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SOME BOUNDARY VALUE PROBLEMS ON THE UPPER HALF PLANE

ARUN CHAUDHARY* and RAVINDRA KUMAR

Department of Mathematics
Rajdhani College, University of Delhi
Delhi 110 015, India
E-mail: arunchaudhary@rajdhani.du.ac.in
ravindra.kumar@rajdhani.du.ac.in

Abstract

The present work provides explicit representation of Half-Neumann boundary value problem on the upper half plane. The solution of inhomogeneous polyanalytic equation arising from the combination of $(n-1)$ Dirichlet and 1-Half Neumann boundary conditions is also determined on the upper half plane $\mathbb{H}$.

1. Introduction

In this article, we introduce the concept of Half-Neumann boundary value problem on the upper half plane and provide its solution. In case of Neumann boundary value condition the derivative is obtained via outer normal derivative which is defined as $\partial_y = i(\partial_z - \partial_{\bar{z}})$ for the upper half plane [1, 2]. In case of Half Neumann it is taken as ∂_z, which makes it different from Neumann boundary problems. In the last section, we will also provide the solution of inhomogeneous polyanalytic equation on $\mathbb{H}$ arising from the combination of Dirichlet and Half-Neumann boundary conditions. These type of problems for polyanalytic functions are also studied on regular domains see [7, 8, 9, 10, 11, 12, 13, 14]. The complex form of Cauchy-Pompeiu formula and Gauss theorem on $\mathbb{H}$ were deduced by taking limiting case of their corresponding representation formulas on regular domain see [4, 5]. The area

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*Corresponding author.
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integral given in the Cauchy-Pompeiu formula is known as the Pompeiu operator, was studied by Vekua [6]. For a regular domain D, if $f \in L_p(D, \mathbb{C}),\ p > 1,$ then the Pompeiu operator Tf possesses weak derivatives and

$$\frac{\partial}{\partial \bar{z}}(Tf) = f,\ \frac{\partial}{\partial z}(Tf) = \Pi f,$$

where Πf (in the principal value sense) is a singular integral. In case of Upper half plane if $w : \mathbb{H} \to \mathbb{C}$ satisfies $|w(x)| \leq C|x|^{-\epsilon}$ for $|x| > k$ and $w_{\bar{z}} \in L_1(\mathbb{H}; \mathbb{C}),$ then the Cauchy-Pompeiu formula is given by

$$w(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} w(t)\frac{dt}{t - z} - \frac{1}{\pi}\int_{0<\mathrm{Im}\,\zeta} w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

$$w(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} w(t)\frac{dt}{t - \bar{z}} - \frac{1}{\pi}\int_{0<\mathrm{Im}\,\zeta} w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}},$$

where $z \in \mathbb{H}$ upper half plane, the Pompeiu operator T has the following form:

$$Tf(z) = -\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

T satisfies the properties $\partial\bar{z}(Tf) = f,\ \partial z(Tf) = \Pi f$ where

$$\Pi f(z) = -\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}.$$

Here the derivatives are taken in distributional sense. Higher order operators are well studied by Hile [3].

2. Half-Neumann Boundary Value Problem

For analytic functions, solution of the Half-Neumann boundary value problem on $\mathbb{H}$ may be written as:

Theorem 2.1. *$w_{\bar{z}} = 0$ in $\mathbb{H}$, $\partial_z w = \gamma$ on $\mathbb{R}$, $w(i) = c$ is uniquely solvable for $\gamma \in L_2(\mathbb{R}; \mathbb{C}) \cap C(\mathbb{R}; \mathbb{C}),\ c \in \mathbb{C}$ if and only for $z \in \mathbb{H}$*

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma(t)\frac{dt}{t-\overline{z}} = 0 \tag{2.1}$$

solution then is given as

$$w(z) = c + \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t))\log\left(\frac{t-i}{t-z}\right)dt. \tag{2.2}$$

