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ON CAUCHY PROBLEMS FOR INITIAL VALUE PROBLEMS

T. JEYARAJ

PG and Research, Department of Mathematics
T. B. M. L. College (Affiliated to Annamalai
University, Chidambaram)
Porayar-609 307, Mayiladuthurai District
Tamil Nadu, South India
E-mail: jrajkadali@gmail.com

Abstract

In this paper, we deal with and prove some uniqueness theorems on the Cauchy problem for the first-order initial value problem.

1. Introduction

Chang and Zadeh [3] introduced the concept of fuzzy differentiable equations in 1972, Dubois and Prade [4], who apply the extension principle of Chang and Zadeh [3] concept in their work. The derivatives of fuzzy differential equations with the initial condition are presented by S. Seikkala [12] and also S. Seikkala [12] used the Hukuhara derivative and the fuzzy integral in the study. Most of the researchers have studied fuzzy differential and integral equations for getting unique solutions to fuzzy initial value problems. This paper deals with and proves some uniqueness theorems on the Cauchy problem for the first-order fuzzy initial value problem.

2. Preliminaries

A nonempty set $D \in R^n$ is said to be a convex set if and only if

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$$(1-\gamma)x+\gamma y \in D \text{ for every } x, y \in D \text{ and } 0 \le \gamma \le 1.$$

Let $CoV(R^n)$ be the family of all nonempty compact convex subsets of R^n, and let $C = [a, b] \subset R$ be a compact interval, $T = [t_0, t_0 + \in]$, $\in > 0$, P, $S \subset C$.

Let $A, B \in CoV(R^n)$, the Hausdorff metric is defined by

$$d(A, B) = \max\{sub_{a\in A}\inf_{b\in B}\| a-b \|, sub_{b\in B}\inf_{a\in A}\| a-b \|\}.$$

The fuzzy set in R^n is $F^n = \{v : R^n \to [0, 1]/v$ satisfies (i)-(iv) below$\}$,

(i) v is normal, i.e., there exists an $x_0 \in R^n$ such that $v(x_0) = 1$

(ii) v is fuzzy convex, i.e., $v((1-\gamma)x+\gamma y) \ge \min\{v(x), v(y)\}$

(iii) v is upper semi continuous

(iv) $[v]^c = \overline{\{x \in R^n/v(x) > 0\}}$ is compact

For $0 < \alpha \le 1$, represent $[v]^\alpha = \{x \in R^n/v(x) \ge \alpha\}$. Then from (i)-(iv), the α-level set $[v]^\alpha \in CoV(R^n)$, $\forall\ 0 < \alpha \le 1$.

Let $D : F^n \times F^n \to [0, \infty)$ defined by

$$D(u, v) = \sup\min\{u(x), v(y)\}$$

and $[D(u, v)]^\alpha = D([u]^\alpha, [v]^\alpha)$, $u, v \in F^n$.

For all $u, v \in F^n$, $0 \le \alpha \le 1$ and D is continuous.

(i) $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$,

(ii) $[ku]^\alpha = k[u]^\alpha$, $k \in R$.

Definition 1. Let $D : F^n \times F^n \to R_+ \cup \{0\}$ defined by

$$D(u, v) = \sup_{0\le\alpha\le 1}\{d([u]^\alpha, [v]^\alpha)\},$$

where d is the Hausdorff metric for $CoV(R^n)$.

Clearly (F^n, D) is a complete metric space, and D satisfies the following

(i) $D(u+w, v+w) = D(u, v), \forall u, v, w \in F^n$

(ii) $D(cu, cv) = |c| D(u, v), \forall u, v \in F^n, c \in R.$

Definition 2. A function $E : C \to F^n$ is differentiable at $t_0 \in C$ if there exists $E'(t_0) \in F^n$ such that the limits, $\lim_{\varepsilon \to 0^+} \frac{E(t_0 + \in) - E(t_0)}{\in}$ and $\lim_{\varepsilon \to 0^+} \frac{E(t_0) - E(t_0 - \in)}{\in}$ exists and equal to $E'(t_0)$.

If a function $E : C \to F^n$ is differentiable at $t_0 \in C$ then for any $\alpha \in [0, 1]$, the set valued function $E_\alpha(t) = [E(t)]^\alpha$ is Hukuhara differentiable at point t_0 with $DE_\alpha(t) = [E'(t)]^\alpha$.

