = [A_L^+, A_U^+]$ *is an interval valued* $s - \kappa$ *range symmetric*

(4) $VKA = [VKA_L, VKA_U]$ *is normal.*

Lemma 3.15. *For* $A = [A_L, A_U] \in IVFM_{nn}$ *the following statements are equal*

(1) $A = [A_L, A_U]$ *is an interval valued* $s - \kappa$ *range symmetric*

(2) $A_L A_L^+ KV = KVA_L^+ A_L, A_U A_U^+ KV = KVA_U^+ A_U$

(3) $VKA_L A_L^+ = A_L^+ A_L VK, VKA_U A_U^+ = A_U^+ A_U VK$

Proof. (1) $\Leftrightarrow$ (2)

Since $A^+ = [A_L^+, A_U^+]$ exits By Corollary (3.14)

$A = [A_L, A_U]$ is an interval valued $s - \kappa$ range symmetric

$\Leftrightarrow VKA = [VKA_L, VKA_U]$ is normal

$\Leftrightarrow (VKA_L)(VKA_L)^T = (VKA_L)^T (VKA_L), (VKA_U)(VKA_U)^T$

$= (VKA_U)^T (VKA_U)$

$\Leftrightarrow VKA_L A_L^T KV = A_L^T KVVKA_L, VKA_U A_U^T KV = A_U^T KVVKA_U$

$\Leftrightarrow VKA_L A_L^T KV = A_L^T A_L, VKA_U A_U^T KV = A_U^T A_U$ [By C 2.2.1 and C 2.2.2]

$\Leftrightarrow A_L A_L^T KV = KVA_L^T A_L, A_U A_U^T KV = KVA_U^T A_U$

$\Leftrightarrow A_L A_L^+ KV = KVA_L^+ A_L, A_U A_U^+ KV = KVA_U^+ A_U$

(2) $\Leftrightarrow$ (3)

Since By [C 2.2.1 and C 2.2.2], $K^2 = I$ and $V = I$ As a result, equivalent is maintained by pre- and post-multiplying $A_L A_L^+ KV = KVA_L^+ A_L, A_U A_U^+ KV = KVA_U^+ A_U$ by VK.

Theorem 3.16. *For* $A = [A_L, A_U] \in IVFM_{n \times n}$ *then any two of the conditions below imply the other*

1 $A = [A_L, A_U]$ *is an interval valued* κ- *range symmetric*

2 $A = [A_L, A_U]$ *is an interval valued* $s - \kappa$ *range symmetric*

3 $R(A_L^T) = R((VKA_L)^T), R(A_U^T) = R((VKA_U)^T)$.

Proof. (1) and (2) $\Rightarrow$ (3)

$A = [A_L, A_U]$ is an interval valued $s - \kappa$ range symmetric

$\Rightarrow R(A_L) = R(A_L^T VK), R(A_U) = R(A_U^T VK)$ [By Theorem 3.1]

$\Rightarrow R(KA_L K) = R(KA_L^T K), R(KA_U K) = R(KA_U^T K)$ [By Lemma 2.2]

$\Rightarrow R(A_L^T) = R((VA_L K)^T), R(A_U^T) = R((VA_U K)^T)$

Therefore (1) and (2)

As a result (3) is true

(1) and (3) $\Rightarrow$ (2)

$A = [A_L, A_U]$ is an interval valued κ-range symmetric

$\Rightarrow R(A_L) = R(KA_L^T K), R(A_U) = R(KA_U^T K)$

$\Rightarrow R(KA_L K) = R((A_L)^T), R(KA_U K) = R((A_U)^T)$ [By Lemma 2.5]

$\therefore$ (1) and (3) $\Rightarrow R(KA_L K) = R((VA_L K)^T), R(KA_U K) = R((VA_U K)^T)$

$\Rightarrow R(A_L) = R(A_L^T VK), R(A_U) = R(A_U^T VK)$

$\Rightarrow R(A_L) = R((KVA_L)^T), R(A_U) = R((KVA_U)^T)$

$\Rightarrow A = [A_L, A_U]$ is an interval valued $s-\kappa$ range symmetric [By Theorem 3.10]

As a result (2) is true

(2) and (3) $\Rightarrow$ (1)

$A = [A_L, A_U]$ is an interval valued $s-\kappa$ range symmetric

$\Rightarrow R(A_L) = R(A_L^T VK), R(A_U) = R(A_U^T VK)$

$\Rightarrow R(KA_L K) = R(KA_L^T K), R(KA_U K) = R(KA_U^T K)$ [By C 2.2.5]

$\therefore$ (2) and (3) $\Rightarrow R(KA_L K) = R(A_L^T), R(KA_U K) = R(A_U^T)$

$\Rightarrow R(A_L) = R(KA_L^T K), R(A_U) = R(KA_U^T K)$

$\Rightarrow A = [A_L, A_U]$ is an interval valued κ-range symmetric

As a result (1) is true.

