Advances and Applications in Mathematical Sciences
Volume 22, Issue 1, November 2022, Pages 1-11

SECURE DOMINATION IN BIPOLAR FUZZY GRAPHS

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Abstract

In this paper, we established and investigated the concept of secure domination in bipolar fuzzy graphs and denoted it as $\gamma_{bs}(G)$. The definitions of 2-secure dominating set and its domination number in bipolar fuzzy graphs are defined and some results are derived with suitable examples.

1. Introduction

Zhang [15] established the concept of bipolar fuzzy set in 1994. It was initiated as a generalisation of fuzzy sets which is an expansion of fuzzy sets with membership range $[-1, 1]$. Later, M. Akram [1] introduced Bipolar fuzzy graphs and various related notions. A. Somasundaram and S. Somasundaram [12] discussed about domination in fuzzy graph. The idea of domination in bipolar fuzzy graphs was introduced by M. G. Karunambigai [5]. The notion of secure dominating set and 2-dominating set in graphs was introduced by Merouane and Chellali [7]. The secure and 2-secure domination in fuzzy and intuitionistic fuzzy graphs was discussed by M. G.

2020 Mathematics Subject Classification: Primary 47H10; Secondary 54H25.
Keywords: Bipolar domination, secure domination number, 2-domination, 2-secure domination number.
Received May 3, 2021; Accepted March 25, 2022

Karunambigai, et al., [6]. Motivated by these domination concepts, we aim to establish the concept of secure domination in bipolar fuzzy graph (BFG), also discuss some definitions and properties related to 2-secure domination in BFG with examples.

2. Preliminaries

Definition 2.1 [5]. Let $\mathcal{X}$ be a non empty set. A bipolar fuzzy set $\mathcal{M}$ in $\mathcal{X}$ is an object having the form $B = \{(x, \mu_B^+(x), \mu_B^-(x)) \mid x \in \mathcal{X}\}$ where, $\mu_B^+ : \mathcal{X} \to [0, 1]$ and $\mu_B^- : \mathcal{X} \to [-1, 0]$ are mappings.

Definition 2.2 [5]. A Bipolar fuzzy graph (BFG) is of the form $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where

(1) $\mathcal{V} = v_1, v_2, \ldots, v_n$ such that $\mu_1^+ : X \to [0, 1]$ and $\mu_1^- : X \to [-1, 0]$

(2) $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ where $\mu_2^+ : \mathcal{V} \times \mathcal{V} \to [0, 1]$ and $\mu_2^- : \mathcal{V} \times \mathcal{V} \to [-1, 0]$ such that

$$\mu_{2ij}^+ = \mu_2^+(v_i, v_j) \leq \min(\mu_1^+(v_i), \mu_1^+(v_j))$$

and

$$\mu_{2ij}^- = \mu_2^-(v_i, v_j) \geq \max(\mu_1^-(v_i), \mu_1^-(v_j))$$

for all $(v_i, v_j) \in \mathcal{E}$.

Definition 2.3 [5]. A BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called strong if $\mu_2^+ = \min(\mu_1^+(v_i), \mu_1^+(v_j))$ and $\mu_2^- = \max(\mu_1^-(v_i), \mu_1^-(v_j)) \; \forall v_i, v_j \in \mathcal{V}$.

Definition 2.4 [5]. A BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called complete if

$$\mu_2^+(v_i, v_j) \leq \min(\mu_1^+(v_i), \mu_1^+(v_j))$$

$$\mu_2^-(v_i, v_j) \geq \max(\mu_1^-(v_i), \mu_1^-(v_j))$$

for all $v_i, v_j \in \mathcal{V}$.

Definition 2.5 [5]. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a BFG, then cardinality of $\mathcal{G}$ is defined as

$$|\mathcal{G}| = \sum_{v_i \in \mathcal{V}} \frac{(1 + \mu_1^+(v_i) + \mu_1^-(v_i))}{2} + \sum_{(v_i, v_j) \in \mathcal{E}} \frac{(1 + \mu_2^+(v_i, v_j) + \mu_2^-(v_i, v_j))}{2}.$$

Definition 2.6 [5]. The cardinality of $\mathcal{V}$, i.e., amount of nodes is termed as the order of a BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and is signified by $|\mathcal{V}|$ (or $O(\mathcal{G})$) and determined by

$$O(\mathcal{G}) = |\mathcal{V}| = \sum_{v_i \in \mathcal{V}} \frac{(1 + \mu_1^+(v_i) + \mu_1^-(v_i))}{2}$$

The no. of elements in a set of S, i.e., amount of edges is termed as size of BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and signified as $|S|$ (or $S(\mathcal{G})$) and determined by

$$O(\mathcal{G}) = |S| = \sum_{v_i \in \mathcal{V}} \frac{(1 + \mu_2^+(v_i) + \mu_2^-(v_i, v_j))}{2}$$

for all $(v_i, v_j) \in \mathcal{E}$.

