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FIXED-POINTS THEOREMS FOR EXPANSIVE TYPE MAPPING ON *F*-CONE METRIC SPACES

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Abstract

Our goal in this article is to prove some fixed-point theorems for expansive maps in an F-cone metric space over Banach algebras, which unify, extend and generalize most of the existing relevant fixed-point theorems from the literature.

1. Introduction

Partial metric spaces were introduced by Matthews [1] in 1994. He studied a partial metric space as a part of the denotational semantics of dataflow networks and showed that the Banach contraction principle can be generalized to the partial metric context for applications in program verification. Especially, it has the property that differentiates it from other spaces, that is, the self-distance of any point may not be zero, also a convergent sequence need not have unique limit in these spaces.

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On the other hand, in 1989, the concept of b-metric spaces was introduced by Bakhtin [2] as a generalization of metric spaces. He showed the contraction mapping principle in a b-metric space that generalizes the prominent Banach contraction principle in metric spaces.

In the same spirit, recently, Huang and Zhang [3] replaced the set of real numbers by ordering Banach space and defined a cone metric space as a generalization of the metric space. The authors proved some fixed-point theorems of contractive mappings on cone metric spaces. They also defined the Cauchy sequence and convergence of a sequence in such spaces in terms of interior points of the underlying cone. After that, in [4], Rezapour and Hamlbarani generalized some results of [5] by omitting the assumption of normality. For fixed-point theorems on cone metric spaces, readers may refer to [6-9] and the references therein.

Malviya et al. [10] introduced the concept of N-cone metric spaces, which is a new generalization of the generalized G-cone metric space [11] and the generalized D^*-metric space [12]. They proved fixed-point theorems for asymptotically regular maps and sequences. Afterwards, in [13], the authors defined contractive maps in N-cone metric spaces and proved various fixed-point theorems for such maps.

Despite these features, some authors demonstrated that the fixed-point results proved on cone metric spaces are the straightforward outcome of the corresponding results of usual metric spaces where the real-valued metric function d^* is defined by a nonlinear scalarization function ξ_e (see [14]) or by a Minkowski functional q_e (see [5]).

Due to the concrete reasons mentioned above, researchers were losing their interest in a cone metric space. Fortunately, Liu and Xu [15] introduced the approach of cone metric spaces over Banach algebras by replacing Banach spaces E by Banach algebras A as the underlying spaces of cone metric spaces. They verified some fixed-point theorems of generalized Lipschitz mappings with weaker conditions on the generalized Lipschitz constant k by means of the spectral radius. Not long ago, Xu and Radenović [16] deleted the normality of cones and greatly generalized the main results of [15]. In particular, some authors have shown recently some fixed-point results given in [17-19].

Following these ideas, very recently, Fernandez et al. [6] introduced the notion of partial cone metric spaces over Banach algebra as a generalization of partial metric spaces and cone metric spaces over Banach algebra, which was selected for Young Scientist Award 2016, M.P., India (see [20]).

Recently, proceeding in this direction, Fernandez et al. introduced the structure of N_p-cone metric space over Banach algebra [21] as a generalization of N-cone metric space over Banach algebra [22] and partial metric space and N_b-cone metric space over Banach algebra [23] as a generalization of N-cone metric space over Banach algebra [22] and b-metric space, respectively.

Inspired and encouraged by the previous works, our goal in this article is to prove some fixed-point theorems for expansive maps in an F-cone metric space over Banach algebras, which unify, extend and generalize most of the existing relevant fixed-point theorems from the literature.

2. Preliminaries

We begin with the following definition as a recall from [15].

Let A always be a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined subject to the following properties $(\forall \eta, \xi, \mu \in A, \alpha \in R)$:

1. $(\eta\xi)\mu = \eta(\xi\mu)$;

2. $\eta(\xi + \mu) = \eta\xi + \eta\mu$ and $(\eta + \xi)\mu = \eta\mu + \xi\mu$;

3. $\alpha(\eta\xi) = (\alpha x)\xi = x(\alpha\xi)$;

4. $\| x\xi \| \leq \| \eta \| \| \xi \|$.

Throughout this paper, we shall assume that a Banach algebra has a unit (i.e., a multiplicative identity) e such that $e\eta = \eta e = \eta, \forall \eta \in A$. An element $\eta \in A$ is said to be invertible if there is an inverse element $\xi \in A$ such that $\eta\xi = \xi\eta = e$. The inverse of η is denoted by η^{-1}. For more details, we refer the reader to [24].

The following proposition is given in [24].

