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THE MIDDLE DOMINATING FUZZY GRAPH OF A FUZZY GRAPH

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Abstract

In this paper we introduce a new type of Dominating Fuzzy Graph such as The Middle Dominating Fuzzy Graph of a Fuzzy Graph. The Middle Dominating Fuzzy graph is denoted by $M_dF(G) : (\sigma_M, \mu_M)$ and is defined to be the intersection graph on the minimal dominating sets of vertices in Fuzzy Graph. And characterizations are given for fuzzy graphs whose dominating fuzzy graph is connected and complete. Some other results are established relating to this new Fuzzy Graph.

1. Introduction

The study of domination set was initiated by Ore [10] and Berge [1]. The domination number and therefore independent domination number were introduced by Cockayne and Hedetniemi [2]. Rosenfeld [11] introduced the notion of fuzzy graph and a number of other fuzzy analogs of graph theoretic concepts like paths, cycles and connectedness. Somasundaram and somasundaram [12] discussed domination in fuzzy graphs. They defined domination using effective edges in fuzzy graphs. Nagoor Gani and Chandrasekaran [8] discussed domination in fuzzy graph using strong arcs. V. R. Kulli et al. [6, 7] introduced various type of dominating graphs which are graph valued functions in the field of domination theory. B.

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Basavanagoud and S. M. Hosamani [4] introduced a new class of intersection graphs in the field of domination theory. In this paper we introduce a new type of fuzzy graph in the field of fuzzy domination theory.

2. Preliminaries

Definition 2.1. A fuzzy graph $G = (\sigma, \mu)$ is a pair of function $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ where $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for $u, v \in V$. The underlying crisp graph of $G = (\sigma, \mu)$ is denoted by $G^* = (V, E)$, where

$$V = \{u \in V : \sigma(u) > 0\} \text{ and } E = \{(u, v) \in V \times V : \mu(u, v) > 0\}.$$

Definition 2.2. The order p and size q of the fuzzy graph $G = (\sigma, \mu)$ are defined by $p = \sum_{v \in V} \sigma(v)$ and $q = \sum_{(u, v) \in E} \mu(u, v)$.

Definition 2.3. A path ρ in a fuzzy graph is a sequence of distinct nodes $x_1, x_2, x_3, \ldots, x_n$ such that $\mu(x_{i-1}, x_i) > 0, 1 \leq i \leq n$, here $n \geq 0$ is called the length of the path.

Definition 2.4. An arc (u, v) in a fuzzy graph $G = (\sigma, \mu)$ is said to be strong if $\mu^{\infty}(u, v) = \mu(u, v)$ and the node v is said to be strong neighbor of u. If $\mu(u, v) = 0$ for every $v \in V$, then u is called isolated node. If a path has length zero, then its strength is defined to be $\sigma(u_0)$. The path ρ is called a cycle if $u_0 = u_n$ and $n \geq 3$.

Definition 2.5. Two nodes of a fuzzy graph are said to be fuzzy independent if there is no strong arc between them. A subset S of V is said to be a fuzzy independent set of G if any two nodes of S are fuzzy independent.

Definition 2.6. The strength of connectedness between two nodes u, v in a fuzzy graph G is $\mu^{\infty}(u, v) = \sup\{\mu^k(u, v); k = 1, 2, 3, \ldots\}$ where $\mu^k(u, v) = \sup\{\mu(u, u_1) \wedge \mu(u, u_2) \wedge \ldots \wedge \mu(u_{k-1}, v)\}$.

Definition 2.7. Let $G = (\sigma, \mu)$ be a fuzzy graph. A subset D of V is said to be fuzzy dominating set of G if for every $v \in V - D$ there exists $u \in D$

such that (u, v) is a strong arc. A dominating set D is called a minimal dominating set if no proper subset of D is a dominating set.