Definition 2.7 [5]. The degree of a vertex v in a BFG, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined to be the sum of the weights of the strong edges incident at v. It is denoted by $d_{\mathcal{G}}(v)$. The minimum degree of $\mathcal{G}$ is $\Delta(\mathcal{G}) = \min(d_{\mathcal{G}}(v) \mid v \in V)$. The maximum degree of $\mathcal{G}$ is $\Delta(\mathcal{G}) = \max(d_{\mathcal{G}}(v) \mid v \in V)$.

Definition 2.8 [5]. Two vertices v_i and v_j are said to be neighbors in BFG, if either one of the following conditions hold

(1) $\mu_2^+(v_i, v_j) > 0$ and $\mu_2^-(v_i, v_j) = 0$

(2) $\mu_2^+(v_i, v_j) = 0$ and $\mu_2^+(v_i, v_j) < 0$

(3) $\mu_2^+(v_i, v_j) > 0$ and $\mu_2^+(v_i, v_j) < 0, v_i, v_j \in \mathcal{V}$.

Definition 2.9 [10]. The strength of connectedness between two nodes a and b is

$$\mu^{\infty}(a, b) = \sup(\mu^{k}(a, b) \mid k = 1, 2, \ldots)$$

whereas $\mu^{k}(a, b) = \sup(\mu(aa_1) \wedge \mu(a_1a_2) \ldots \wedge \mu(a_{k-1}b) \mid a_1, \ldots, a_{k-1} \in \mathcal{V}$.

Definition 2.10 [5]**.** An arc (a, b) is said to be strong edge in a BFG, if

$$\mu_2^{+}(a, b) \geq (\mu_2^{+})^{\infty}(a, b) \text{ and } \mu_2^{-}(a, b) \geq (\mu_2^{-})^{\infty}(a, b)$$

whereas $(\mu_2^{+})^{\infty}(a, b) = \max\{(\mu_2^{+})^{k}(a, b) \mid k = 1, 2, \ldots, n\}$

and $(\mu_2^{-})^{\infty}(a, b) = \max\{(\mu_2^{-})^{k}(a, b) \mid k = 1, 2, \ldots, n\}$.

Definition 2.11 [5]**.** Let u be a vertex in a BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ then $N(u) = \{v : v \in \mathcal{V}\}$ and (u, v) is a strong edge in $\mathcal{G}$ is called neighbourhood of u in $\mathcal{G}$.

Theorem 2.12. *Every arc in a complete BFG is a strong arc*.

Definition 2.13 [5]**.** A vertex $u \in \mathcal{V}$ of a BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be an isolated vertex if $\mu_2^{+}(u, v) = 0$ and $\mu_2^{-}(u, v) = 0 \; \forall v \in \mathcal{V}, u \neq v$. That is, $N(u) = \phi$. Thus an isolated vertex does not dominate any other vertex of $\mathcal{G}$.

Definition 2.14 [5]**.** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a BFG on $\mathcal{V}$, Let $u, v \in \mathcal{V}$, we say that u dominates v in $\mathcal{G}$ if there exists a strong edge between them.

Remark 2.15 [5]**.** (1) For any $u, v \in \mathcal{V}$, if u dominates v then v dominates u and hence domination is a symmetric relation on $\mathcal{V}$.

(2) For any $v \in \mathcal{V}$, $N(v)$ is precisely the set of all vertices in $\mathcal{V}$ which are dominated by v.

(3) If $(\mu_2^{+})(u, v) < (\mu_2^{+})^{\infty}(u, v)$ and $(\mu_2^{-})(u, v) < (\mu_2^{-})^{\infty}(u, v)$ for all $u, v \in \mathcal{V}$, then the dominating set of $\mathcal{G}$ is $\mathcal{V}$.