Proposition 2.1. *Let A be a Banach algebra with a unit e, and $\eta \in A$. If the spectral radius $\rho(\eta)$ of η is less than* 1, *i.e.*,

$$\rho(\eta) = \lim_{n\to\infty} \| \eta^n \|^{\frac{1}{n}} = \inf \| \eta^n \|^{\frac{1}{n}} < 1,$$

then $e - \eta$ is invertible. Actually,

$$(e - \eta)^{-1} = \sum_{i=0}^{\infty} \eta_i.$$

Remark 2.2. From [24] we see that the spectral radius $\rho(\eta)$ of η satisfies $\rho(\eta) \leq \| \eta \|, \forall \eta \in A$ where A is a Banach algebra with a unit e.

Remark 2.3 (see [16])**.** In Proposition 2.1, if the condition '$\rho(\eta) < 1$' is replaced by '$\| \eta \| < 1$,' then the conclusion remains true.

Remark 2.4 (see [16])**.** If $\rho(\eta) < 1$, then $\| \eta \|^n \to 0 (n \to \infty)$.

Lemma 2.5 (see [25])**.** *If E is a real Banach space with a solid cone P and if $\theta \preccurlyeq x \ll c$ for each $\theta \ll c$, then $x = \theta$.*

A subset P of A is called a cone of A if

1. P is nonempty closed and $\{\theta, e\} \subset P$,

2. $\alpha P + \beta P \subset P$ for all nonnegative real numbers α, β;

3. $P^2 = PP \subset P$,

4. $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra A. For a given cone $P \subset A$, we can define a partial ordering $\preccurlyeq$ with respect to P by $\eta \preccurlyeq \xi$ if and only if $\xi - \eta \in P$. $\eta \prec \xi$ will stand for $\eta \preccurlyeq \xi$ and $\eta = \xi$, while $\eta \ll \xi$ will stand for $\xi - \eta \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. If $\operatorname{int} P \neq \emptyset$, then P is called a solid cone.

The cone P is called normal if there is a number $M > 0$ such that,

$\forall \eta, \xi \in A, \theta \preccurlyeq \eta \preccurlyeq \xi \Rightarrow \| \eta \| \leq M \| \xi \|.$

The least positive number satisfying the above is called the normal constant of P [3].

In the following we always assume that A is a Banach algebra with a unit e, P is a solid cone in A and $\preccurlyeq$ is the partial ordering with respect to P.

Definition 2.6 ([3, 5])**.** Let X be a nonempty set. Suppose that the mapping $d : X \times X \to A$ satisfies

1. $\theta \prec d(\eta, \xi), \forall \eta, \xi \in X$ and $d(\eta, \xi) = \theta \Leftrightarrow \eta = \xi$;

2. $d(\eta, \xi) = d(\xi, \eta), \forall \eta, \xi \in X$;

3. $d(\eta, \xi) \preccurlyeq d(\eta, \mu) + d(\mu, \xi), \forall \eta, \xi, \mu \in X.$

Then d is called a cone metric on X, and (X, d) is called a cone metric space over Banach algebra A.

For other definitions and related results on cone metric space over Banach algebra, we refer to [15].

Definition 2.7 ([2])**.** Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric on X if, for all $\eta, \xi, \mu \in X$, the following conditions hold:

1. $d(\eta, \xi) = 0$ iff $\eta = \xi$;

2. $d(\eta, \xi) = d(\xi, \eta)$;

3. $d(\eta, \mu) \leq s[d(\eta, \xi) + d(\xi, \mu)]$.

In this case, the pair (X, d) is called a b-metric space. For more definitions and results on b-metric spaces, we refer the reader to [26].

Definition 2.8 ([1])**.** A partial metric on a nonempty set X is a function $p : X \times X \to R^+$ such that for all $\mu, \xi, \mu \in X$, the following conditions hold:

1. $\eta = \xi \Leftrightarrow p(\eta, \eta) = p(\eta, \xi) = p(\xi, \xi)$;

2. $p(\eta, \eta) \leq p(\eta, \xi)$;

3. $p(\eta, \xi) = p(\xi, \eta)$;

4. $p(\eta, \xi) \leq p(\eta, \mu) + p(\mu, \xi) - p(\mu, \mu)$.

The pair (X, p) is called a partial metric space. It is clear that if $p(\eta, \xi) = 0$, then from (1) and (2) $\eta = \xi$. But if $\eta = \xi$, $p(\eta, \xi)$ may not be 0.

Definition 2.9 ([10])**.** Let X be a nonempty set. A function $N : X^3 \to A$ is called N-cone metric on X if for all $\eta, \xi, \mu, a \in X$, the following conditions hold:

1. $\theta \preccurlyeq (\eta, \eta, \eta)$;

2. $N(\eta, \xi, \mu) = \theta$ iff $\eta = \xi = \mu$;

3. $N(\eta, \xi, \mu) \preccurlyeq N(\eta, \eta, a) + N(\xi, \xi, a) + N(\mu, \mu, a)$.

Then the pair (X, N) is called an N-cone metric space over Banach algebra A.