Definition 2.8. A fuzzy graph $G = (\sigma, \mu)$ with underlying crisp graph $G^* = (\sigma^*, \mu^*)$ be given. Let G^* be (V, E), S is the collection of all minimal dominating set of G. The middle dominating fuzzy graph of G is denoted by $M_dF(G) : (\sigma_M, \mu_M)$ with node set the disjoint union of $V \cup S$, where

$\sigma_M(u) = \sigma(u)$ if $u \in \sigma^*$

$= \mu^{\infty}(u, v)$ if $u, v \in e_i$ and $\forall e_i \in S$

$= 0$ otherwise

$\mu_M(e_i, e_j) = \mu(e_i) \wedge \mu(e_j)$ if $e_i, e_j \in \mu^*$

$= 0$ otherwise

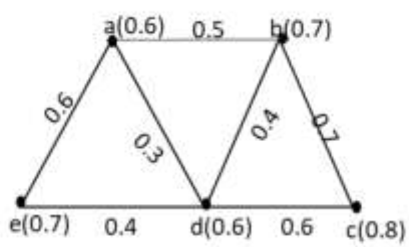
$\mu_M(v_i, v_j) = 0$ if $v_i, v_j \in \sigma^*$

$\mu_M(v_i, e_j) = \mu(e_j)$ if $v_i \in \sigma^*, e_j \in \mu^*$

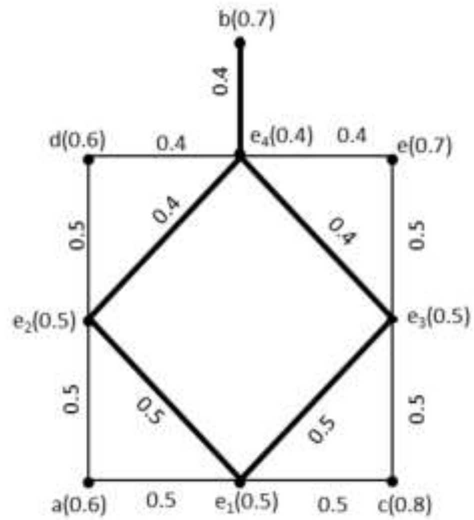
$= 0$ otherwise

As σ_M is defined only through the values of σ and μ, $\sigma_M : V \cup E \to [0, 1]$ is a well-defined fuzzy subset on $V \cup E$. Also μ_M is a fuzzy relation on σ_M and $\mu_M(u, v) \leq \sigma_M(u) \wedge \sigma_M(v)$ $\forall u, v$ in $V \cup E$.

Example 2.9.



Fuzzy graph G.



Middle dominating fuzzy graph $M_dF(G)$

where $e_1 = \{a, c\}$, $e_2 = \{a, d\}$, $e_3 = \{c, e\}$, $e_4 = \{b, d, e\}$ are the minimal dominating sets of G.

Observations 2.10.

(i) $M_dF(G)$ is a strong fuzzy graph.

(ii) For any connected fuzzy graph G,

$$|E(MDF(G))| < |E(M_dF(G))| \text{ and } |E(DF(G))| < |E(M_dF(G))|.$$

Where $MDF(G)$ is minimal dominating fuzzy graph, $DF(G)$ is dominating fuzzy graph.

(iii) In $M_dF(G)$, all the edges are effective edges.

(iv) If G be a connected fuzzy graph, but $M_dF(G)$ need not be connected fuzzy graph.

(v) If G is a complete fuzzy graph, then middle dominating fuzzy graph is totally disconnected.

3. Preliminary Results

Proposition 3.1.

$$order[DF(G)] = order[M_d F(G)] \text{ and } size[DF(G)] \leq size[M_d F(G)].$$

Proposition 3.2.

$$M_d F(G) = DF(G) \cup MDF(G).$$

4. Main Results

Theorem 4.1. *Dominating fuzzy graph is a spanning sub graph of middle dominating fuzzy graph.*

Proof. By the definition of dominating fuzzy graph, $DF(G)$ is disjoint union of $V \cup S$, where V be the node set of G and S be the collection of all minimal dominating set of G.

Here the middle dominating fuzzy graph, $M_d F(G)$ is defined as the disjoint union of $V \cup S$.

The dominating fuzzy graph and the middle dominating fuzzy graph have the same order, they differ only in the size.

Since, $order[DF(G)] = order[M_d F(G)]$ and $size[DF(G)] \leq size[M_d F(G)]$

Therefore, the dominating fuzzy graph is the spanning subgraph of the middle dominating fuzzy graph.

Theorem 4.2. *G is a complete fuzzy graph with p vertices if and only if the middle dominating fuzzy graph, $M_d F(G) = pk_2$.*

Proof. Assume that G is a complete fuzzy graph with p vertices then G has strong arcs between every pair of vertices.

Each vertex in G will form a minimal dominating set.

By the definition of middle dominating fuzzy graph each minimal dominating set is the strong neighbor to the corresponding node of G in the middle dominating fuzzy graph and form a complete fuzzy graph with 2 vertices.

Hence for any complete fuzzy graph with p vertices, $M_dF(G) = pk_2$.

