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A STUDY ON (G, H)-DERIVATIONS OF BH-ALGEBRAS

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Abstract

The notion of BCK-algebras was proposed by Imai and Iseki in 1966. In the same year Iseki introduced the notion of BCI-algebras, which is generalization of a BCK-algebras. Y. B. Jun, E. H. Roh and H. S. Kim defined the notion of BH-algebras. Motivated by some results on derivations on rings and the generalizations of BCK and BCI-algebras. In 2019, P. Ganesan and N. Kandaraj introduced the notion of the various derivations of BH-algebras. In this paper, we study the notion of (G, H)-derivations of BH-algebras and investigate simple, interesting and elegant results.

1. Introduction

Y. Imai and K. Iseki [3, 4] introduced the axiom system of propositional calculi and have been extensively investigated by many researchers. K. Iseki and S. Tanaka [5] introduced the theory of BCK-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Zhang Q, Jun Y. B. and Roh E. H. [14] introduced the notion of BH-algebras. Also Jun Y. B, Kim H. S and Kondo M [7] developed the BH-relations in BH-algebras. They investigated several relations between BH-algebras and BCK-algebras. In 1957, Posner E [13] introduced the notion of derivations in prime rings theory. Also Lee P. H and Lee T. K [11] developed on derivations of prime rings. The notion of derivations in ring theory is quite old and plays an

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important role in algebras. Many research papers have appeared on the derivations of BCI-algebras in different ways. Al-Roqi A. M. [1] introduced the notion of generalized (α, β) derivations in BCI-algebras. S. M. Bawazeer, N. O. Alshehri and Babusail R. S. [2] introduced the notion of generalized derivations of BCC-algebras. Also Kamali Ardikani L. and Davvaz B. [10] developed the properties in generalized derivations of BCI-algebras. Jun Y. B and Xin X. L. [9] introduced the notion of derivations of BCI-algebras. M. A. Javed and M. Aslam [6] introduced the concept of f-derivations on BCI-algebras. G. Muhiuddin and Abdullah Al-roqi M [12] introduced t-derivations of BCI-algebras.

The notion of the derivations is the same as that in ring theory and the usual algebraic theory. Motivated by a lot of work done on derivations of BH-algebras and on derivations of other related abstract algebraic structures such as TM-algebras and d-algebras. In this paper we introduce the notion of (G, H)-derivations of BH-algebras and investigate simple, interesting and elegant results.

2. Basic Facts about BH-Algebras

In this section, we summarize some basic concepts which will be used throughout this paper.

Let U be a set with a binary operation $*$ and a constant 0. Then $(U, *, 0)$ is called BH-algebra, if it satisfies the following axioms [8]

(1) $u * u = 0$

(2) $u * 0 = u$

(3) If $u * v = 0$ and $v * u = 0 \Rightarrow u = v$ for all $u, v \in U$.

Define a binary relation $\leq$ on U by taking $u \leq v$ if and only if $u * v = 0$. In this case $(U, \leq)$ is a partially ordered set [8].

Let $(U, *, 0)$ be a BH-algebra and $u \in U$. Define $u * U = \{u * v \mid v \in U\}$, then U is said to be edge BH-algebra if for any $u \in U$, $u * U = \{u, 0\}$ [8].

Let S be a nonempty subset of a BH-algebra U. Then S is called sub algebra of U, if $u * v \in S$ for all $u, v \in S$ [8].

A subset I of a BH-algebra U is called an ideal of U if it satisfies [8]

1. $0 \in I$

2. $u * v \in I$ and $v \in I$ implies that $u \in I$ for all $u, v \in U$.

In BH-algebra X for all $x, y, z \in U$, the following property holds [7]

1. $((u * v) * (u * w)) * (w * v) = 0$

2. $(u * v) * u = 0$

3. $(u * (u * v)) = v$

For a BH-algebra U, we denote $u \wedge v$ for $v * (v * u)$, $\forall x, y \in U$.

3. Main Results of (G, H)-Derivations on BH-algebras

In this section we introduce the notion of right (G, H)-derivations, left (G, H)-derivations and give some examples and propositions to explain the theory of (G, H)-derivations in BH-algebras.