Definition 2.10 ([23])**.** An N_b-cone metric on a nonempty set X is a function $N_b : X^3 \to A$ such that for all $\eta, \xi, \mu, a \in X$.

1. $\theta \preccurlyeq N_b(\eta, \eta, \eta)$;

2. $N_b(\eta, \xi, \mu) = \theta$ iff $\eta = \xi = \mu$;

3. $N_b(\eta, \xi, \mu) \preccurlyeq s[N_b(\eta, \eta, a) + N_b(\xi, \xi, a) + N_b(\mu, \mu, a)]$.

The pair (X, N_b) is called an N_b-cone metric space over Banach algebra A. The number $s \geq 1$ is called the coefficient of (X, N_b).

Definition 2.11 ([21])**.** An N_p-cone metric on a nonempty set X is a function $N_p : X^3 \to A$ such that for all $\eta, \xi, \mu, a \in X$, the following conditions hold:

1. $\eta = \xi = \mu \Leftrightarrow N_p(\eta, \eta, \eta) = N_p(\xi, \xi, \xi) = N_p(\mu, \mu, \mu) = N_p(\eta, \xi, \mu)$;

2. $\theta \preccurlyeq N_p(\eta, \eta, \eta) \preccurlyeq N_p(\eta, \eta, \xi) \preccurlyeq N_p(\eta, \xi, \mu)$, for all $\eta, \xi, \mu \in X$ with $\eta = \xi = \mu$;

3. $N_p(\eta, \xi, \mu) \preccurlyeq N_p(\eta, \eta, a) + N_p(\xi, \xi, a) + N_p(\mu, \mu, a) - N_p(a, a, a)$.

The pair (X, N_p) is called an N_p-cone metric space over Banach algebra A.

Definition 2.12 ([30]). Let X be a nonempty set. A function $F : X^3 \to A$ is called F-cone metric on X if for any $\eta, \xi, \mu, a \in X$, the following conditions hold:

1. $\eta = \xi = \mu \Leftrightarrow F(\eta, \eta, \eta) = F(\xi, \xi, \xi) = F(\mu, \mu, \mu) = F(\eta, \xi, \mu)$;

2. $\theta \preccurlyeq F(\eta, \eta, \eta) \preccurlyeq F(\eta, \eta, \xi) \preccurlyeq F(\eta, \xi, \mu)$, for all $\eta, \xi, \mu \in X$ with η, ξ, μ;

3. $F(\eta, \xi, \mu) \preccurlyeq s[F(\eta, \eta, a) + F(\xi, \xi, a) + F(\mu, \mu, a)] - F(a, a, a)$.

Then the pair (X, F) is called an F-cone metric space over Banach algebra A. The number $s \geq 1$ is called the coefficient of (X, F).

Remark 2.13 ([30]). In an F-cone metric space over Banach algebra (X, F), if $\eta, \xi, \mu \in X$ and $F(\eta, \xi, \mu) = \theta$, then $\eta = \xi = \mu$, but the converse may not be true.

Example 2.14 ([30]). Let $A = C_1^R[0, 1]$ and define a norm on A by $\| \eta \| = \| \eta \|_\infty + \| \eta' \|_\infty$ for $\eta \in A$. Define multiplication in A as just point wise multiplication. Then A is a real unit Banach algebra with unit $e = 1$. Set $P = \{\eta \in A : \eta \geq 0\}$ is a cone in A. Moreover, P is not normal (see [4]). Let $X = [0, \infty)$. Define a mapping $F : X^3 \to A$ by

$$F(\eta, \xi, \mu)(t) = (\max\{\eta, \mu\})^2 + (\max\{\xi, \mu\})^2, (\max\{\eta, \mu\})^2 + (\max\{\xi, \mu\}^2)e^t$$

for all $\eta, \xi, \mu \in X$, and let $\alpha > 0$ be any constant. Then (X, F) is an F-cone metric space over Banach algebra A with the coefficient $s = 2$. But, it is not an N_p-cone metric space over Banach algebra since the triangle inequality is not satisfied; neither it is an N_b-cone metric space over Banach algebra A since for any $\eta > 0$, we have $N_b(\eta, \eta, \eta)(t) = 2x^2e^t \neq \theta$.

Lemma 2.15 ([30]). *Let (X, F) be an F-cone metric space over Banach algebra A. Then*

(a) *if* $(\eta, \xi, \mu) = \theta$, *then* $\eta = \xi = \mu$.

(b) *if* $\eta = \xi$, *then* $F(\eta, \eta, \xi) > \theta$.

Proposition 2.16 ([30]). *If* (X, F) *is an F-cone metric space over Banach algebra, then for all* $\eta, \xi \in X$, *we have* $F(\eta, \eta, \xi) = F(\xi, \xi, \eta)$.

