Advances and Applications in Mathematical Sciences
Volume 21, Issue 10, August 2022, Pages 5779-5788

CAPUTO FRACTIONAL DERIVATIVE OF SINE AND COSINE HYPERBOLIC FUNCTIONS

RAJBIR SINGH[1], DIMPLE SINGH[2] and REKHA[3]

[1]School of Engineering and Sciences
GD Goenka University Gurgaon
Haryana, India
E-mail: rajbir.singh@gdgu.org

[2,3]Amity School of Applied Sciences
Amity University Haryana, Gurgaon, India

Abstract

In this paper we intend to aim at calculating fractional derivative of sine and cosine hyperbolic functions with Caputo approach. Hyperbolic functions have many applications in various sciences and in the field of engineering they are used to express the form of the loop established between two towers by high power lines. They may also be used to find distance in non-Euclidean geometry.

1. Introduction

Fractional calculus is generalized idea of derivative and integral of integer order [2]. Fractional order derivates discussed preliminarily in 1695 in a correspondence of Leibniz to L'Hospital explaining about the derivative of arbitrary order. After that many mathematicians laid base for fractional derivatives. Abel, Liouville, Riemann, Euler, and Caputo are supposed to be pioneer in the theory of fractional calculus [3, 4]. Consequently, the fractional calculus finds immense applications various sciences mathematics, and engineering. See [5] for deterministic fractional order models which occur in the field bioengineering and branch of nanotechnology.

2020 Mathematics Subject Classification: 26A33.

Keywords: Fractional Calculus, Fractional Derivatives, Caputo Fractional Derivative, Sine and Cosine Hyperbolic Functions.

[2]Corresponding author; E-mail: dsingh@ggn.amity.edu

Received December 15, 2021; Accepted January 20, 2022

Further, the derivatives of fractional order have significantly found applications in defining the physical and chemical behavior of plenty of real materials such as polymer, rocks, and other different form of matter [6]. The fractional-order models were found more robust tool in divulging the properties lying between the given two integers there are several definitions of fractional derivative and fractional integrals, see. [7, 8]. Some of the definitions of fractional derivatives are enlisted below.

1.1. Some Special Functions and definition of Fractional derivatives.

In this section, we discuss some elementary definitions of some special functions and fractional derivates [8, 5].

1.1.1 Gamma Function.

Gamma function is one of the basic functions which is frequently used in fractional calculus. It is basically generalization of factorial!m.it allows m to take real and even complex values. The formula for gamma function is given below:

$$\Gamma(m) = \int_0^\infty e^{-y} \cdot y^{m-1} dy$$

The gamma function can also be defined by the limit representation as:

$$\Gamma(m) = \lim_{n\to\infty} \frac{m!\, m^y}{y(y+1)(y+2)\ldots(y+m)},\ re(y) > 0.$$

some basic properties of gamma function are defined as:

$$\Gamma(m+1) = m\Gamma(m) = m \cdot (m-1)! = m!$$

Gamma function has simple pole at the points $y = m,\ m = 0, 1, 2, \ldots,$

1.1.2 Beta Function.

Beta function is expressed in terms of a definite integral and given by

$$\beta(r, s) = \int_0^1 m^{r-1}(1-m)^{s-1} dm,\ r, s \in R^+$$

Relation between Beta function and Gamma function is defined as:

$$\beta(r, s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}, \ r, s \in R^{+}$$

1.1.3 Mitta Leffler Function.

Mittag-Leffler function has found a significant place in the field of fractional order differential equations. one-parameter Mittag-Leffler Function is generalization of exponential function, which is given as:

$$E_{\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^{n}}{\Gamma(\alpha x + 1)}$$

Mittag-Leffler function (MLF) for two parameters is defined as:

$$E_{\alpha, \beta}(x) = \sum_{n=0}^{\infty} \frac{x^{n}}{\Gamma(\alpha x + \beta)}, \ \alpha > 0, \beta > 0$$

One of the most important form of the MLF which is used widely in the theory of fractional order systems is given as:

$$E_{x}(v, a) = t^{v} \sum_{n=0}^{\infty} \frac{(ax)^{n}}{\Gamma(v + n + 1)} = t^{v} E_{1, v+1}(ax)$$

Where v is representing a fraction and a be any constant.