Definition 3.1. Let U be a BH-algebra. A right (G, H)-derivation of U is a self-map θ of U satisfying the identity $\theta(u * v) = (G(u) * \theta(v)) \wedge (H(v) * \theta(u))$ for all $u, v \in U$. Where G and H are endomorphism of U.

If θ satisfies the condition $\theta(u * v) = (\theta(u) * G(v)) \wedge (\theta(v) * H(u))$ for all $u, v \in U$, then θ is called a left (G, H)-derivation of U.

Moreover, if θ is both right (G, H) and left (G, H)-derivation, then θ is called (G, H)-derivation of BH-algebra U.

Example 3.2. Let $U = \{0, a, e, i, p, q\}$ be a BH-algebra with the following Cayley table.

*	0	a	e	i	p	q
0	0	0	e	e	e	e
a	a	0	e	e	e	e
e	e	e	0	0	0	0
i	i	e	a	0	0	0
p	p	e	a	a	0	a
q	q	e	a	a	a	0

Define a map $\theta : U \to U$ and $G, H : U \to U$ such that $H(u) = 0$ for all $u \in U$.

$$\theta(u) = \begin{cases} e & \text{if } u = 0, a \\ 0 & \text{if otherwise} \end{cases}$$

$$G(u) = \begin{cases} 0 & \text{if } u = 0, a \\ e & \text{if otherwise} \end{cases}$$

It is easily to verify that θ is both right (G, H)-derivation and left (G, H)-derivation.

Therefore θ is an (G, H)-derivation.

Example 3.3. Consider the above example 3.2, define $\theta(u) = \begin{cases} e & \text{if } u = 0, a \\ 0 & \text{if otherwise} \end{cases}$ such that $G(u) = 0$ and $H(u) = I$, then it is verify that θ is not a right (G, H)-derivation.

Since $\theta(e * i) = e$

But $(G(e) * \theta(i)) \wedge (H(i) * \theta(e)) = 0 \wedge i = 0$

Also θ is not a left (G, H)-derivation

Since $\theta(e * i) = e.$ But $(\theta(e) * G(i)) \wedge (\theta(i) * H(e)) = 0 \wedge e = 0.$

Example 3.4. Let $U = \{0, a, e\}$ be a BH-algebra with the following Cayley table.

*	0	a	e
0	0	0	0
a	a	0	a
e	e	e	0

Then $(U, *, 0)$ is a commutative BCK-algebra

Define a map $\theta, G, H : U \to U$ such that $\theta(u) = 0$ for all $u \in U, G = I$ and

$$H(u) = \begin{cases} 0 & \text{if } u = 0 \\ e & \text{if } u = a \\ a & \text{if } u = e \end{cases}$$

Where G and H are endomorphism. It is verified that θ is a left (G, H)-derivation. But θ is not a right (G, H)-derivation.

Since $\theta(e * a) = 0$

Here $(G(e) * \theta(a)) \wedge (H(a) * \theta(e)) = (e * 0) \wedge (e * 0) = e \wedge e = e * 0 = e \neq 0.$

Definition 3.5. Let θ be a (G, H)-derivation of a BH-algebra U, then θ is said to be regular if $\theta(0) = 0.$ If $\theta(0) \neq 0,$ then θ is called irregular.

Theorem 3.6. *Every right (G, H)-derivation of a BH-algebra is regular.*

Proof. Let U be a BH-algebra and θ is a right (G, H)-derivation of U.

Now $\theta(0) = \theta(0 * 0)$

$$= (G(0) * \theta(0)) \wedge (H(0) * \theta(0)) = 0$$

Therefore θ is regular.

Theorem 3.7. *Let θ be a right (G, H)-derivation of a BH-algebra U. Then $\theta(u) \in H(U)$ for all $u \in H(U)$.*

Proof. Let $u \in H(U)$

Now $\theta(u) = \theta(0 * u)$

$$= (G(0) * \theta(u)) \wedge (H(u) * \theta(0))$$

$$\leq G(0) * \theta(u)$$

$$= 0 * \theta(u) \in C_p(U).$$

Thus $\theta(u) = 0 * \theta(u)$

Hence $\theta(u) \in H(U)$.