If $E : C \to F^n$ is differentiable at $t_0 \in C$ then $E'(t_0)$ is the fuzzy derivative of E at the point t_0.

Theorem 3. *If $E : C \to F^n$ is differentiable then E is continuous.*

Proof. Let $t, t + \in$ in C and $\in > 0$.

$$D(E(t+\in), E(t)) = D(E(t+\in) - E(t), 0)$$

$$\leq \in D\left(\frac{(E(t+\in) - E(t))}{\in}, E'(t)\right) + \in D(E'(t), 0) \to 0 \text{ as } \in \to 0^+$$

where $\in$ is very small and $E(t+\in) - E(t)$ exists.

Hence E is right continuous, and also we can prove the E is left continuity.

3. Initial Value Problem

Let $f : R^{n+1} \to R^n, n > 0$ be a function and the ordinary differential equation

$$y'(t) = f(t,\ y(t)),\ y(t_0) = y_0, \tag{1}$$

where t_0 is a given constant, $y_0 \in R^n$, is said to be an initial value problem. Then the function $y(t)$, which satisfies the equation (1) is the solution of the differential equation (1).

Theorem 4. *A mapping* $y : S \to F^n$ *is a solution of the initial value problem* $y'(t) = f(t,\ y(t))$, *satisfying the initial condition* $y(t_0) = y_0$ *if and only if it is continuous and satisfies the following integral equation*

$$y(t) = y_0 + \int_{t_0}^{t} f(s,\ y(s))ds,\ \forall\, t \in P.$$

Proof. Let $y : S \to F^n$ be the solution of (1)

$\Rightarrow y$ satisfying the initial condition $y(t_0) = y_0$.

By integrating on both sides of the equation $y'(t) = f(t,\ y(t))$ from t_0 to t,

$$\int_{t_0}^{t} y'(t)\, ds = \int_{t_0}^{t} f(s,\ y(s))\, ds$$

$$y(t) - y(t_0) = \int_{t_0}^{t} f(s,\ y(s))\, ds$$

$$y(t) = y_0 + \int_{t_0}^{t} f(s,\ y(s))\, ds,\ \forall t \in P \tag{2}$$

Conversely, let $y : S \to F^n$ be a function and satisfies (2), and by setting $t = t_0$ yields $y(t_0) = y_0$, i.e., y satisfies the initial condition. Thus, the integral, $y(t)$ are continuous. it follows that the integrand $f(s,\ y(s))$ is continuous. Hence, by fundamental theorem of calculus y is differentiable and $y'(t) = f(t,\ y(t))$.

By the method of successive approximation, let us consider the sequence $\{y_n(t)\}$ such that $y_n(t) = y_0 + \int_{t_0}^{t} f(s,\ y_{n-1}(s))\, ds,\ n = 1,\ 2,\ \ldots,\ y(t_0) = y_0$.

4. Fuzzy Cauchy Problem

Let $f : C \times F^n \to F^n$ be a continuous function and the fuzzy Cauchy problem is defined by

$$y'(t) = f(t, y(t)), \; y(0) = y_0 \tag{3}$$

Let $y : [0, C] \to F^n$ be a solution of (3), if y is a primitive of $f(y)$ starting at y_0.

Define the function $Ey : [0, C] \to F^n$ by $[Ey](t) = f(y(t))$ and denote by Gy the unique continuous primitive of Ey starting at y_0.

A function $y \in C([0, T], F^n)$ is a solution of the initial value problem (3) if and only if y coincides with Gy.

Lemma 5. *Let $y_1, y_2 : S \to F^n$ with primitives x_1, x_2.*

Suppose the function $S \to d(y_1(s), y_2(s))$ is integrable on S. Then

$$d(x_1(t), x_2(t)) \leq d(x_1(0), x_2(0)) + \int_{t_0}^{t} d(y_1(s), y_2(s))\, ds$$

Proof. Assume $y(t) = d(x_1(t), x_2(t)), \; t \in T$,

$$\begin{aligned} y(t + \varepsilon) - y(t) \leq\; & d(x_1(t + \varepsilon), x_1(t) + \varepsilon y_1(t)) \\ & + d(x_1(t) + \varepsilon y_1(t), x_2(t) + \varepsilon y_1(t)) \\ & + d(x_2(t) + \varepsilon y_1(t), x_2(t) + \varepsilon y_2(t)) \\ & + d(x_2(t) + \varepsilon y_2(t), x_2(t + \varepsilon)) - d(x_1(t), x_2(t)) \\ & = d(x_1(t + \varepsilon), x_1(t) + \varepsilon y_1(t)) + d(x_1(t), x_2(t)) \\ & + \varepsilon d(y_1(t), y_2(t)) + d(x_2(t), \varepsilon y_2(t), y_2(t + \varepsilon)) \\ & - d(x_1(t), x_2(t)). \end{aligned}$$