The solution of inhomogeneous Half-Neumann boundary value problem is given as follows:

Theorem 2.2. *The Half Neumann problem* $w_{\overline{z}} = f$ *in* $\mathbb{H}$, $\partial_z w = \gamma$ *on* $\mathbb{R}$, $w(i) = c$ *is uniquely solvable for* $f \in L_{p,2}(\mathbb{H};\mathbb{C}) \cap C^1(\overline{\mathbb{H}};\mathbb{C}) \cap L_2(\mathbb{R};\mathbb{C})$, $\gamma \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$, $c \in \mathbb{C}$ *if and only for* $z \in \mathbb{H}$

$$\frac{\mathrm{i}}{2\pi i}\int_{-\infty}^{\infty}\gamma(t)\frac{dt}{t-\overline{z}} - \frac{1}{\pi}\int_{\mathbb{H}} f_\zeta(\zeta)\frac{d\xi d\eta}{\zeta-\overline{z}} = 0 \tag{2.3}$$

solution then is given as

$$w(z) = c + \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t) + f(t))\log\left(\frac{t-i}{t-z}\right)dt$$

$$-\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{z-i}{(\zeta-z)(\zeta-i)}d\xi d\eta. \tag{2.4}$$

Proof. Representing $w = \phi + Tf$ with analytic ϕ, where ϕ satisfies

$$\phi' = \partial_z w - \partial_z(Tf) = \partial_z w - \pi f \text{ in } \mathbb{H}$$

where

$$\pi f(z) = -\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{(\zeta-z)^2}$$

Representing,

$$\pi f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} f(t)\frac{ds}{s-z} - \frac{1}{\pi}\int_{\mathbb{H}} f_\zeta(\zeta)\frac{d\xi d\eta}{\zeta-z}$$

we see for $t \in \mathbb{R}$,

$$(\pi f)^+(t) = \lim_{\substack{z\to t\\ z\in\mathbb{H}}}\pi f(z) = -\frac{1}{2}f(s) - \frac{1}{2\pi i}\int_{-\infty}^{\infty} f(s)\frac{ds}{s-t} - \frac{1}{\pi}\int_{\mathbb{H}} f_\zeta(\zeta)\frac{d\xi d\eta}{\zeta-z}. \tag{2.5}$$

Thus, the boundary values of ϕ' on $\mathbb{R}$ are

$$\phi' = \gamma - (\pi f)^{+} = \gamma + \frac{1}{2} f + \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(s) \frac{ds}{s-t} + \frac{1}{\pi} \int_{\mathbb{H}} f_{\zeta}(\zeta) \frac{d\xi d\eta}{\zeta - z}.$$

Now, applying the result of Dirichlet theorem on $\mathbb{H}$ see [4], then the solution to this Dirichlet problem is

$$\phi'(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\gamma(t) + \frac{1}{2} f(t) + \frac{1}{2\pi i} \int_{\infty}^{\infty} f(s) \frac{ds}{s-t} + \frac{1}{\pi} \int_{\mathbb{H}} f_{\zeta}(\zeta) \frac{d\xi d\eta}{\zeta - t} \right] \frac{ds}{t-z} \tag{2.6}$$

if and only if

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\gamma(t) + \frac{1}{2} f(t) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(s) \frac{ds}{s-t} + \frac{1}{\pi} \int_{\mathbb{H}} f_{\zeta}(\zeta) \frac{d\xi d\eta}{\zeta - t} \right] \frac{ds}{t - \bar{z}} = 0 \tag{2.7}$$

because for $z \in \mathbb{H}$

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\pi i} \left(\int_{-\infty}^{\infty} f(s) \frac{ds}{s-t} \right) \frac{dt}{t-z} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(s)}{2} \frac{ds}{s-z}$$

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\pi i} \left(\int_{\mathbb{H}} f_{\zeta}(\zeta) \frac{d\xi d\eta}{\zeta - t} \right) \frac{dt}{t-z} = 0$$