Theorem 3.8. *Let* θ *be a right* (G, H)*-derivation of a BH-algebra U. Then the following results are hold*:

(a) $\theta(u) \in C_p(U)$, *for all* $u \in U$

(b) $G(v) * (G(v) * \theta(u)) = \theta(u)$ *for all* $u, v \in U$

(c) $\theta(u) * G(v) = 0 * (G(v) * \theta(u))$ *for all* $u, v \in U$

(d) $\theta(U) * G(v) \in C_p(U)$, *for all* $u, v \in U$.

Proof. (a) Let $u \in U$

Now $\theta(u) = \theta(u * 0)$

$$= (G(u) * \theta(0)) \wedge (H(0) * \theta(u))$$

$$= (0 * \theta(u)) * ((0 * \theta(u)) * (G(u) * \theta(u)))$$

$$= (0 * \theta(u)) * ((0 * (G(u) * \theta(0))) * \theta(u))$$

$$\leq 0 * (0 * (G(u) * \theta(0)))$$

Hence $\theta(u) \in C_p(U)$

Similarly we can prove (b), (c) and (d).

Theorem 3.9. *Let U be BH-algebra. Then*

(a) *If* θ *is a left-right-*(G, H)*-derivation of U, then* θ *is a left* (G, H)*-derivation of U.*

(b) *If* θ *is a right-left-*(G, H)*-derivation of U, then* θ *is a right* (G, H)*-derivation of U.*

Proof. (a) Let $u, v \in U$ and θ be a left-right-(G, H)-derivation.

Then $\theta(u * v) = (\theta(u) * G(v)) \wedge (H(u) * \theta(v))$

$\leq \theta(u) * G(v)$

Now $\theta(u * v) = (\theta(u) * G(v))$

$= (\theta(u) * G(v)) \wedge (\theta(v) * H(u))$

$\Rightarrow \theta$ is a left (G, H)-derivation of U.

(b) Let $u, v \in U$ and θ is a right-left (G, H)-derivation.

$\theta(u * v) = (G(u) * \theta(v)) \wedge (\theta(u) * H(v))$

$\leq (G(u) * \theta(v))$

Now $\theta(u * v) = (G(u) * \theta(v))$

$= (G(u) * \theta(v)) \wedge (H(v) * \theta(u))$

$\Rightarrow \theta$ is a right (G, H)-derivation of U.

Theorem 3.10. *Let* θ *be a self-map and let* $X(u_0)$ *be any branch of a BH-algebra U. If for any* $u \in X(u_0), \theta(u) = G(u_0)$, *then* θ *is a regular left* (G, H)-*derivation.*

Proof. Let $u, v \in U, u \in X(u_0)$ and $v \in X(u_0)$

It is not necessary that $u_0 \neq v_0$. then $u * v \in X(u_0 * v_0)$

Using hypothesis, we get $\theta(u * v) = G(u_0 * v_0)$ (1)

Also we have by using hypothesis

$(\theta(u) * G(v)) \wedge (\theta(v) * H(u))$

$= (G(u_0) * G(v)) \wedge (G(v_0) * H(u))$

$\leq G(u_0) * G(v)$

$= G(u_0 * v) = G(u_0 * v_0)$ (2)

Here $u_0, v_0 \in C_p(U) \Rightarrow G(u_0 * v_0) \in C_p(U)$

Using (1) and (2), $\theta(u * v) = (\theta(u) * G(v)) \wedge (\theta(v) * H(u))$

Therefore θ is a left (G, H)-derivation.

Hence θ is regular.

Theorem 3.11. *A self-map θ of a BH-algebra U defined by* $\theta(x) = 0 * (0 * G(u)) = Gu$ *for all* $u \in U$ *is a left* (G, H)*-derivation.*

Proof. Let $u, v \in U$, we have $\theta(u * v) = G_{u*v} = G_u * G_v$ and

$$(\theta(u) * G(v)) \wedge (\theta(v) * H(u))$$

$$= (G_u * G(v)) \wedge (G_v * H(u))$$

$$\leq G_u * G(v)$$

Now $(\theta(u) * G(v)) \wedge (\theta(v) * H(u)) * (G_u * G_v)$

$$\leq (G_u * G(v)) \wedge (G_u * G_v)$$

$$\leq G_v * G(v) = 0.$$

Hence $(\theta(u) * G(v)) \wedge (\theta(v) * H(u)) * (G_u * G_v)$

Hence $(\theta(u) * G(v)) \wedge (\theta(v) * H(u)) * (G_u * G_v) = \theta(u * v)$

Therefore θ is a left (G, H)-derivation.