Definition 2.17 ([30]). Let (X, F) be an F-cone metric space over Banach algebra A. Then, for $\eta \in X$ and $c > \theta$, the F-balls with center x and radius $c > \theta$ are

$$B_F(\eta, c) = \{\xi \in X : F(\eta, \eta, \xi) \ll F(\eta, \eta, \eta) + c\}$$

Definition 2.18 ([30]). Let (X, F) be an F-cone metric space over Banach algebra A. A sequence $\{\eta_n\}$ in (X, F) converges to a point $\eta \in X$ whenever for every $c \gg \theta$ there is a natural number N such that $F(\eta_n, \eta, \eta) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} \eta_n = \eta$ or $\eta_n \to \eta(\eta \to \infty)$.

Definition 2.19 ([30]). Let (X, F) be an F-cone metric space over Banach algebra A. A sequence $\{\eta_n\}$ in X is said to be a θ-Cauchy sequence in (X, F) if $\{F(\eta_n, \eta_m, \eta_m)\}$ is a c-sequence in A, i.e., if for every $c \gg \theta$ there exists $n_0 \in N$ such that $F(\eta_n, \eta_m, \eta_m) \ll c$ for all $n, m \geq n_0$.

Definition 2.17 ([30]). Let (X, F) be an F-cone metric space over Banach algebra A. Then X is said to be θ-complete if every θ-Cauchy sequence $\{\eta_n\}$ in (X, F) converges to $\eta \in X$ such that $F(\eta, \eta, \eta) = \theta$.

Definition 2.18 ([30]). Let (X, F) be an F-cone metric space over Banach algebra A and P be a cone in A. A map $T : X \to X$ is said to be a generalized Lipschitz mapping if there exists a vector $h \in P$ with $\rho(h) < 1$ for all $\eta, \xi \in X$ such that

$$F(f\eta, f\eta, f\xi) \preccurlyeq hF(\eta, \eta, \xi).$$

Now we review some facts on c-sequence theory.

Definition 2.19 ([27]). Let P be a solid cone in a Banach space E. A sequence $\{x_n\} \subset P$ is said to be a c-sequence if for each $c \gg \theta$ there exists a natural number N such that $x_n \ll c$ for all $n > N$.

Lemma 2.20 ([28]). *If E is a real Banach space with a solid cone P and* $\{x_n\} \subset P$ *is a sequence with* $\| x_n \| \to 0 \, (n \to \infty)$, *then* x_n *is a c-sequence.*

Lemma 2.21 ([24]). *Let A be a Banach algebra with a unit* $e, h \in A$, *then* $\lim_{n\to\infty} \| h^n \|^{\frac{1}{n}}$ *exists and the spectral radius* $\rho(h)$ *satisfies*

$$\rho(h) = \lim_{n\to\infty} \| h^n \|^{\frac{1}{n}} = \inf \| h^n \|^{\frac{1}{n}}$$

If $\rho(h) < |\omega|$, *then* $(\omega e - h)$ *is invertible in A*; *moreover*,

$$(\omega e - h)^{-1} = \sum_{i=0}^{\infty} \frac{h^i}{\omega^{i+1}}$$

where ω *is a complex constant.*

Lemma 2.22 ([24]). *Let A be a Banach algebra with a unit* $e, h, k \in A$. *If h commutes with k*, *then*

$$\rho(h + k) \leq \rho(h) + \rho(k),$$

$$\rho(hk) \leq \rho(h)\rho(k).$$

Lemma 2.23 ([28]). *If E is a real Banach space with a solid cone P*

(1) *If* $a_1, a_2, a_3 \in E$ *and* $a_1 \leq a_2 \ll a_3$, *then* $a_1 \ll a_3$.

(2) *If* $a_1 \in P$ *and* $a_1 \ll a_3$ *for each* $a_3 \gg \theta$, *then* $a_1 = \theta$.

Lemma 2.24 ([16]). *Let P be a solid cone in a Banach algebra A. Suppose that* $h \in P$ *and* $\{x_n\} \subset P$ *is a c-sequence. Then* $\{hx_n\}$ *is a c-sequence.*

Lemma 2.25 ([28]). *Let A be a Banach algebra with a unit e and* $h \in A$. *If* ω *is a complex constant and* $\rho(h) < |\omega|$, *then*

$$\rho((\omega e - h)^{-1}) \leq \frac{1}{|\omega| - \rho(h)}$$

Lemma 2.26 ([28]). *Let A be a Banach algebra with a unit e and P be a solid cone in A. Let* $h, k, l \in P$ *hold* $l \preccurlyeq k$ *and* $h \preccurlyeq lh$. *If* $\rho(k) < 1$, *then* $h = \theta$.

3. Main Results

Now, we give a simple but a useful Lemma.