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\pi i} \left(\int_{-\infty}^{\infty} f(s) \frac{ds}{s-t} \right) \frac{dt}{t-z} = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(s)}{2} \frac{ds}{s - \bar{z}}$$

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\pi} \left(\int_{\mathbb{H}} f_{\zeta}(\zeta) \frac{d\xi d\eta}{\zeta - t} \right) \frac{dt}{t - \bar{z}} = \frac{1}{\pi} \int_{\mathbb{H}} f_{\zeta}(\zeta) \frac{d\xi d\eta}{\zeta - \bar{z}} = Tf\zeta(\bar{z}).$$

then

$$\phi'(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\gamma(t) + \frac{1}{2} f(t) \right] \frac{ds}{t-z} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(s)}{2} \frac{ds}{s-z}$$

i.e.

$$\phi'(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} [\gamma(t) + f(t)] \frac{dt}{t-z} \tag{2.8}$$

if and only if

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\gamma(t)+\frac{1}{2}f(t)\right]\frac{ds}{t-\bar{z}}-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(s)}{2}\frac{ds}{s-\bar{z}}+Tf_{\zeta}(\bar{z})=0$$

i.e.

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma(t)\frac{dt}{t-\bar{z}}-\frac{1}{\pi}\int_{\mathbb{H}}f_{\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-\bar{z}}=0 \tag{2.9}$$

which is (2.3).

Integrating (2.8) leads to

$$\phi(z)=\phi(i)+\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t)+f(t))\log\left(\frac{t-i}{t-z}\right)dt$$

so that

$$w(z)=c+\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t)+f(t))\log\left(\frac{t-i}{t-z}\right)dt$$

$$-\frac{1}{\pi}\int_{\mathbb{H}}f(\zeta)\frac{z-i}{(\zeta-z)(\zeta-i)}d\xi d\eta$$

which is (2.4).

It can be easily verified at $z = i$ that $w(i) = c$. Taking derivative of (2.4) w.r.t. we will get

$$\partial_z w(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}[\gamma(t)+f(t)]\frac{dt}{t-z}+\pi f(z). \tag{2.10}$$

Subtracting (2.3) from (2.10), we have

$$w_z(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\gamma(t)\frac{ydt}{|t-z|^2}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(t)\frac{dt}{t-z}+\pi f(z)-Tf_{\zeta}(\bar{z}).$$

Now, letting z tend to $t \in \mathbb{R}$

$$w_z^{+}(t)=\gamma(t)+(\pi f)^{+}(t)-Tf_{\zeta}(t)+\lim_{\substack{z\to t\\ z\in\mathbb{H}}}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(t)\frac{dt}{t-z}\right].$$

Using Plemelj-Sokhotzki formula [4], we have

$$\lim_{\substack{z\to t\\ z\in\mathbb{H}}}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(s)\frac{ds}{s-z}\right] = \frac{1}{2}f(t) + \int_{-\infty}^{\infty} f(s)\frac{ds}{s-t}$$

$$= -(\pi f)^{+}(t) + Tf_{\zeta}(t) \quad \text{[using (2.5)]}$$

therefore $w_z^+(t) = \gamma(t)$.

3. Dirichlet-Half Neumann mix Problem on the Upper Half Plane

We require complex form of Cauchy-Pompeiu formula and Gauss theorem on $\mathbb{H}$. In order to solve boundary value problems for higher orders. Let $\mathcal{F}_k$ be the space of functions w in $W^{k,1}(\mathbb{H}, \mathbb{C})$ for which $\lim_{R\to\infty} R^{v} M(\partial_{\bar{z}}^{v} w, R) = 0$, $0 \le v \le k-1$ where $M(\partial_{\bar{z}}^{v} w, R) = \max_{\substack{|z|=R\\ 0<\operatorname{Im} z}} |\partial_{\bar{z}}^{v} w(z)|$ and $\bar{z}^{k-2}\partial_{\bar{z}}^{k} w \in L^1(\mathbb{H}, \mathbb{C})$. Using (Theorem 4 from [4]), it follows that for each $w \epsilon \mathcal{F}_k$ it may be written as