3. Secure Domination in Bipolar Fuzzy Graphs

Definition 3.1. Let $\mathcal{G}$ be a BFG and $u, v \in \mathcal{V}$. A subset $\mathcal{D}$ of $\mathcal{V}$ is called dominating set in $\mathcal{G}$ if for every $u \in \mathcal{V} - \mathcal{D}$, there exists $u \in \mathcal{D}$ such that u dominates v. The minimum cardinality taken over all dominating sets of $\mathcal{G}$ is called the domination number of $\mathcal{G}$ and denoted by $\gamma_b(\mathcal{G})$.

Definition 3.2. Let $\mathcal{G}$ be a BFG without isolated vertices. A Total dominating set $\mathcal{D}$ of a BFG $\mathcal{G}$ is a dominating set in which the subgraph $\langle \mathcal{D} \rangle$ induced by $\mathcal{D}$ has no isolated vertices. The minimum cardinality taken over all total dominating sets is called the total domination number of G and is denoted as $\gamma_{bt}(\mathcal{G})$.

Definition 3.3. In a BFG $\mathcal{G}$. A Secure dominating set $\mathcal{S} \subseteq \mathcal{V}$ is a dominating set, if for every vertex $u \in \mathcal{V} - \mathcal{S}$ is adjacent to a vertex $v \in \mathcal{S}$ such that $(\mathcal{S} - \{v\}) \cup \{u\}$ is also a dominating set. The minimum cardinality taken over all secure dominating sets of $\mathcal{G}$ is called the secure domination number of $\mathcal{G}$ and is expressed as $\gamma_{bs}(\mathcal{G})$.

From the above graph (Figure 1), $\{v_1, v_2, v_7, v_8\}$, $\{v_2, v_3, v_7, v_9\}$, $\{v_2, v_4, v_7, v_{10}\}$ are the secure dominating sets and the secure domination number, $\gamma_{bs}(\mathcal{G}) = 1.8$.

Definition 3.4. Consider a BFG $\mathcal{G}$ without isolated vertices. A total secure dominating set is a secure dominating set $\mathcal{S}$ in which the subgraph $\langle \mathcal{S} \rangle$ induced by $\mathcal{S}$ has no isolated vertices. The minimum fuzzy cardinality taken over all secure total dominating sets of $\mathcal{G}$ is called the total secure domination number of $\mathcal{G}$ and is denoted by $\gamma_{bst}(\mathcal{G})$.

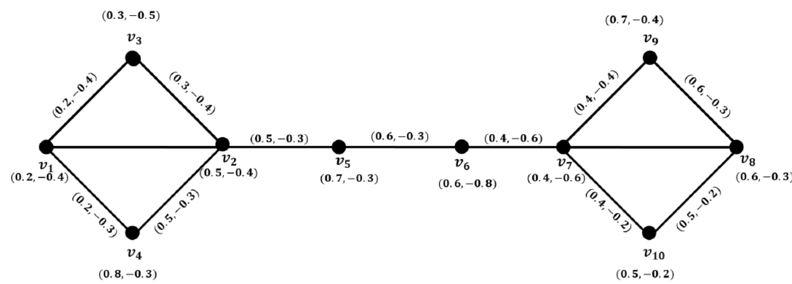

Figure 1. Secure domination in BFG.

Definition 3.5. A subset $\mathcal{D}^*$ of $\mathcal{V}$ is called a 2-dominating set in $\mathcal{G}$ if every vertex of $\mathcal{V} - \mathcal{D}^*$ has at least two neighbour in $\mathcal{D}^*$.

The minimum cardinality taken over all 2-dominating sets of $\mathcal{G}$ is called the 2-domination number of $\mathcal{G}$ and is denoted by $\gamma_2(\mathcal{G})$.

Definition 3.6. A subset $\mathcal{D}^*$ of $\mathcal{V}$ is called a 2-total dominating set in $\mathcal{G}$, if $\mathcal{D}^*$ is a 2-dominating set and the subgraph induced by $\mathcal{D}^*$ has no isolated vertices. The minimum cardinality taken over all 2-total dominating sets of $\mathcal{G}$ is called the 2-total domination number of $\mathcal{G}$ and is denoted by $\gamma_{2t}(\mathcal{G})$.

Definition 3.7. In a BFG $\mathcal{G}$. A secure 2-dominating set is a 2-dominating set $S^* \subseteq \mathcal{V}$, if for every vertex $u \in \mathcal{V} - \mathcal{S}^*$ is adjacent to a vertex $v \in S^*$ such that $(\mathcal{S}^* - \{v\}) \cup \{u\}$ is 2-dominating set. The minimum cardinality taken over all 2-secure dominating sets of $\mathcal{G}$ is called the 2-secure domination number of $\mathcal{G}$ and is expressed as $\gamma_{2bs}(\mathcal{G})$.