Let $f(t)$ be a given real valued function. Then for a given rational number p, the Grunwald-Letnikov approach of the p-th order fractional derivative of $f(t)$ is defined as:

1.1.4 Grunwald-Letnikov Fractional Derivative.

For a real valued function $g(s)$. Then for a given rational number q, the Grunwald-Letnikov approach of the q-th order fractional derivative of $g(s)$ is defined as:

$${}_{a}D_{s}^{q} g(s) = \lim_{\substack{k \to 0 \\ mk = s - a}} k^{-q} \sum_{l=0}^{m} (-1)^{l} \binom{q}{l} g(s - lk),$$

where k is size of the step and a be any fixed number belongs to the set of real number.

1.1.5 Riemann-Liouville approach of Fractional Derivative.

The Riemann-Liouville order fractional derivative of the q-th with respect to the variable 't' for a real valued function $f(t)$ is described by formula given below:

$$_{a}D_t^q f(t) = \left(\frac{d}{dt}\right)^{r+1} \int_a^t (t-\tau)^{r-q} f(\tau) d\tau,\ r \leq q < r+1$$

1.1.6 Euler's Fractional Derivative.

The Euler's fractional derivative of $f(t) = t^{\beta}$ is be expressed in the following formula:

$$\frac{d^{\propto}}{dt^{\propto}}[t^{\beta}] = D_t^{\alpha}[t^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha},\ \alpha \in \mathbb{R}$$

Where $\Gamma(r)$ is standard Gamma function for a given $r \in \mathbb{R}$ and α is the order of the derivative.

In this sub-section, we discuss on Euler's definition of fractional order derivatives. It is found that that arbitrary order derivatives satisfy nearly all the properties that are carried by the derivatives with integral orders. Some of the general properties [5] of the fractional-order derivatives are given as under below. If f, g be the pair of real-valued functions, then:

- $D_s^{\propto}[f(s)g(s)] = \sum_{j=0}^{\infty} \binom{\propto}{j} D_s^{\propto-j}[f(s)] D_s^{j}[g(s)],$ where $\binom{\propto}{j} = \frac{\Gamma(\propto+1)}{\Gamma(j+1)\Gamma(\propto+1-j)}$.
- $D_s^{\propto}[f(s)H] = \sum_{j=0}^{\infty} \binom{\propto}{j} D_s^{\propto-j}[f(s)] D_s^{j}[H] = D_s^{\propto}[f(s)H]$ where H is an arbitrary constant
- $D_s^{\propto}[h(s)+g(s)] = \sum_{j=0}^{\infty} \binom{\propto}{j} D_s^{\propto-j}[S^0] D_s^{j}[h(s)+g(s)]$ $= D_s^{\propto}[h(s)] + D_s^{\propto} g[h(s)].$

- $D_s^{\propto}[h(bs)] = b^{\propto} D_y^{\propto}[h(y)]$ under the scaling $y = bs$.

- $D_s^{\propto}[s^{-r}] = (-1)^{\propto} \frac{\Gamma(r+\propto)}{\Gamma(r)} s^{-(r+\propto)}$ for a given $r \in \mathbb{R}$.

- $D_s^{\mu+\nu}[f(s)] = D_s^{\mu}[D_s^{\nu}(f(s))] = D_s^{\nu}[D_s^{\mu}(f(s))]$ under the composition of D_s^{ν} and D_s^{μ} on $f(s)$.

- $D_s^{-1}[s^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+1)} s^{\beta+1} = \frac{s^{\beta+1}}{\beta+1}$, where $\beta \in \mathbb{R}$ corresponding to a negative order derivative.

2.1.7 Caputo Fractional-Order Derivative Operator.

Another approach for evaluating fractional derivative is given by Caputo in one of his articles which was introduced in 1967. Contrary to the R-L approach of fractional derivative, while solving fractional differential equation using Caputo approach, fractional order initial condition are not necessary required. Caputo's definition is given as follows:

Suppose that $\alpha > 0$, $t > a; \alpha, a, t \in R$, and then the fractional derivative operator is

$$cD_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^m(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m, m \in N \\ f^m(t) & \alpha = m \end{cases},$$

where $cD^{\alpha} f(t)$ is called the Caputo fractional derivative of order α.

2. Caputo Fractional Derivative of Hyperbolic Functions

The hyperbolic function appears in several problems of mathematics and mathematical physics. For example, series of hyperbolic sine occurs in the gravitational potential of a cylinder, and they are used in the calculation of the Roche limit [1]. The hyperbolic cosine function describes the shape of suspended cable. Hyperbolic functions also occur in hyperbolic spaces to study the theory of triangles. The process of decay of physical entity such as light, velocity, and electricity is expressed in terms of hyperbolic functions. Hyperbolic functions are also used to study the ocean waves.

In this part of the paper, we have calculated Caputo Fractional Derivatives of Sine and cosine hyperbolic functions.