$$w(z) = \sum_{v=0}^{k-1}\frac{1}{2\pi i}\frac{1}{v!}\int_{-\infty}^{\infty}\frac{(\overline{z-\zeta})^{v}}{(\zeta - z)}\partial_{\bar{\zeta}}^{v} w(\zeta)d\zeta$$

$$-\frac{1}{(k-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\frac{(\overline{z-\zeta})^{k-1}}{(\zeta - z)}\partial_{\bar{\zeta}}^{k} w(\zeta)d\xi d\eta = 0 \tag{3.1}$$

for $z \in \mathbb{H}$. Moreover

$$\frac{(-1)^{n}}{(n-1)!\,\pi}\frac{1}{\pi}\int_{\mathbb{H}}\frac{(\bar{\zeta} - z)^{n-1}}{(\zeta - \bar{z})}\partial_{\bar{\zeta}}^{n} w(\zeta)d\xi d\eta$$

$$= -\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{1}{\lambda!}\int_{-\infty}^{\infty}\frac{(t-z)^{\lambda}}{(t-\bar{z})}\partial_{\bar{\zeta}}^{\lambda} w(t)dt. \tag{3.2}$$

Theorem 3.1. *For the inhomogeneous polyanalytic equation having* $(n-1)$ *Dirichlet-Half Neumann boundary conditions on the upper half plane* $\mathbb{H}$,

$$\partial_{\bar{z}}^{n} w = f \; in \; \mathbb{H}, \; \partial_{\bar{z}}^{\lambda} w = \gamma_{\lambda} \; on \; \mathbb{R}, 1 \leq \lambda \leq n-2,$$

$$\partial_{z}(\partial_{\bar{z}}^{n-1} w(z)) = \gamma_{n-1} \; on \; \mathbb{R}, \; on \; \mathbb{R}\, \partial_{\bar{z}}^{n-1} w(i) = c, \tag{3.3}$$

is uniquely solvable for $f \in L_{p,2}(\mathbb{H}; \mathbb{C}) \cap C(\overline{\mathbb{H}}; \mathbb{C}), \; p \geq 2, \; t^{\lambda}\gamma_{\lambda} \in L^{p}(\mathbb{R}; \mathbb{C}) \cap C(\mathbb{R}; \mathbb{C}), \; 0 \leq \lambda \leq n-1$ *if and only for* $0 \leq v \leq n-2$,

$$\frac{(-1)^{n-v}}{(n-v-1)!} \frac{1}{\pi} \int_{\mathbb{H}} f(\zeta) \left\{ \frac{(\bar{\zeta} - z)^{n-v-1}}{(\zeta - \bar{z})} - \frac{z^{n-v-1}}{(\zeta - i)} \right\} d\xi d\eta$$

$$+ \frac{(-1)^{n-v-1}}{(n-v-1)!} z^{n-v-1} c + \sum_{\lambda=v}^{n-2} \frac{1}{2\pi i} \frac{(-1)^{\lambda-v}}{(\lambda-v)} \int_{-\infty}^{\infty} \gamma_{v}(t)(t-z)^{\lambda-v} \frac{dt}{t-\bar{z}}$$

$$+ \frac{1}{2\pi i} \frac{(-1)^{n-v}}{(n-v-1)!} \int_{-\infty}^{\infty} [\gamma_{n-1}(t) + f(t)] \{h_{v}(t, z) - z^{n-v-1} \log(t-i)\} dt = 0 \tag{3.4}$$