Lemma 3.1. *Let* (X, F) *be a* θ*-complete F-cone metric space over Banach algebra A. Let* $\{u_n\} \subset X$ *be a sequence such that*

$$F(\eta_n, \eta_n, \eta_{n+1}) \preccurlyeq hF(\eta_{n-1}, \eta_{n-1}, \eta_n) \tag{3.1}$$

where $\rho(h) < 1$ *and* $n \in \mathbb{N}$. *Then* $\{\eta_n\}$ *is a* θ*-Cauchy sequence in X.*

Proof. By the simple induction with the condition (3.1), we have

$$\begin{aligned} F(\eta_n, \eta_n, \eta_{n+1}) &\preccurlyeq hF(\eta_{n-1}, \eta_{n-1}, \eta_n) \\ &\preccurlyeq h^2 F(\eta_{n-2}, \eta_{n-2}, \eta_{n-1}) \\ &\vdots \\ &\preccurlyeq h^n F(\eta_0, \eta_0, \eta_1) \end{aligned} \tag{3.2}$$

Now, if $n < m$, we have

$$F(\eta_n, \eta_n, \eta_m) \preccurlyeq s\begin{bmatrix} F(\eta_n, \eta_n, \eta_{n+1}) + F(\eta_n, \eta_n, \eta_{n+1}) + F(\eta_m, \eta_m, \eta_{n+1}) \\ - F(\eta_{n+1}, \eta_{n+1}, \eta_{n+1}) \end{bmatrix}$$

$$\preccurlyeq s[2F(\eta_n, \eta_n, \eta_{n+1}) + F(\eta_m, \eta_m, \eta_{n+1})]$$

$$= 2sF(\eta_n, \eta_n, \eta_{n+1}) + sF(\eta_{n+1}, \eta_{n+1}, \eta_m)$$

$$\preccurlyeq 2sF(\eta_n, \eta_n, \eta_{n+1}) + s^2\begin{bmatrix} F(\eta_{n+1}, \eta_{n+1}, \eta_{n+2}) \\ + F(\eta_{n+1}, \eta_{n+1}, \eta_{n+2}) \\ + F(\eta_m, \eta_m, \eta_{n+2}) \\ - F(\eta_{n+2}, \eta_{n+2}, \eta_{n+2}) \end{bmatrix}$$

$$\preccurlyeq 2sF(\eta_n, \eta_n, \eta_{n+1}) + s^2[2F(\eta_{n+1}, \eta_{n+1}, \eta_{n+2}) + F(\eta_m, \eta_m, \eta_{n+2})]$$

$= 2sF(\eta_n, \eta_n, \eta_{n+1}) + 2s^2F(\eta_{n+1}, \eta_{n+1}, \eta_{n+2})$

$+ s^2F(\eta_{n+2}, \eta_{n+2}, \eta_m)$

$\preccurlyeq 2sF(\eta_n, \eta_n, \eta_{n+1}) + 2s^2F(\eta_{n+1}, \eta_{n+1}, \eta_{n+2})$

$+ 2s^3F(\eta_{n+2}, \eta_{n+2}, \eta_{n+3}) + \ldots + 2s^{m-n}F(\eta_{m-1}, \eta_{m-1}, \eta_m)$

$\preccurlyeq 2s^3h^{n+2}F(\eta_0, \eta_0, \eta_1) + \ldots + 2s^{m-n}h^{m-n}F(\eta_0, \eta_0, \eta_1)$

$= (2sh^n + 2s^2h^{n+1} + 2s^3h^{n+2} + \ldots + 2s^{m-n}h^{m-1})F(\eta_0, \eta_0, \eta_1)$

$\preccurlyeq (2(sh)^n + 2(sh)^{n+1} + 2(sh)^{n+2} + \ldots + 2(sh)^{m-1})F(\eta_0, \eta_0, \eta_1)$

$= 2s^nh^n(e + (sh) + (sh)^2 + \ldots + (sh)^{m-n-1})F(\eta_0, \eta_0, \eta_1)$

$$\preccurlyeq 2(sh)^n(e - sh)^{-1}F(\eta_0, \eta_0, \eta_1) \tag{3.3}$$

By Remark 2.4, $\| (sh)^n F(\eta_0, \eta_0, \eta_1) \| \leq \| (sh)^n \| \| F(\eta_0, \eta_0, \eta_1) \| \to 0$. By Lemma 2.20, we have $\{(sh)^n F(\eta_0, \eta_0, \eta_1)\}$ is a c-sequence. Next, by using Lemmas 2.23 and 2.24, we conclude that $\{\eta_n\}$ is a θ-Cauchy sequence in X.