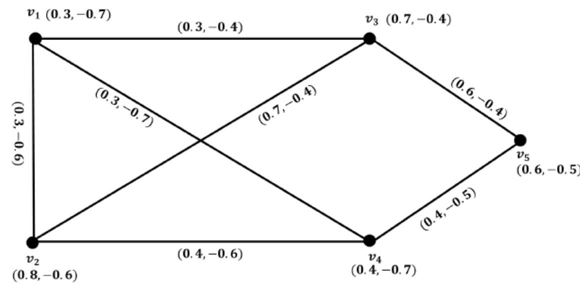

Figure 2. 2-secure domination in BFG.

From the above graph (Figure 2), $\{v_1, v_4, v_5\}$, $\{v_2, v_3, v_5\}$, $\{v_1, v_3, v_5\}$ are secure 2-dominating sets and 2-secure domination number is $\gamma_{2bs}(\mathcal{G}) = 1.2$.

Definition 3.8. Consider a BFG $\mathcal{G}$ without isolated vertices. A 2-secure total dominating set is a 2-secure dominating set in which the subgraph $\langle \mathcal{S}^* \rangle$ induced by $\mathcal{S}^*$ has no isolated vertices. The minimum fuzzy cardinality taken over all 2-secure total dominating sets of $\mathcal{G}$ is called the 2-secure total domination number of $\mathcal{G}$ and is denoted by $\gamma_{2bst}(\mathcal{G})$.

Theorem 3.9. *Let $\mathcal{G}$ be a complete BFG. If $\mathcal{S}$ is a minimal dominating set in $\mathcal{S}$ then*

(1) *$\mathcal{S}$ is a secure dominating set.*

(2) *$\mathcal{S}$ is not a secure total dominating set.*

Proof. Given $\mathcal{S}$ is a minimal dominating set of a complete BFG $\mathcal{G}$. By Theorem (2.12), every arc in a complete bipolar fuzzy graph is a strong arc, then minimal dominating set $\mathcal{S}$ contains only one vertex v, i.e., $\mathcal{S} = \{v\}$. Now for any vertex $v_i \in \mathcal{V} - \mathcal{S}$ and v_i is adjacent to v. Then $(\mathcal{S} - \{v\}) \cup \{v_i\} = \{v_i\}$ is a dominating set. Thus, $\mathcal{S}$ is secure dominating set.

Since any secure dominating set of a complete BFG contains a vertex v_i, by the definition of total dominating, $\mathcal{S}$ is not a secure total dominating set. □

Theorem 3.10. *Let $\mathcal{G}$ be a complete BFG. If $\mathcal{D}$ is a minimal dominating set in $\mathcal{G}$ then*

(1) *$\mathcal{D}$ is not a 2-dominating set,*

(2) *$\mathcal{D}$ is not a 2-total dominating set.*

Proof. Consider a complete BFG $\mathcal{G}$, If $\mathcal{D}$ is a minimal dominating set in $\mathcal{G}$, then $\mathcal{D}$ contains a vertex of minimum cardinality but 2-dominating set should contain at least two vertices. Therefore, $\mathcal{D}$ is not a 2-dominating set. Similarly, $\mathcal{D}$ is not a 2-total dominating set. □

Theorem 3.11. *For a complete BFG $\mathcal{G}$,*

$$\gamma_{bs}(\mathcal{G}) = \gamma_b(\mathcal{G}).$$

Proof. Let us consider a complete BFG $\mathcal{G}$. Let $\mathcal{S}$ be a minimal dominating set of $\mathcal{G}$. Then $\mathcal{S}$ contains a vertex $\{v\}$, i.e., $\mathcal{S} = \{v\}$. The minimum cardinality of $\mathcal{S}$ is denoted by $\gamma_b(\mathcal{G})$. By Theorem (3.9), $\mathcal{S}$ is also secure dominating set and the minimum cardinality of secure dominating set is denoted by $\gamma_{bs}(\mathcal{G})$. Hence, $\gamma_{bs}(\mathcal{G}) = \gamma_b(\mathcal{G})$.