Theorem 3.12. *Let θ be a left* (G, H)*-derivation of a BH-algebra U with commutative. Then* $u \leq v$ *implies that* $\theta(u)$ *and* $\theta(v)$ *belong to the same branch of U.*

Proof. Let $u, v \in U$ and $u \leq v$.

Since U is commutative, $v * (v * u) = u$

Also θ is a left (G, H)-derivation.

Therefore $\theta(u) = \theta(v * (v * u))$

$= (\theta(v) * G(v * u)) \wedge (\theta(v * u) * H(v))$ (1)

$\leq \theta(v) * G(v * u) = \theta(v) * (G(v) * G(u))$

Since $u \leq v$, $0 = G(0) = G(u * v) = G(u) * G(v)$

$G(u)$ and $G(v)$ are contained in the same branch.

Hence $G(v) * G(u) \in U_+$

Using (1), we have $\theta(u) * \theta(v) \leq (\theta(v) * (\theta(v) * \theta(u))) * \theta(v)$

$$= (\theta(v) * \theta(v)) * (G(v) * G(u))$$

$$= 0 * (G(v) * G(u)) = 0$$

Hence $\theta(u) * \theta(v) = 0$.

So $\theta(u) \leq \theta(v)$.

Therefore $\theta(u)$ and $\theta(v)$ are contained in the same branch of U.

Theorem 3.13. *Let* θ *be a* (G, H)*-derivation of a BH-algebra U. Also let* $X(u_0)$ *and* $X(v_0)$ *be two arbitrary branches of U with* $G(u_0) \in X(u_0)$ *and for all* $v \in X(u_0)$, $G(v) \in X(v_0)$. *Then* $\theta(u) = v_0$ *implies* $\theta(v) \in X(u_0)$.

Proof. Let $u \in X(u_0)$, $v \in X(v_0)$ and $\theta(u) = v_0$

Since θ is a left (G, H)-derivation, we have

$\theta(u * v) = (\theta(u) * G(v)) \wedge (\theta(v) * H(u))$ (1)

$\leq \theta(u) * G(v) = v_0 * G(v)$

On the other side, $v_0 * G(v) = 0$

Since $G(v) \in X(v_0)$ Hence $\theta(u * v) = 0$.

Since θ is a right (G, H)-derivation,

$0 = \theta(u * v) = (G(u) * \theta(v)) \wedge (H(v) * \theta(u))$

$\leq G(u) * \theta(v)$

Hence $0 * G(u) \leq (G(u) * \theta(v)) * G(u) = 0 * \theta(v)$

So, $0 * (0 * \theta(v)) \leq 0 * (0 * G(u)) = G(u_0)$

Since $u_0 \in C_p(U)$, $G(u_0) \in C_p(U)$,

Therefore $G(u_0) = 0 * (0 * \theta(v)) \leq \theta(v)$

Hence $G(u_0)$ and $\theta(v)$ belong to the same branch of U.

Then $\theta(v) \in X(u_0)$, since $G(u_0) \in X(u_0)$.

4. Conclusion

An algebraic structure that arises from the study of algebraic formulations of propositional logic. Taking different theorems or statements of propositional logic, different algebraic structures could be obtained. The BH-algebras is one of such algebras. The derivations concept is an important and very interesting area of research in the theory of algebraic structures in mathematics. The deep theory has been developed for derivations in BCI-algebras, BCC-algebras, d-algebras and BP-algebras. It plays an important role in algebra, algebraic geometry and linear differential equations. We have considered the concept of (G, H)-derivations on BH-algebras. Finally, we investigated the notion of the regular (G, H)-derivations on BH-algebras. In future any researcher can study the notion of (G, H)-derivations in different algebraic structures which may have a lot of applications in various fields. This work is a foundation for the further study of the researcher on derivations of algebras.

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