Where

$$\{h_{v}(t, z) = (t-z)^{n-1-v} \log(t-\bar{z})$$

$$+ \sum_{\lambda=v}^{n-2} (-1)^{\mu-1}(n-v-1), \ldots, (n-v-\mu-1)(t-z)^{n-v-\mu} I_{\mu-1}(t, z),$$

$$I_{\mu}(t, z) = \frac{(t-z)^{\mu}}{\mu!} \left(\log(t-\bar{z}) - \sum_{r=1}^{\mu} \frac{1}{r} \right) \tag{3.5}$$

and

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \gamma_{n-1}(t) \frac{dt}{t-\bar{z}} - \frac{1}{\pi} \int_{\mathbb{H}} f_{\zeta}(\zeta) \frac{d\xi d\eta}{(\zeta - \bar{z})} = 0. \tag{3.6}$$

Then, the solution is given by

$$w(z) = \frac{(-1)^{n-1}}{(n-1)!} \bar{z}^{n-1} c + \sum_{\lambda=0}^{n-2} \frac{1}{2\pi i} \frac{1}{\lambda!} \int_{-\infty}^{\infty} \gamma_{\lambda}(t) (\overline{z-t})^{\lambda} \frac{dt}{t-z}$$

$$- \frac{1}{2\pi i} \frac{(-1)^{n-1}}{(n-1)!} \int_{-\infty}^{\infty} [\gamma_{n-1}(t) + f(t)](t-\bar{z})^{n-v-1} \log(t-z) dt$$

$$-\frac{1}{2\pi i}\frac{(-1)^{n-1}}{(n-1)!}\bar{z}^{n-1}\int_{-\infty}^{\infty}[\gamma_{n-1}(t)+f(t)]\{h_0(t,\bar{z})-\bar{z}^{n-1}\log(t-i)\}dt$$

$$+\frac{(-1)^{n-1}}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{H}}f(\zeta)\left\{\frac{(\overline{\zeta-z})^{n-1}}{(\zeta-z)}-\frac{\bar{z}^{n-1}}{(\zeta-i)}\right\}d\xi d\eta = 0. \tag{3.7}$$

Proof. Applying Cauchy-Pompeiu formula and (3.1) representation formula, we have

$$\frac{(-1)^{n-1}}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{H}}f(\zeta)\left\{\frac{(\overline{\zeta-z})^{n-1}}{(\zeta-z)}-\frac{\bar{z}^{n-1}}{(\zeta-i)}\right\}d\xi d\eta$$

$$=\sum_{\lambda=0}^{n-2}\frac{1}{2\pi i}\frac{1}{v!}\int_{-\infty}^{\infty}\gamma_{\lambda}(t)(\overline{z-t})^{\lambda}\frac{dt}{t-z}+w(z)-\frac{(-1)^{n-1}}{(n-1)!}\bar{z}^{n-1}c-I_1$$

$$+\frac{(-1)^{n-1}}{(n-1)!}\bar{z}^{n-1}I_2 \tag{3.8}$$

where

$$I_1=\frac{(-1)^{n-1}}{2\pi i}\frac{1}{(n-1)!}\int_{-\infty}^{\infty}\partial_{\bar{\zeta}}^{n-1}w(t)\frac{(\overline{t-z})^{n-1}}{t-z}dt$$

and

$$I_2=\frac{1}{2\pi i}\int_{\infty}^{\infty}\frac{\partial_{\bar{\zeta}}^{n-1}w(t)}{t-i}dt.$$

Let $h_0(t,\bar{z})$ be as in (3.5) so that $\frac{d}{dt}h_o(t,\bar{z})=\frac{(\overline{t-z})^{n-1}}{t-z}$. Integrating by parts and using regularity conditions of $\partial_{\bar{\zeta}}^{n-1}w$, we have

$$I=\frac{(-1)^{n-1}}{2\pi i}\frac{1}{(n-1)!}\int_{-\infty}^{\infty}(\partial_{\zeta}\partial_{\bar{\zeta}}^{n-1}w(t)+\partial_{\bar{\zeta}}^{n-1}w(t))h_0(t,\bar{z})dt$$

$$=\frac{(-1)^{n-1}}{2\pi i}\frac{1}{(n-1)!}\int_{-\infty}^{\infty}(f(t)+\gamma_{n-1}(t))h_0(t,\bar{z})dt$$

and

$$I_2=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}(f(t)+\gamma_{n-1}(t))\log(t-i)dt.$$