Definition 3.2 ([30])**.** Suppose (X, F) is an F-cone metric space over a Banach algebra A and P be a cone in A. A mapping $f : X \to X$ is said to be an expansive mapping where $h, h^{-1} \in P$ are called the generalized Lipschitz constants with $\rho(h^{-1}) < 1$ for all $\eta, \xi \in X$ such that

$$F(f\eta, f\eta, f\xi) \succcurlyeq hF(\eta, \eta, y)$$

Now, we give some fixed-point results for expansive mappings in an θ-complete F-cone metric space over a Banach algebra A.

Theorem 3.3. *Let (X, F) be a θ-complete F-cone metric space over Banach algebra, and let P be an underlying solid cone, where $k_1, k_2, k_3, -k_1 \in P$ are generalized Lipschitz constants with $\rho[(e - k_1)(k_2 + k_3)^{-1}] < \frac{1}{s}$. Suppose that $f : X \to X$ is a surjective mapping satisfying the following condition*:

$$F(f\eta,\ f\eta,\ f\xi) + k[F(\eta,\ \eta,\ f\xi) + F(\xi,\ \xi,\ f\eta)]$$

$$\succcurlyeq k_1 F(\eta,\ \eta,\ f\eta) + k_2 F(\xi,\ \xi,\ f\xi) + k_3 F(\eta,\ \eta,\ \xi) \tag{3.4}$$

for all $\eta,\ \xi \in X$. *Then f has a unique fixed-point in X.*

Proof. Let us denote the inverse mapping of f by g. Let $\eta_0 \in X$ and define the sequence $\{\eta_n\}$ as follows:

$$\eta_1 = g\eta_0,\ \eta_2 = g\eta_1 = g^2\eta_0,$$

$$\eta_3 = g\eta_2 = gg^2\eta_0 = g^3\eta_0,\ \ldots,\ \eta_{n+1} = g\eta_n = g^{n+1}\eta_0 \tag{3.5}$$

Suppose that $\eta_n \neq \eta_{n+1}$ for all n. Using (3.4) and (3.5), we have

$$\begin{aligned}
F(\eta_{n-1},\ \eta_{n-1},\ \eta_n) &= F(ff^{-1}\eta_{n-1},\ ff^{-1}\eta_{n-1},\ ff^{-1}\eta_n) \\
&\succcurlyeq k_1 F(f^{-1}\eta_{n-1},\ f^{-1}\eta_{n-1},\ ff^{-1}\eta_{n-1}) \\
&+ k_2 F(f^{-1}\eta_n,\ f^{-1}\eta_n,\ ff^{-1}\eta_n) \\
&+ k_3 F(f^{-1}\eta_{n-1},\ f^{-1}\eta_{n-1},\ f^{-1}\eta_n) \\
&- k\begin{bmatrix} F(f^{-1}\eta_{n-1},\ f^{-1}\eta_{n-1},\ ff^{-1}\eta_n) \\ + F(f^{-1}\eta_n,\ f^{-1}\eta_n,\ ff^{-1}\eta_{n-1}) \end{bmatrix} \\
&= k_1 F(g\eta_{n-1},\ g\eta_{n-1},\ \eta_{n-1}) \\
&+ k_2 F(g\eta_n,\ g\eta_n,\ \eta_n) \\
&+ k_3 F(g\eta_{n-1},\ g\eta_{n-1},\ g\eta_n) \\
&- k\begin{bmatrix} F(g\eta_{n-1},\ g\eta_{n-1},\ \eta_n) \\ + F(g\eta_n,\ g\eta_n,\ \eta_{n-1}) \end{bmatrix}
\end{aligned}$$

$$\succcurlyeq k_1 F(\eta_n,\ \eta_n,\ \eta_{n-1}) + k_2 F(\eta_{n+1},\ \eta_{n+1},\ \eta_n) + k_3 F(\eta_n,\ \eta_n,\ \eta_{n+1})$$

$$- k[F(\eta_n,\ \eta_n,\ \eta_n) + F(\eta_{n+1},\ \eta_{n+1},\ \eta_{n-1})]$$

$$\succcurlyeq k_1 F(\eta_{n-1},\ \eta_{n-1},\ \eta_n) + k_2 F(\eta_n,\ \eta_n,\ \eta_{n+1}) + k_3 F(\eta_n,\ \eta_n,\ \eta_{n+1})$$

$$-k\begin{bmatrix} F(\eta_n, \eta_n, \eta_{n-1}) + sF(\eta_{n+1}, \eta_{n+1}, \eta_n) + sF(\eta_{n+1}, \eta_{n+1}, \eta_n) \\ + sF(\eta_{n-1}, \eta_{n-1}, \eta_n) - sF(\eta_n, \eta_n, \eta_n) \end{bmatrix}$$

$$= k_1 F(\eta_{n-1}, \eta_{n-1}, \eta_n) + k_2 F(\eta_n, \eta_n, \eta_{n+1}) + k_3 F(\eta_n, \eta_n, \eta_{n+1})$$

$$-k\begin{bmatrix} F(\eta_{n-1}, \eta_{n-1}, \eta_n) + 2sF(\eta_n, \eta_n, \eta_{n+1}) \\ + sF(\eta_{n-1}, \eta_{n-1}, \eta_n) - sF(\eta_n, \eta_n, \eta_{n+1}) \end{bmatrix}$$