Theorem 3.12. *Every 2-secure dominating set of a BFG $\mathcal{G}$ is a secure dominating set of $\mathcal{G}$.*

Proof. Let $\mathcal{G}$ be a BFG and $\mathcal{S}$ be a 2-secure dominating set of $\mathcal{G}$. Then every vertex $u \in \mathcal{V} - \mathcal{S}$ is adjacent to a vertex $v \in \mathcal{S}$ such that $(\mathcal{S} - \{v\}) \cup \{u\}$ is 2-dominating set. Since $\mathcal{S}$ is a 2-secure dominating set then by definition, $\mathcal{S}$ is a 2-dominating set and every 2-dominating set is a dominating set. Thus every vertex $u \in \mathcal{V} - \mathcal{S}$ is adjacent to a vertex $v \in \mathcal{S}$ such that $(\mathcal{S} - \{v\}) \cup \{u\}$ is a dominating set. Thus $\mathcal{S}$ is a secure dominating set of $\mathcal{G}$. □

Theorem 3.13. *For a bipolar fuzzy graph* $\mathcal{G}$,

$$\gamma_{2bs}(\mathcal{G}) \geq \gamma_{bs}(\mathcal{G}).$$

Proof. By Theorem (3.12), every 2-secure dominating set of a BFG $\mathcal{G}$ is a secure dominating set of $\mathcal{G}$. Thus every minimum 2-secure dominating set of $\mathcal{G}$ is also a secure dominating set of $\mathcal{G}$. Thus, $\gamma_{2bs}(\mathcal{G}) \geq \gamma_{bs}(\mathcal{G})$. □

Theorem 3.14. *Let* $\mathcal{G}$ *be a BFG. If* $\mathcal{S}$ *is a* 2*-dominating set of a path of* $\mathcal{G}$, *then* $\mathcal{S}$ *is not* 2*-secure dominating set.*

Proof. Consider a BFG $\mathcal{G}$. Let P_n be a path of $\mathcal{G}$ and $\mathcal{S}$ is a 2-dominating set of a path P_n. Then $\mathcal{S}$ contain two pendent vertices v_i and v_j. Now for some $u \in \mathcal{V} - \mathcal{S}$ and u is adjacent to v_i. Thus $(\mathcal{S} - \{v_i\}) \bigcup u$ is not 2-dominating set. Thus $\mathcal{S}$ is not 2-secure dominating set. □

Theorem 3.15. *Let* $\mathcal{G}_{m,n}$ *be a complete bipartite BFG. If* $\mathcal{S}$ *is a dominating set of* $\mathcal{G}$, *then* $\mathcal{S}$ *is not a secure dominating set.*

Proof. Given that $\mathcal{S}$ is a dominating set of a complete bipartite BFG $\mathcal{G}_{m,n}$. Then $\mathcal{S}$ should contain a vertex in $\mathcal{V}_1$ say u and a vertex in V_2 say v. Now for some $v_i \in \mathcal{V} - \mathcal{S}$ and v_i is adjacent to $u \in \mathcal{V}_1$. Thus $\{\mathcal{S} - \{v_i\}\} \bigcup \{v_i\}$ is not dominating set. So $\mathcal{S}$ is not a secure dominating set.

Theorem 3.16. *Let* $\mathcal{G}$ *be a BFG with only strong edges and without isolated vertices and* $\mathcal{S}$ *is a minimal secure dominating set. Then* $\mathcal{V} - \mathcal{S}$ *is a secure dominating set of* $\mathcal{G}$.

Proof. Consider a BFG $\mathcal{G}$ with only strong edges and without isolated vertices. Given that $\mathcal{S}$ is a minimal secure dominating set of $\mathcal{G}$. Then by definition, every vertex $u \in \mathcal{V} - \mathcal{S}$ is adjacent to a vertex $v \in \mathcal{S}$ such that $(\mathcal{S} - \{v\}) \bigcup \{u\}$ is dominating set.

Claim: prove that $\mathcal{V} - \mathcal{S}$ is a secure dominating set of $\mathcal{G}$.

Assume that $\mathcal{V} - \mathcal{S}$ is not secure dominating set. Then there exist vertex $w \in \mathcal{S}$ is adjacent to a vertex $x \in \mathcal{V} - \mathcal{S}$ such that $(\mathcal{S} - \{x\}) \cup \{w\}$ is not dominating set. Thus x is not dominated by any vertex in $\mathcal{S}$ which is contradiction to our assumption that $\mathcal{S}$ is minimal secure dominating set and $\mathcal{G}$ has no isolated vertices and has only strong edges. So $\mathcal{V} - \mathcal{S}$ is a secure dominating set of $\mathcal{G}$. □

4. Conclusion

In this paper, we have discussed about secure domination in bipolar fuzzy graphs and obtained definitions as well as some results related to 2-secure domination in BFG.

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