Conversely, assume that $M_dF(G) = pk_2$.

To prove G is a complete fuzzy graph with p vertices.

Suppose G is not a complete fuzzy graph, then there exists at least one minimal dominating set S containing two vertices of G.

By the definition of middle dominating fuzzy graph, the minimal dominating set S will form a path with 3 vertices in $M_dF(G)$ which is a contradiction to our assumption.

Hence G must be a complete fuzzy graph with p vertices.

Theorem 4.3. *For a (p, q) fuzzy graph G, the middle dominating fuzzy graph is of the type $(2p, p)$ if and only if G is a complete fuzzy graph.*

Proof. Consider $M_dF(G) - (2p, p)$.

Since pK_2 is the only fuzzy graph which satisfies $2p$ number of nodes and p number of edges, i.e., $M_dF(G) = pk_2$

By theorem 4.2, G is a complete fuzzy graph with p vertices.

Conversely, suppose G is a complete fuzzy graph with p vertices.

By theorem 4.2, $M_dF(G) = pk_2$ since $pK_2 = (2p, p)$.

Therefore the middle dominating fuzzy graph of fuzzy graph G is of the type $(2p, p)$.

Theorem 4.4. *For any connected Fuzzy graph G with at least two vertices, the middle dominating fuzzy graph of G is connected iff $\Delta_S(G) < p - 1$.*

Proof. Let $\Delta_S(G) < p - 1$. Consider the following cases:

Case (i). Suppose there is no minimal dominating set containing both x and y. Then there exists a vertex z in $V(G)$, such that which is not strong neighbor to both x and y.

Let S and S' be two maximal fuzzy independent sets containing (x, z) and (y, z) respectively.

Since every maximal fuzzy independent set in a fuzzy graph G is a minimal dominating set in G.

Therefore, x and y are connected in $M_d F(G)$ by the path $xSyS'$.

Thus $M_d F(G)$ is connected.

Case (ii). Suppose there exists a minimal dominating set S containing both x and y or there exists two non-disjoint minimal dominating sets S_1 and S_2 containing both x and y respectively.

From the definition of middle dominating fuzzy graph, $M_d F(G)$ is connected by a path xSy (or) xS_1S_2y.

Conversely, suppose $M_d F(G)$ is connected. On the contrary assume that $\Delta_S(G) = p - 1$.

Let $x \in V(G)$ and $|N_s(x)| = p - 1$.

Then $S = \{x\}$ is a minimum dominating set of G. Since every minimum dominating set of a fuzzy graph is a minimal dominating set.

That is x is a strong neighbor to every other nodes of G, and G has no isolated nodes. W. K. T $V - \{x\}$ contains a minimal dominating set S'.

Since $\{x\} \cap S' = \emptyset$, in $M_d F(G)$ there is no path connecting x to any node of S' which implies that $M_d F(G)$ is disconnected, a contradiction to our assumption.

Hence $\Delta_S(G) < p - 1$.

Theorem 4.5. *For any connected fuzzy graph G, the middle dominating fuzzy graph $M_d F(G)$ is a fuzzy bipartite graph.*

Proof. The Middle dominating fuzzy graph $M_d F(G)$ is a disjoint union of $V \cup S$, where S be the collection of all minimal dominating set of G.

Then the vertex set V can be partitioned into two nonempty sets V and S such that V and S are fuzzy independent sets.

Every arc in $M_d F(G)$ is a strong arc since $M_d F(G)$ is a strong fuzzy graph.

Thus every strong arc of $M_dF(G)$ has one end in V and the other end in S.

Hence $M_dF(G)$ is a fuzzy bipartite graph.

Theorem 4.6. *For any connected fuzzy graph G,*

(i) $\beta_0(M_dF(G)) = p$

(ii) $\alpha_0(M_dF(G)) = |S(G)|$, $S(G)$ *is the set of all minimal dominating sets of a fuzzy graph G.*

Proof. (i) Consider a fuzzy graph of order p. By the definition of middle dominating fuzzy graph, each node u_i, $i = 1, 2, \ldots, p$ of G are not a neighbor in $M_dF(G)$.

And these nodes will form together maximum independent set of $M_dF(G)$. Therefore, the maximum independent set of $M_dF(G)$ is the number of nodes in G.

(ii) The middle dominating fuzzy graph is defined as $V \cup S$, where V is the vertex set of G and S be the minimal dominating set of G.

Sum of maximal independent set and vertex covering number is the number of nodes in $M_dF(G)$.