Hence,

$$\frac{y(t+\varepsilon)-y(t)}{\varepsilon} \le \frac{1}{\varepsilon} d(x_1(t+\varepsilon), x_1(t)+\varepsilon x_1(t))$$

$$+\frac{1}{\varepsilon} d(x_1(t)+\varepsilon y_1(t), x_2(t)+\varepsilon y_1(t)) + d(y_1(t), y_2(t))$$

$$+\frac{1}{\varepsilon} d(x_2(t)+\varepsilon y_2(t), x_2(t+\varepsilon)).$$

Therefore, $D^{+}y(t) \le d(y_1(t), y_2(t); t \in T$, and

$$y(t) \le y(0) + \int_{t_0}^{t} d(y_1(s), y_2(s))ds.$$

Theorem 6. *Suppose that* $f : F^n \to F^n$ *is such that there exists* $\rho \ge 0$ *with*

$$\frac{d(f(x), f(y))}{d(x, y)} \le \rho; x, y \in F^n.$$

Then the fuzzy initial value problem (2) *has a unique solution.*

Proof. $C([0, T], F^n)$, is a complete metric space with

$$D(x, y) = \sup_{t\in[0,T]} e^{-\rho t} d(x(t), y(t))$$

By lemma 5, for any $t \in T$.

$$d([Gx](t), [Gy](t)) \le \int_0^t d([Fx](s), [Fy](s))\, ds$$

$$= \int_0^t d(f(x(s)), f(y(s)))\, ds \le \rho \int_0^t d(x(s), y(s))\, ds$$

$$= \rho \int_0^t e^{\rho s} e^{-\rho s} d(x(s), y(s))\, ds \le \rho \int_0^t e^{\rho s} d(x, y)\, ds$$

$$= (e^{\rho t} - 1) D(x, y)$$

$$D(Gx, Gy) \le [1 - e^{-\rho T}] D(x, y),$$

By the Banach contraction principle, G has a unique solution for the initial value problem (2).

Theorem 7. S*uppose that* $f : F^n \to F^n$ *is such that there exists* $\rho, \delta \geq 0$, *with*

$$d(f(x), f(y)) \leq \rho d(fx, y) + \delta d(x, fy),\ x, y \in F^n,\ \rho + \delta < 1.$$

Then the initial value problem (2) *has a unique solution.*

Proof. Let $C([0, T], F^n)$ be a complete metric space with

$$D(x, y) = \sup_{t \in [0, T]} e^{-(\rho+\delta)t} d(x(t), y(t))$$

By lemma 5, for any $t \in T$.

$$d([Gx](t), [Gy](t)) \leq \int_0^t d([Fx](s), [Fy](s))\, ds$$

$$= \int_0^t d(f(x(s)), f(y(s)))\, ds$$

$$\leq \int_0^t \{\rho d(f(x(s)), y(s)) + \delta d(x(s), f(y(s)))\}\, ds$$

$$\leq \rho \int_0^t d(f(x(s)), y(s))\, ds + \delta \int_0^t d(x(s), f(y(s)))\, ds$$

$$\leq \rho \int_0^t d(x(s), y(s))\, ds + \delta \int_0^t d(x(s), y(s))\, ds$$

$$\leq (\rho + \delta) \int_0^t d(x(s), y(s))\, ds$$

$$= (\rho + \delta) \int_0^t e^{(k+l)s} e^{-(k+l)s} d(x(s), y(s))\, ds$$

$$\leq (\rho + \delta) \int_0^t e^{(k+l)s} D(x, y)\, ds$$

$$\leq (e^{(k+l)l} - 1) D(x, y)$$

$$D(Gx,\ Gy) \le [1 - e^{-(\rho+\delta)T}]\, D(x,\ y).$$

By the Banach contraction principle, G has a unique solution for the initial value problem (2).