Hence the theorem.

4. Interval valued $s-\kappa$ range symmetric regular fuzzy matrices

This section revealed the existence of several generalized inverses of a matrix in $IVFM$. It is also established what are the equivalent criteria for various g-inverses of an interval valued $s-\kappa$ range symmetric fuzzy matrix to be an interval valued $s-\kappa$ range symmetric. The generalized inverses of an interval valued $s-\kappa$ range symmetric A corresponding to the sets $A\{1, 2\}$, $A\{1, 2, 3\}$ and $A\{1, 2, 4\}$ are characterized.

When A is an interval valued $s-\kappa$ range symmetric matrix under certain criteria, any $X \in A\{1, 2\}$ is proved an interval valued $s-\kappa$ range symmetric matrix in the subsequent manner.

Theorem 4.1. *Let us assume* $A \in IVFM_{n\times n}$, $X \in A\{1, 2\}$ *and* AX, XA *are an interval valued* $s-\kappa$ *range symmetric. Then A is an interval valued* $s-\kappa$ *range symmetric* $\Leftrightarrow$ X *is an interval valued* $s-\kappa$ *range symmetric.*

Proof. $R(KVA_L) = R(KVA_LXA_L) \subseteq R(XA_L)$ Since $A = AXA$

$$= R(XVVA_L) \subseteq R(XVKKVA_L) \subseteq R(KVA_L)$$

Hence,

$$R(KVA_L) = R(XA_L)$$

$$= R(KV(XA_L)^T VK), \text{ [XA is interval valued } s-\kappa \text{ range symmetric]}$$

$$= R(A_L^T X_L^T VK)$$

$$= R(X_L^T VK)$$

$$= R((KVX_L)^T)$$

$$R((KVA_L)^T) = R(A_L^T VK)$$

$$= R(X_L^T A_L^T VK)$$

$$= R((KVA_LX_L)^T)$$

$= R(KVA_L X_L)$ [VA is an interval valued $s-\kappa$ range symmetric]

$$= R(KVX_L)$$

Similarly we can prove that $R(KVX_U) = R((KVA_U)^T)$ KVX is an interval valued range symmetric.

$$\Leftrightarrow R(KVA_L) = R((KVA_L)^T),\ R(KVA_U) = R((KVA_U)^T)$$

$$\Leftrightarrow R(KVX_L) = R((KVX_L)^T),\ R(KVX_U) = R((KVX_U)^T)$$

$\Leftrightarrow KVX = [KVX_L,\ KVX_U]$ is an interval valued range symmetric $\Leftrightarrow X$ is an interval valued $s-\kappa$ range symmetric.

Theorem 4.2. *Let the matrix* $[A_L, A_U] \in IVFM_{n\times n}, X = [X_L, X_U] \in A\{1, 2, 3\}, R(KVA_L) = (R(KVX_L)^T).\ R(KVA_U) = R(KVX_U)^T).$ *Then* $A = [A_L, A_U]$ *is an interval valued* $s-\kappa$ *range symmetric* $\Leftrightarrow X = [X_L,\ X_U]$ *is an interval valued* $s-\kappa$ *range symmetric.*

Proof. Since $X \in A\{1,\ 2,\ 3\}$. We acquire $A_L X_L A_L = A_L,\ X_L A_L X_L = X_L,$ $(A_L X_L)^T = A_L X_L$

$A_U X_U A_U = A_U,\ X_U A_U X_U,\ (A_U X_U)^T = A_U X_U$

Consider $R((KVA_L)^T) = R(X_L^T A_L^T VK)$ [By C 2.2.5]

$$= R(KV(A_L X_L)^T)$$

$= R((A_L X_L)^T)$ [By C 2.2.5]

$= R(A_L X_L)$ $[(AX)^T = AX]$

$= R(X_L)$ By using $X = XAX$

$= R(KVX_L)$ [By C 2.2.5]