By (i) V be the maximal independent set of G, obviously set of all minimal dominating set is the vertex covering number of $M_dF(G)$.

Therefore, $\alpha_0(M_dF(G)) = |S(G)|$.

Theorem 4.7. *For any connected fuzzy graph G with p vertices and q edges, $M_dF(G)$ is either connected or it has at least one component which is K_2.*

Proof. Let G be a (p, q) connected fuzzy graph, consider the following cases:

Case (i). If $\Delta_S(G) < p - 1$, then by theorem 4.4, $M_dF(G)$ is connected.

Case (ii). If $\Delta_s(G) = \delta_s(G) = p - 1$, then the fuzzy graph is complete with p vertices. By Theorem 4.2 $M_dF(G) = pK_2$.

Case (iii). If $\delta_s(G) < \Delta_s(G) = p-1$.

Consider $v_1, v_2, \ldots, v_n$ be the only nodes of cardinality of strong neighborhood with $p-1$ in fuzzy graph G.

Let $H = G - \{v_1, v_2, \ldots, v_n\}$ which implies $\Delta_S(H) < p-1$. Then by theorem 4.4, $M_d F(G)$ is connected.

Since,

$$M_d F(G) = \Omega(V(M_d(H))) \cup (\{v_1\} + v_1) \cup (\{v_2\} + v_2) \cup \ldots \cup (\{v_n\} + v_n).$$

Therefore which implies that at least one component of $M_d F(G)$ is K_2.

Theorem 4.8. *For any connected fuzzy graph G,*

$$\kappa(M_d F(G)) = \min\{\min_{1 \le i \le n} |S_i|, \min(\deg_{M_d F(G)} v_j, 1 \le j \le p)\}$$

Where S_i's are the minimal dominating sets of G.

Proof. Let G be a (p, q) connected fuzzy graph. Consider the following cases:

Case (i). Let $u \in S_i$ for some i, having minimum of weight of strong edges incident among all nodes of S_i's.

If the degree of u is less than any other nodes in $M_d F(G)$.

Then by removing those nodes of $M_d F(G)$ which are strong neighbor with u, the graph will be disconnected.

Case (ii). Let $v \in V_j$ for some j, having minimum of weight of strong edges incident among all v_j's in the middle dominating fuzzy graph.

If the degree of v is less than any other nodes in $M_d F(G)$.

Then by removing those nodes of $M_d F(G)$ which are strong neighbor with v, the graph will be disconnected.

Hence the vertex connectivity of $M_d F(G)$ is the minimum of these two cases.

Theorem 4.9. *For any connected fuzzy graph G,*

$$\lambda(M_d F(G)) = \min\{\min_{1 \le i \le n} |S_i|, \min(\deg_{M_d F(G)} v_j, 1 \le j \le p)\}$$

Where S_i's are the minimal dominating sets of G.

Proof. Let G be a connected fuzzy graph with p nodes q edges. Consider the following cases:

Case (i). Let $u \in S_i$ for some i, having minimum of weight of strong edges incident among all nodes of S_i's.

If the degree of u is less than any other nodes in $M_d F(G)$.

Then by removing those edges of $M_d F(G)$ which are strong edge incident with u, the graph will be disconnected.

Case (ii). Let $v \in V_j$ for some j, having minimum of weight of strong edges incident among all v_j's in the middle dominating fuzzy graph. If the degree of v is less than any other nodes in $M_d F(G)$.

Then by removing those edges of $M_d F(G)$ which are strong edge incident with v, the graph will be disconnected.

Hence the edge connectivity of $M_d F(G)$ is the minimum of these two cases.

Theorem 4.10. *For any connected fuzzy graph G,* $d(M_d F(G)) = 2$ *if* $G = K_p$, *where* $d(M_d F(G))$ *is the domatic number of* $M_d F(G)$.

Proof. If $G = K_p$, then by Theorem 4.2, $M_d F(G) = pK_2$.

Since the graph is complete fuzzy graph with p vertices then maximum cardinality of domatic partition is two, i.e., $d(pK_2) = 2$, therefore $d(M_d F(G)) = 2$.

5. Conclusion

We defined middle dominating fuzzy graph of a fuzzy graph. We have given some properties of middle dominating fuzzy graph. Further works are to define some other parameters of dominating fuzzy graph and introduce a replacement class of intersection graphs within the field of fuzzy domination theory and also to develop some real life applications on dominating fuzzy graph.

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