Theorem 8. *Let* $f : C \times F^n \to F^n$ *be a continuous function and assume that there exists* $\lambda > 0,\ \beta \in [0,\ 2]$ *such that*

$$d([f(t,\ y(t))]^{\frac{\beta}{2}},\ [f(t,\ x(t))]^{\frac{\beta}{2}}) \le \lambda d([y(t)]^{\frac{\beta}{2}},\ [x(t)]^{\frac{\beta}{2}}) \text{ for all } 0 \le \beta \le 2.$$

Then the initial value problem $y'(t) = f(t,\ y(t)),\ y(t_0) = y_0$ *has a unique solution.*

Proof. Let $f : C \times F^n \to F^n$ be a continuous function,

$$D(f(t,\ y(t)),\ f(t,\ x(t))) = Sub\{d([f(t,\ y(t))]^{\frac{\beta}{2}},\ [f(t,\ x(t))]^{\frac{\beta}{2}})\}$$

$$\le Sub\{\lambda d([y(t)]^{\frac{\beta}{2}},\ [x(t)]^{\frac{\beta}{2}})\}$$

$$= \lambda D(y(t),\ x(t)),\ \forall t \in T \text{ and } x,\ y \in F^n.$$

Thus, the initial value problem $y'(t) = f(t,\ y(t)),\ y(t_0) = y_0$ has a unique solution.

Corollary 9. *Let* $f : C \times F^n \to F^n$ *be a continuous function and assume that there exists* $\lambda > 0$ *such that*

$$d(\sqrt{f(t,\ y(t))},\ \sqrt{f(t,\ x(t))}) \le \lambda d(\sqrt{y(t)},\ \sqrt{x(t)}).$$

Then the initial value problem $y'(t) = f(t,\ y(t)),\ y(t_0) = y_0$ *has a unique solution on* C.

Proof. The corollary is follows from the above theorem by taking by taking $\beta = 1$.

Example 10. Let $y,\ (\mu - \lambda) : T \to F^n$ be continuous maps and $k > 0$. Then the initial value problem $y'(t) = ky(t) + (\mu - \lambda)(t),\ y(t_0) = \theta_1,\ t_0 \in T$ has a unique solution.

Let $f : T \times F^n \to F^n$ be such that $f(t, y(t)) = ky(t) + (\mu - \lambda)(t)$, $y(t_0) = \theta_1$, $t_0 \in T$ and $y(t) \in F^n$. Clearly, the map f is continuous and

$$d([f(t, y(t))]^{\frac{\beta}{2}}, [f(t, x(t))]^{\frac{\beta}{2}})$$

$$\leq d([ky(t) + (\mu - \lambda)(t)]^{\frac{\beta}{2}}, [kx(t) + (\mu - \lambda)(t)]^{\frac{\beta}{2}})$$

$$\leq d([ky(t)]^{\frac{\beta}{2}} + [(\mu - \lambda)(t)]^{\frac{\beta}{2}}, [kx(t)]^{\frac{\beta}{2}} + [(\mu - \lambda)(t)]^{\frac{\beta}{2}})$$

$$\leq d([ky(t)]^{\frac{\beta}{2}}, [kx\,(t)]^{\frac{\beta}{2}}) \leq kd([y(t)]^{\frac{\beta}{2}}, [x(t)]^{\frac{\beta}{2}})$$

for all $t \in T$, $x, y \in F^n$, and $\beta \in [0, 2]$.

Hence $y'(t) = ky(t) + (\mu - \lambda)(t)$, $y(t_0) = \theta_1$, $t_0 \in T$ has a unique solution.

Theorem 11. *Let $f : C \times F^n \to F^n$ be a continuous function and assume that there exists $\rho, \delta > 0$ and $\rho + \delta < 1$ such that*

$$D(f(y(t)), f(x(t))) \leq \rho D(f(y(t)), x(t)) + \delta D(y(t), f(x(t))), \forall t \in T,$$

$x, y : C \to F^n$. Then the initial value problem $y'(t) = f(t, y(t))$, $y(t_0) = y_0$, has a unique solution.

Proof. Let $S(C, F^n)$ be set of all continuous mappings from C to F^n, $C \in R$.

$S(C, F^n)$ is a metric space by setting

$$H(x, y) = \sup\{D(x(t), y(t))\} : t \in C\}, \forall x, y \in S(C, F^n).$$

Clearly $(S(C, F^n), H)$ is a complete metric space.

Let $(x_1(t), x_2(t)) \in C \times F^n$ be arbitrary, and let $n > 0$ be such that $(\rho + \delta)n^2 < 1$.