2.1 Caputo Fractional Derivative of sine hyperbolic function.

We know that Caputo fractional derivative is given by

$$D_*^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}}{(t-\tau)^{\alpha+1-m}} d\tau,\ m-1 < \alpha < m,\ m \in N$$

where $\alpha > 0 > a$, a, $\alpha \in R$.

We define here Sine hyperbolic function in terms of exponential function as Caputo Fractional derivative of Sinhx is given by: $\text{Sinhx} = \frac{e^x - e^{-x}}{2}$

$$D_*^\alpha (Sinhx) = D_*^\alpha \left(\frac{e^x - e^{-x}}{2} \right)$$

$$= D_*^\alpha \left(\frac{e^x}{2} \right) - D_*^\alpha \left(\frac{e^{-x}}{2} \right)$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{d^m}{d\tau^m} \left(\sum_{j=0}^\infty \frac{\tau^j}{j!}\right) \frac{1}{(t-\tau)^{\alpha+m+1}} d\tau$$

$$- \frac{1}{2\Gamma(m-\alpha)} \int_0^t \frac{d^m}{d\tau^m} \left(\sum_{j=0}^\infty \frac{(-1^j)\tau^j}{j!}\right) \frac{1}{(t-\tau)^{\alpha+m+1}} d\tau$$

$$= \frac{1}{2\Gamma(m-\alpha)} \sum_{j=0}^\infty \frac{1}{j!} \int_0^t \frac{d^m}{d\tau^m} (\tau^j)(t-\tau)^{m-\alpha-1} d\tau$$

$$- \frac{1}{2\Gamma(m-\alpha)} \int_0^t \frac{d^m}{d\tau^m} \left(\sum_{j=0}^\infty \frac{-1^j}{j!}\right) \frac{1}{(t-\tau)^{\alpha+m+1}}$$

$$= \frac{1}{2\Gamma(m-\alpha)} \sum_{j=0}^\infty \frac{1}{j!} (j)(j-1)(j-2)\ldots(j-m) \int_0^t \tau^{j-m} (t-\tau)^{m-\alpha-1} d\tau$$

$$- \frac{1}{2\Gamma(m-\alpha)} \sum_{j=0}^\infty \frac{-1^j}{j!} (j)(j-1)(j-2)\ldots(j-m) \int_0^t \tau^{j-m} (t-\tau)^{m-\alpha-1} d\tau$$

$$= \frac{1}{2\Gamma(m-\alpha)} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{(j)(j-1)(j-2)\ldots(j-m)(j-m-1)\ldots 3.2.1}{(j-m-1)\ldots 3.2.1}$$

$$\int_0^t t^{m-\alpha-1} u^{m-\alpha-1} (1-u)^{m-\alpha-1} du -$$

$$\frac{1}{\Gamma(m-\alpha)} \sum_{j=0}^{\infty} \frac{-1^j}{j!} \frac{(j)(j-1)(j-2)\ldots(j-m)(j-m-1)\ldots 3.2.1}{(j-m-1)\ldots 3.2.1}$$

$$\int_0^t t^{m-\alpha-1} u^{m-\alpha-1} (1-u)^{m-\alpha-1} du$$

$$= \frac{1}{2\Gamma(m-\alpha)} \sum_{j=0}^{\infty} \frac{-1^j}{j!} \frac{\Gamma(j+1)}{\Gamma(j-m)} t^{j-\alpha} \int_0^1 u^{j-m} (1-u)^{m-\alpha-1} du - \frac{1}{2\Gamma(m-\alpha)}$$

$$\sum_{j=0}^{\infty} \frac{-1^j}{j!} \frac{\Gamma(j+1)}{\Gamma(j-m)} t^{j-\alpha} \int_0^1 u^{j-m} (1-u)^{m-\alpha-1} du$$

$$= \frac{1}{2\Gamma(m-\alpha)} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\Gamma(j+1)}{\Gamma(j-m)} t^{j-\alpha} \beta(j-m-1,\, m-\alpha) - \frac{1}{2\Gamma(m-\alpha)}$$

$$\sum_{j=0}^{\infty} \frac{-1^j}{j!} \frac{\Gamma(j+1)}{\Gamma(j-m)} t^{j-\alpha} \beta(j-m-1,\, m-\alpha)$$

$$= \frac{1}{2\Gamma(m-\alpha)} \sum_{j=0}^{\infty} [1-(-1^j)] \frac{1}{j!} \frac{\Gamma(j+1)}{\Gamma(j-m)} t^{j-\alpha} \beta(j-m-1,\, m-\alpha)$$

