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ON SOPHIE GERMAIN PRIMES

NECHEMIA BURSHTEIN

117 Arlozorov street
Tel Aviv 6209814, Israel

Abstract

A Sophie Germain prime is a prime number P such that $2P + 1$ is also prime. Some observations are made on the three sub-classes of the Sophie Germain prime pairs $(P, 2P + 1)$.

1. Introduction

Sophie Germain (1776-1831) was a French lady mathematician, physicist and philosopher. Among other fields, she was also known in Number Theory for her work on Fermat's Last Theorem, and for the Sophie Germain prime numbers.

A Sophie Germain prime is a prime number $\boldsymbol{P}$ such that $\boldsymbol{2P + 1}$ is also prime. The prime $\boldsymbol{P}$ is also called a "Sophie Germain number", whereas $\boldsymbol{2P + 1}$ is called a "safe prime". The first few Sophie Germain primes are $\boldsymbol{P} = 2, 3, 5, 11, 23, 29, \ldots$.

Twin primes are pairs of primes of the form $(p, p + 2)$. Such are (3, 5), (5, 7), (11, 13), (17, 19), and so on.

Numerous articles have been written on the Sophie Germain primes and the Twin primes. Only a very small abstract of authors is provided here, for example [1, 2, 3, 4].

It is conjectured that there are an infinite number of Twin primes, and also an infinite number of Sophie Germain pairs $(\boldsymbol{P}, \boldsymbol{2P + 1})$.

From [1] we cite the following conjecture.

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Conjecture. *The number of Sophie Germain primes* $\boldsymbol{P}$ *with* $\boldsymbol{P} \leq N$ *is approximately*

$$2C_2 \int_2^N \frac{dx}{\log x \log 2x} \sim \frac{2C_2 N}{(\log N)^2}$$

where $C_2 = 0.66016\ldots$ is the twin prime constant.

The above two conjectures are related, and it is extremely difficult to prove them.

From [5] we also cite: As of 29.2.2016, the largest known proven Sophie Germain prime $\boldsymbol{P}$ is

$$\boldsymbol{P} = 2618163402417 \cdot 2^{1290000} - 1$$

having 388342 decimal digits.

2. Some Observations on the pairs $(\boldsymbol{P}, \mathbf{2}\boldsymbol{P} + \mathbf{1})$

Evidently, the two Sophie Germain primes 2 and 5 which yield the pairs (2, 5) and (5, 11) are excluded from our discussion. Hence, all other pairs $(\boldsymbol{P}, \mathbf{2}\boldsymbol{P} + \mathbf{1})$ may be classified into three sub-classes as shown in the following Table 1.

Table 1.

Type of $(\boldsymbol{P}, \mathbf{2}\boldsymbol{P} + \mathbf{1})$	$\boldsymbol{P}$ last digit	$\mathbf{2}\boldsymbol{P} + \mathbf{1}$ last digit
Type 1	**1**	**3**
Type 2	**3**	**7**
Type 3	**9**	**9**

The three types of pairs appearing in Table 1 are considered in the forthcoming Tables 2, 3 and 4. In these tables we shall employ the following notation:

N-a power of 10.

A-the number of pairs of each type contained in N.

B-the total number of pairs contained in N.

C-the ratio A/B.

The values $N = 10^3$, 10^4 and 10^5 will be used respectively in Tables 2, 3 and 4 which are now exhibited in sequence.

Table 2. $\boldsymbol{P} < N = 10^3$.

Type of $(\boldsymbol{P}, \mathbf{2}\boldsymbol{P} + \mathbf{1})$	$\boldsymbol{P}$ last digit	$\mathbf{2}\boldsymbol{P} + \mathbf{1}$ last digit	A	B	C
Type 1	**1**	**3**	**11**	**35**	0.3142857
Type 2	**3**	**7**	**14**	**35**	0.4
Type 3	**9**	**9**	**10**	**35**	0.2857142

Table 3. $\boldsymbol{P} < N = 10^4$.

Type of $(\boldsymbol{P}, \mathbf{2}\boldsymbol{P} + \mathbf{1})$	$\boldsymbol{P}$ last digit	$\mathbf{2}\boldsymbol{P} + \mathbf{1}$ last digit	A	B	C
Type 1	**1**	**3**	**60**	**188**	0.3191489
Type 2	**3**	**7**	**66**	**188**	0.3510638
Type 3	**9**	**9**	**62**	**188**	0.3297872

Table 4. $\mathbf{P} < N = 10^5$.

Type of $(\boldsymbol{P}, \mathbf{2}\boldsymbol{P} + \mathbf{1})$	$\boldsymbol{P}$ last digit	$\mathbf{2}\boldsymbol{P} + \mathbf{1}$ last digit	A	B	C
Type 1	**1**	**3**	**382**	**1169**	0.326775
Type 2	**3**	**7**	**396**	**1169**	0.338751
Type 3	**9**	**9**	**391**	**1169**	0.3344739

The three tables yield some facts, and some questions may arise, such as: on average, are the three types of pairs equally distributed on the line of all pairs $(\boldsymbol{P}, \mathbf{2}\boldsymbol{P} + \mathbf{1})$? on the average, do these three types appear consecutively, i.e., a pair of Type 1 is followed by a pair of Type 2 and then by a pair of Type 3 ? For a fixed value of N, which of the three types contains the minimal/maximal number of pairs ? and so on.

It is clearly seen from Tables 2, 3 and 4 that in each table the number of pairs of Type 2 is the largest one among the three types. However, for values of $N > 10^5$ which are not considered here, this is probably not always the case. Moreover, in Table 4, it is observed that the smallest value A corresponds to pairs of Type 1. Nevertheless, the pairs of Type 1 contain a chain of six consecutive pairs. These are: (85931, 171863), (86111, 172223), (86171, 172343), (86291, 172583), (86441, 172883), (86771, 173543). This is the longest possible chain of pairs of any type in Table 4. Obviously, chains of 5, 4, 3, 2 consecutive pairs occur for all primes $\boldsymbol{P}$ in Table 4. The above six pairs imply that at short intervals of numbers as in the interval 85931-86771, and also in the smaller chains of consecutive pairs, the difference between the types is erratic. And yet, there are consecutive pairs which "behave as expected", i.e., a pair of Type 1 is followed by a pair of Type 2 and then by a pair of Type 3. Such are for example: (11, 23), (23, 47), (29, 59) and also (7541, 15083), (7643, 15287), (7649, 15299).

From Tables 2, 3 and 4, we also have for

Pairs of Type 1: $C_{1000} < C_{10000} < C_{100000} \cong 0.326 < 1/3$,

Pairs of Type 2: $C_{1000} > C_{10000} > C_{100000} \cong 0.338 > 1/3$,

Pairs of Type 3: $C_{1000} < C_{10000} < C_{100000} \cong 0.334 > 1/3$.

The above inequalities show that when N gets larger, then the deviations from $1/3$ get smaller and smaller.

Conclusion

Though we do not pursue this matter any further, we expect and conjecture that for sufficiently very large values of N, then for each of the Types 1, 2 and 3, the corresponding value A converges to $B/3$. The heuristic is based upon the above data.

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