Now, our claim is to show following initial value problem

$y'(t) = f(t, y(t)), y(t_1) = y_1, t_1 \in T$ has a unique solution on T.

For $y \in S(C, F^n)$, define Gy on by

$$Gy = y_1(t) + \int_{t_1}^{t} f(s, y(s))\, ds + y_0.$$

Then $Gy \in S(C, F^n)$. Furthermore, by Lipchitz condition on f,

$$H(Gy, Gx) = \sup\{D(Gy(t), Gx(t)) : t \in T\}$$

$$= \sup\left\{D\left(\int_{t_1}^{t} f(s, y(s))\, ds, \int_{t_1}^{t} f(s, x(s))\, ds\right) : t \in T\right\}$$

$$= \int_{t_1}^{t_1+n} D(f(s, y(s)), f(s, x(s)))\, ds$$

$$= \int_{t_1}^{t_1+n} [\rho D(f(y(t)), x(t)) + \delta D(y(t), f(x(t)))]\, ds$$

$$= \int_{t_1}^{t_1+n} (\rho + \delta) D(y(t), x(t)) ds$$

$$\leq (\rho + \delta) n^2 H(x, y), \text{ for all } x, y \in S(C, F^n).$$

Thus, by Banach's contraction principle,

$\Rightarrow G$ has a unique solution for the initial value problem.

Corollary 12. *Let* $f : C \times F^n \to F^n$ *be a continuous function and assume that there exists* $k > 0$ *such that*

$$D(f(y(t)), f(x(t))) \leq k[D(f(y(t)), x(t)) + D(y(t), f(x(t)))], \forall\, t \in T,$$

$x, y : C \to F^n$. *Then the initial value problem* $y'(t) = f(t, y(t)), y(t_0) = y_0$, *has a unique solution*.

Proof. The corollary is follows from the above theorem by taking $k = \rho = \delta$.

Theorem 13. *Let* $f : C \times F^n \to F^n$ *be a continuous function and assume* *that there exists* $\delta > 0$ *such that*

$$D(f(y(t)), f(x(y))) \leq \delta D(y(t), x(t)) \text{ for all } t \in C, x, y : C \to F^n.$$

Then $y'(t) = f(t, y(t)), y(t_0) = y_0$, *has a unique solution on C.*

Proof. Let $S(C, F^n)$ be set of all continuous mappings from C to F^n, $C \in R$.

$S(C, F^n)$ is a metric space by setting

$H(y, x) = \sup\{D(x(t), y(t)) : t \in C\}$ for all $x, y \in S(C, F^n)$.

$\Rightarrow (S(C, F^n), H)$ is a complete metric space.

Now, let $(x_1(t), x_2(t)) \in C \times F^n$ be arbitrary and let $n > 0$ be such that $\delta n^2 < 1$. For $y \in S(C, F^n)$, define Gy on T by the equation

$$Gy = y_1(t) + \int_{t_1}^{t} f(s, y(s))\, ds + y_0$$

Then $Gy \in S(C, F^n)$. Furthermore, by Lipschitz condition on f,

$$H(Gy, Gx) = \sup\{D(Gy(t), Gx(t)) : t \in T\}$$

$$= \sup\left\{D\left(\int_{t_1}^{t} f(s, y(s))\, ds, \int_{t_1}^{t} f(s, x(s))\, ds\right) : t \in T\right\}$$

$$= \int_{t_1}^{t_1+n} D(f(s, y(s)), f(s, x(s)))\, ds$$

$$= \int_{t_1}^{t_1+n} \delta D(y(t), x(t)) ds$$

$$\leq \delta n^2 H(x, y) \leq H(x, y) \text{ for all } x, y \in S(C, F^n).$$

Hence, by Banach's contraction mapping theorem, G has a unique solution of the initial value problem.

Corollary 14. *Let* $f : C \times F^n \to F^n$ *be a continuous function and assume that there exists* $k > 0, \beta < 2$ *such that*

$$D(f(y(t)), f(x(t))) \leq (\sqrt{2-\beta})D(y(t), x(t)) \text{ for all } t \in C, x, y : C \to F^n.$$

Then $y'(t) = f(t, y(t)), y(t_0) = y_0$, *has a unique solution on C.*

Proof. The corollary follows from above theorem by taking $k = \delta$ and $\beta = 1$.

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