Similarly, we can consider $R((KVA_U)^T) = R(X_U^T A_U^T VK)$ [By C 2.2.5]

$$= R(KV(A_U X_U)^T)$$

$$= R((X_U A_U)^T) \text{ [By C 2.2.5]}$$

$$= R(X_U A_U) \; [(AX)^T = AX]$$

$$= R(X_U) \text{ By using } X = XAX$$

$$= R(KVX_U) \text{ [By C 2.2.5]}$$

If KVA is an interval valued range symmetric

$$\Leftrightarrow R(KVA_L) = R((KVA_L)^T),\; R(KVA_U) = R((KVA_U)^T)$$

$$\Leftrightarrow R(KVX_L) = R((KVX_L)^T),\; R(KVX_U) = R((KVX_U)^T)$$

$\Leftrightarrow KVX = [KVX_L, KVX_U]$ is an interval valued range symmetric $\Leftrightarrow X = [X_L, X_U]$ is a $s - \kappa$ is an interval valued range symmetric.

Theorem 4.3. *Let* $[A_L, A_U] \in IVFM_{nn}, X \in A\{1, 2, 4\}, R((KVA_L)^T) = R(KVX_L), R((KVA_U)^T) = R(KVX_U)$. *Then* KVA *is an interval valued* $s - \kappa$ *range symmetric* $\Leftrightarrow X = [X_L, X_U]$ *is an interval valued* $s - \kappa$ *range symmetric.*

Proof. Since $X \in A\{1, 2, 4\}$ We acquire $A_L X_L A_L = A_L, X_L X_L, (X_L A_L)^T = X_L A_L$

$A_U X_U A_U = A_U, X_U A_U X_U, (X_U A_U)^T = X_U A_U$

$R(KVA_L) = R(A_L)$ [By C2.2.5]

$$= R(X_L A_L)$$

$$= R(A_L^T X_L^T)$$

$$= R((X_L)^T)$$

$$= R(KVX_L)^T \text{ [By C 2.2.5]}$$

Similar manner $= R(KVA_U) = R(A_U)$ [By C 2.2.5]

$$= R(X_U A_U)$$

$$= R(X_U^T A_U^T)$$

$$= R(X_U^T)$$

$$= R(KVX_U)^T \text{ [By C 2.2.5]}$$

$KVA = [KVA_L, KVA_U]$ is an interval valued range symmetric

$$\Leftrightarrow R(KVA_L) = R((KVA_L)^T), R(KVA_U) = R(KVA_U)^T)$$

$$\Leftrightarrow R(KVX_L) = R((KVX_L)^T), R(KVX_U) = R(KVX_U)^T)$$

$\Leftrightarrow KVX = [KVX_L, KVX_U]$ is an interval valued range symmetric

$\Leftrightarrow X = [X_L, X_U]$ is an interval valued $s - \kappa$ range symmetric

[Theorem 2.3]

The following theorem reduces various g-inverses of an interval valued s-range symmetric fuzzy matrix to interval valued secondary range symmetric fuzzy matrix to similar requirements for $K = I$.

Corollary 4.4. *Let* $[A_L, A_U] \in IVFM_{nn}, X \in A\{1, 2\}$ *and* $AX = [A_L X_L, A_U X_U], XA = [X_L A_L, X_U A_U]$ *are interval valued s-range symmetric then A is an interval valued s-range symmetric* $\Leftrightarrow X$ *is an interval valued s-range symmetric.*

Corollary 4.5. *Let* $[A_L, A_U] \in IVFM_{nn}, X \in A\{1, 2, 3\}$ *and* $R(KVA_L) = R((VX_L)^T), R(KVA_U) = R((VX_U)^T)$ *then A is an interval valued s-range symmetric* $\Leftrightarrow X$ *is an interval valued s-range symmetric.*

Corollary 4.6. *Let* $[A_L, A_U] \in IVFM_{nn}, X \in A\{1, 2, 4\}$ *and* $R((VA_L)^T) = R(VX_L), R((VA_U)^T) = R(VX_U)$ *then A is an interval valued s-range symmetric* $\Leftrightarrow X$ *is an interval valued s-range symmetric.*

5. Conclusion

The classification of interval valued secondary κ-range symmetric fuzzy matrices has been defined. In addition, we investigated into various cases of Proposition of interval valued $s - \kappa$ range symmetric fuzzy matrices.

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