Hence

$$(k_2 + k_3 - 2sk)F(\eta_n, \eta_n, \eta_{n+1}) \preccurlyeq (e + k + sk - k_1)F(\eta_{n-1}, \eta_{n-1}, \eta_n)$$

Put $r = k_2 + k_3 - 2sk$, then

$$rF(\eta_n, \eta_n, \eta_{n+1}) \preccurlyeq (e + k + sk - k_1)F(\eta_{n-1}, \eta_{n-1}, \eta_n) \tag{3.6}$$

Since r is invertible, to multiply r^{-1} on both sides of (3.6), we have

$$F(\eta_n, \eta_n, \eta_{n+1}) \preccurlyeq hF(\eta_{n-1}, \eta_{n-1}, \eta_n)$$

where $h = (e + k + sk - k_1)(k_2 + k_3 - 2sk)^{-1}$ and note that $\rho(h) < \frac{1}{s}$. Hence by Lemma 3.1, the sequence $\{\eta_n\}$ is a θ-Cauchy sequence. Moreover, by the θ-completeness of X, there exists $\eta^* \in X$ such that

$$\lim_{n\to\infty} F(\eta_n, \eta_n, \eta^*) = \lim_{n,\, m\to\infty} F(\eta_n, \eta_n, \eta_m) = F(\eta^*, \eta^*, \eta^*) = \theta$$

Since f is surjective map and hence there exists a point ξ in X such that $\eta^* = f\xi$.

Consider

$$F(\eta_n, \eta_n, \eta^*) = F(ff^{-1}\eta_n, ff^{-1}\eta_n, f\xi)$$

$$\succcurlyeq k_1 F(f^{-1}\eta_n, f^{-1}\eta_n, ff^{-1}\eta_n) + k_2 F(\xi, \xi, f\xi)$$

$$+ k_3 F(f^{-1}\eta_n, f^{-1}\eta_n, \xi)$$

That is,

$$F(\eta_n, \eta_n, \eta^*) \succcurlyeq k_1 F(g\eta_n, g\eta_n, \eta_n) + k_2 F(\xi, \xi, \eta^*) + k_3 F(g\eta_n, g\eta_n, \xi)$$

$$- kF(g\eta_n, g\eta_n, \eta^*) - kF(\xi, \xi, \eta_n)$$

$$= k_1 F(\eta_{n+1}, \eta_{n+1}, \eta_n) + k_2 F(\xi, \xi, \eta^*) + k_3 F(\eta_{n+1}, \eta_{n+1}, \xi)$$

$$- kF(\eta_{n+1}, \eta_{n+1}, \eta^*) - kF(\xi, \xi, \eta_n)$$

Since

$$F(\xi, \xi, \eta^*) \preccurlyeq s[2F(\xi, \xi, \eta_{n+1}) + F(\eta^*, \eta^*, \eta_{n+1}) - F(\eta_{n+1}, \eta_{n+1}, \eta_{n+1})]$$

So

$$k_1 F(\eta_{n+1}, \eta_{n+1}, \eta_n) + k_3 F(\eta_{n+1}, \eta_{n+1}, \xi) \preccurlyeq F(\eta_n, \eta_n, \eta^*) - k_2 F(\xi, \xi, \eta^*)$$

$$- kF(\eta_{n+1}, \eta_{n+1}, \eta^*) - kF(\xi, \xi, \eta_n)$$

$$\preccurlyeq F(\eta_n, \eta_n, \eta^*) - k_2 s[2F(\xi, \xi, \eta_{n+1}) + F(\eta^*, \eta^*, \eta_{n+1}) - F(\eta_{n+1}, \eta_{n+1}, \eta_{n+1})]$$

$$- kF(\eta_{n+1}, \eta_{n+1}, \eta^*) - kF(\xi, \xi, \eta_n)$$

$$\preccurlyeq s[2F(\eta_n, \eta_n, \eta_{n+1}) + F(\eta^*, \eta^*, \eta_{n+1}) - F(\eta_{n+1}, \eta_{n+1}, \eta_{n+1})]$$

$$- k_2 s[2F(\xi, \xi, \eta_{n+1}) + F(\eta^*, \eta^*, \eta_{n+1}) - F(\eta_{n+1}, \eta_{n+1}, \eta_{n+1})]$$