2.2 The Caputo fractional derivative of cosine hyperbolic function.

Here we have, $Coshx = \frac{e^x + e^{-x}}{2}$

Taking Fracional derivaive of Coshx we get, $D_*^{\alpha}(Coshx) = D_*^{\alpha}\left(\frac{e^x + e^{-x}}{2}\right)$

$$= D_*^{\alpha}\left(\frac{e^x}{2}\right) + D_*^{\alpha}\left(\frac{e^{-x}}{2}\right)$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{d^m}{d\tau^m} \left(\sum_{j=0}^{\infty} \frac{\tau^j}{j!}\right) \frac{1}{(t-\tau)^{\alpha+m+1}} d\tau$$

$$+\frac{1}{2\Gamma(m-\alpha)}\int_0^t \frac{d^m}{d\tau^m}\left(\sum_{j=0}^{\infty}\frac{(-1^j)\tau^j}{j!}\right)\frac{1}{(t-\tau)^{\alpha+m+1}}\,d\tau$$

$$=\frac{1}{2\Gamma(m-\alpha)}\sum_{j=0}^{\infty}\frac{1}{j!}\int_0^t \frac{d^m}{d\tau^m}(\tau^j)(t-\tau)^{m-\alpha-1}d\tau$$

$$+\frac{1}{2\Gamma(m-\alpha)}\int_0^t \frac{d^m}{d\tau^m}\left(\sum_{j=0}^{\infty}\frac{-1^j}{j!}\right)\frac{1}{(t-\tau)^{\alpha+m+1}}\,d\tau$$

$$=\frac{1}{2\Gamma(m-\alpha)}\sum_{j=0}^{\infty}\frac{1}{j!}(j)(j-1)(j-2)\ldots(j-m)\int_0^t \tau^{j-m}(t-\tau)^{m-\alpha-1}d\tau$$

$$+\frac{1}{2\Gamma(m-\alpha)}\sum_{j=0}^{\infty}\frac{-1^j}{j!}(j)(j-1)(j-2)\ldots(j-m)\int_0^t \tau^{j-m}(t-\tau)^{m-\alpha-1}d\tau$$

$$=\frac{1}{2\Gamma(m-\alpha)}\sum_{j=0}^{\infty}\frac{1}{j!}\frac{(j)(j-1)(j-2)\ldots(j-m)(j-m-1)\ldots 3.2.1}{(j-m-1)\ldots 3.2.1}$$

$$\int_0^t t^{m-\alpha-1}u^{m-\alpha-1}(1-u)^{m-\alpha-1}du+$$

$$\frac{1}{\Gamma(m-\alpha)}\sum_{j=0}^{\infty}\frac{-1^j}{j!}\frac{(j)(j-1)(j-2)\ldots(j-m)(j-m-1)\ldots 3.2.1}{(j-m-1)\ldots 3.2.1}$$

$$\int_0^t t^{m-\alpha-1}u^{m-\alpha-1}(1-u)^{m-\alpha-1}du$$

$$=\frac{1}{2\Gamma(m-\alpha)}\sum_{j=0}^{\infty}\frac{-1^j}{j!}\frac{\Gamma(j+1)}{\Gamma(j-m)}t^{j-\alpha}\int_0^1 u^{j-m}(1-u)^{m-\alpha-1}du+\frac{1}{2\Gamma(m-\alpha)}$$

$$\sum_{j=0}^{\infty}\frac{-1^j}{j!}\frac{\Gamma(j+1)}{\Gamma(j-m)}t^{j-\alpha}\int_0^1 u^{j-m}(1-u)^{m-\alpha-1}du$$

$$=\frac{1}{2\Gamma(m-\alpha)}\sum_{j=0}^{\infty}\frac{1}{j!}\frac{\Gamma(j+1)}{\Gamma(j-m)}t^{j-\alpha}\beta(j-m-1,\, m-\alpha)+\frac{1}{2\Gamma(m-\alpha)}$$

$$\sum_{j=0}^{\infty}\frac{-1^j}{j!}\frac{\Gamma(j+1)}{\Gamma(j-m)}t^{j-\alpha}\beta(j-m-1,\, m-\alpha)$$

$$= \frac{1}{2\Gamma(m-\alpha)} \sum_{j=0}^{\infty} [1 + (-1^{j})] \frac{1}{j!} \frac{\Gamma(j+1)}{\Gamma(j-m)} t^{j-\alpha} \beta(j-m-1,\ m-\alpha).$$

3. Conclusions

In this paper, we have calculated fractional derivative of Cosine and Sine hyperbolic functions with Caputo approach. As a future scope, one may also calculate the fractional derivative of Cosine and Sine hyperbolic functions with other approaches such as extended Caputo fractional derivative, RL approach of fractional derivatives and extended RL fractional derivative etc. We leave to readers to calculate Caputo fractional derivative of Tanh, Sech and cosech hyperbolic functions.