$$- kF(\eta_{n+1}, \eta_{n+1}, \eta^*) - kF(\xi, \xi, \eta_n)$$

That is,

$$k_1 F(\eta_{n+1}, \eta_{n+1}, \eta_n) + k_3 F(\eta_{n+1}, \eta_{n+1}, \xi)$$

$$\preccurlyeq s[2F(\eta_{n+1}, \eta_{n+1}, \eta_n) + F(\eta_{n+1}, \eta_{n+1}, \eta^*)]$$

$$- (k_2 s)[2F(\eta_{n+1}, \eta_{n+1}, \xi) + F(\eta_{n+1}, \eta_{n+1}, \eta^*)]$$

which implies

$$(2sk_2 + k_3 - sk) F(\eta_{n+1}, \eta_{n+1}, \xi)$$

$$\preccurlyeq (2sk + 2s - k_1) F(\eta_{n+1}, \eta_{n+1}, \eta_n)$$

$$+ (s - sk_2 + k) F(\eta_{n+1}, \eta_{n+1}, \eta^*)$$

Since $2sk_2 + k_3 - sk = r$ is invertible, we have

$$F(\eta_{n+1}, \eta_{n+1}, \xi)$$

$$\preccurlyeq r^{-1}\{(2sk + 2s - k_1)F(\eta_{n+1}, \eta_{n+1}, \eta_n)$$

$$+ (s - sk_2 + k)F(\eta_{n+1}, \eta_{n+1}, \eta^*)\}$$

Now that $\{F(\eta_{n+1}, \eta_{n+1}, \eta_n)\}$ and $\{F(\eta_{n+1}, \eta_{n+1}, \eta^*)\}$ are c-sequences, then by using Lemmas 2.23 and 2.24, we acquire that $\{F(\eta_{n+1}, \eta_{n+1}, \xi)\}$ is a c-sequence, thus $\eta_{n+1} \to \xi (n \to \infty)$. Hence $\xi = \eta^* = f\xi$. In the following we shall show T has a unique fixed-point.

Finally, we prove the uniqueness of the fixed-point. In fact, if η is another fixed-point,

$$F(\eta, \eta, \xi) = F(f\eta, f\eta, f\xi)$$

$$\succcurlyeq k_1 F(\eta, \eta, f\eta) + k_2 F(\xi, \xi, f\xi) + k_3 F(\eta, \eta, \xi)$$

$$- kF(\eta, \eta, f\xi) - kF(\xi, \xi, f\eta)$$

$$= k_1 F(\eta, \eta, \eta) + k_2 F(\xi, \xi, \xi) + k_3 F(\eta, \eta, \xi)$$

$$- kF(\eta, \eta, \xi) - kF(\xi, \xi, \eta)$$

$$= k_3 F(\eta, \eta, \xi) - 2kF(\eta, \eta, \xi)$$

The last inequality gives

$$(k_3 - 2k - e)F(\eta, \eta, \xi) \preccurlyeq \theta$$

which means $F(\eta, \eta, \xi) = \theta$ which implies that $\eta = \xi$, a contradiction. Hence the fixed-point is unique.

Corollary 3.4. *Let (X, F) be a θ-complete F-cone metric space over Banach algebra, and let P be an underlying solid cone, where $k \in P$ are generalized Lipschitz constants with $\rho[(k)^{-1}] < \frac{1}{s}$. Suppose that $f : X \to X$ is a surjective mapping satisfying the following condition*:

$$(f\eta, f\eta, f\xi) \succcurlyeq kF(\eta, \eta, \xi) \tag{3.7}$$

for all $\eta, \xi \in X$. *Then f has a unique fixed-point in X.*

Proof. If we put $k_1 = k_2 = \theta$, $3 = k$ Theorem 3.3, then we get the above Corollary 3.4. which is an extension of Theorem 1 of Wang et al. [29] in an F-cone metric space over Banach algebra.

Corollary 3.5. *Let* (X, F) *be a* θ*-complete F-cone metric space over Banach algebra, and let P be an underlying solid cone, where* $k \in P$ *are generalized Lipschitz constants with* $\rho[(k)^{-1}] < \frac{1}{s}$. *Suppose that* $f : X \to X$ *is a surjective mapping and suppose that there exists a positive integer n satisfying*

$$F(f^n\eta, f^n\eta, f^n\xi) \succcurlyeq kF(\eta, \eta, \xi) \tag{3.8}$$

for all $\eta, \xi \in X$. *Then f has a unique fixed-point in X.*

Proof. From Corollary 3.4, f^n has a unique fixed-point η. But $f^n(f\eta) = f(f^n\eta) = f\eta$, so $f\eta$ is also a fixed-point of f^n. Hence $f\eta = \eta$, η is a fixed-point of. Since the fixed-point of *f* is also a fixed-point of f^n, the fixed-point of *f* is unique.

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