5 References

[1] hps://www.briannica.com/science/hyperbolic-funcions.

[2] I. Podlubny, Fractional differential equations: An introduction to fractional derivatives, Fractional Differential Equations, to Methods of Their Solution, vol. 198 of Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA, 1999.

[3] J. T. Machado, V. Kiryakova and F. Mainardi, Recent history of fractional calculus, Communications in Nonlinear Science and Numerical Simulation 16(3) (2011), 1140-1153.

[4] K. S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, John Wiley and Sons, New York, NY, USA, (1993).

[5] K. B. Oldham and J. Spanier, The fractional calculus: theory and application of differentiation and integration to arbitrary order, Academic Press, New York, NY, USA, (1974).

[6] Prateek Agrawal, Vishu Madaan, Aditya Roy, Ranjna Kumari and Harshal Deore, FOCOMO: Forecasting and monitoring the worldwide spread of COVID-19 using machine learning methods, Journal of Interdisciplinary Mathematics 1-25. DOI:10.1080/09720502.2021.1885812.

[7] Vishu Madaan, Aditya Roy, Charu Gupta, Prateek Agrawal, Anand Sharma, Christian Bologa and Radu Prodan, XCOVNet: Chest X-ray image classification for COVID-19 Early Detection Using Convolutional Neural Networks, New Generation Computing 1-15. DOI: 10.1007/s00354-021 00121-7.

[8] Prateek Agrawal, Anatoliy Zabrovskiy, Adithyan Ilagovan, Christian Timmerer and Radu Prodan, FastTTPS: Fast Approach for Video Transcoding Time Prediction and Scheduling for HTTP Adaptive Streaming Videos, Cluster Computing (CLUS) 1-17, https://doi.org/10.1007/s10586-020-03207-x.

[9] Prateek Agrawal, Deepak Chaudhary, Vishu Madaan, Anatoliy Zabrovskiy, Radu Prodan, Dragi Kimovski and Christian Timmerer, Automated Bank Cheque Verification Using Image Processing and Deep Learning Methods, Multimedia tools and applications (MTAP) 80(1) 5319-5350. https://doi.org/10.1007/s11042-020-09818-1

[10] Charu Gupta, Prateek Agrawal, Rohan Ahuja, Kunal Vats, Chirag Pahuja and Tanuj Ahuja, Pragmatic Analysis of Classification Techniques based on Hyperparameter Tuning for Sentiment Analysis, International Semantic Intelligence Conference (ISIC'21), Delhi, (2021), 453-459.

[11] C. Gupta, D. Gaur, P. Agrawal and D. Virmani, HuDA_COVID Human disposition analysis during COVID-19 using machine learning, International Journal of E-Health and Medical Communications (IJEHMC) 13(2) (2021), 1-15.

[12] B. Sharma, A. Hashmi, C. Gupta, O. I. Khalaf, G. M. Abdulsahib and M. M. Itani, Hybrid Sparrow Clustered (HSC) Algorithm for Top-N Recommendation System, Symmetry 14(4) (2022), 793.

[13] A. A. Abdulhussein, O. A. Hassen, C. Gupta, D. Virmani, A. Nair and P. Rani, Health monitoring catalogue based on human activity classification using machine learning, International Journal of Electrical and Computer Engineering (2088-8708), 12(4) (2022).

[14] H. Pandey, R. Goyal, D. Virmani and C. Gupta, Ensem_SLDR: Classification of Cybercrime using Ensemble Learning Technique, International Journal of Computer Network and Information Security 14(1) (2022).

[15] M. Rivero, J. J. Rujillo and M. P. Velasco, On Deterministic Fractional Models, Edied by Dumiru Baleanu, Ziya B. Guvenc, J. A. Enreiro Machado New Trends in Nano Technology and fractional Calculus Applications, Springer Netherlands, Online ISBN: 978-90-481-3293-5, (2010), 123-150.

[16] R. Hilfer (Editors), Applications of Fractional Calculus in Physics, World Scientific Publishing Co. Pie. Ld., ISBN: 9810234570 (2000).

[17] R. Figueiredo Camargo, A. O. Chiacchio and E. Capelas de Oliveira, Differentiation to fractional orders and the fractional telegraph equation, Journal of Mathematical Physics 49(3) (2008), Article ID 033505.