Substituting these values in (3.8), we obtain (3.7). For $0 \leq v \leq n-2$, using (3.2) and Cauchy- Pompeiu formula, we have

$$\frac{(-1)^{n-v}}{(n-v-1)!\,\pi}\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\left\{\frac{(\overline{\zeta}-z)^{n-1}}{(\zeta-\overline{z})}-\frac{z^{n-v-1}}{(\zeta-i)}\right\}d\xi d\eta$$

$$= -\sum_{\lambda=v}^{n-2}\frac{1}{2\pi i}\frac{(-1)^{\lambda-v}}{(\lambda-v)!}\int_{-\infty}^{\infty}\gamma_{\lambda}(t)\frac{(t-z)^{\lambda-v}}{t-\overline{z}}dt$$

$$-\frac{(-1)^{n-1-v}}{(n-1-v)!}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_{\overline{\zeta}}^{n-1}w(t)\frac{(t-z)^{n-1-v}}{t-\overline{z}}dt$$

$$-\frac{(-1)^{n-v}}{(n-1-v)!}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_{\overline{\zeta}}^{n-1}w(t)\frac{1}{t-i}dt+\partial_{\overline{\zeta}}^{n-1}w(i). \qquad (3.9)$$

Again integrating by parts and considering that $\frac{d}{dt}h_v(t, z) = \frac{(t-z)^{n-1-v}}{t-\overline{z}}$, we have

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_{\overline{\zeta}}^{n-1}w(t)\frac{(t-z)^{n-1-v}}{t-\overline{z}}dt = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}(f(t)+\gamma_{n-1}(t))h_v(t, z)dt$$

and

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\partial_{\overline{\zeta}}^{n-1}w(t)}{t-i}dt = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}(f(t)+\gamma_{n-1}(t))\log(t-i)dt.$$

Substituting above values in (3.9), we have (3.4). Now, making use of Gauss theorem, we get

$$\frac{1}{\pi}\int_{\mathbb{H}} f_{\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-\overline{z}} = \frac{1}{2\pi i}\int_{-\infty}^{\infty}f_{\zeta}(\zeta)\frac{\partial_{\zeta}(\partial_{\overline{\zeta}}^{n-1}w)}{\zeta-\overline{z}}d\zeta = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma_{n-1}(t)\frac{dt}{t-\overline{z}}.$$

Hence the solvability condition (3.6).

Let $T_{0,n}$ be higher order Cauchy Pompeiu operators on $\mathbb{H}$ [3, 5]. Since $\partial_{\overline{z}}T_{0,n}w = T_{0,n-1}w$, it follows that (3.7) indeed satisfy $\partial_{\overline{z}}^{n}w = f$. For any fixed k, $0 \leq k \leq n-2$, note that $\partial_{\bar{z}}^{k} h_0(t, \bar{z}) = h_k(t, z)$. Subtracting (3.4) (for $v = k$) from $\partial_{\bar{z}}^{k} w$, we have

$$\partial_{\bar{z}}^{k} w(z) = \frac{(-1)^{n-k-1}}{(n-k-1)!} c[\bar{z}^{n-k-1} - z^{n-k-1}]$$

$$+ \sum_{\lambda=k}^{n-2} \frac{1}{2\pi i} \frac{(-1)^{\lambda-k}}{(\lambda-k)!} \int_{-\infty}^{\infty} \gamma_\lambda(t) \left(\frac{(t-\bar{z})^{\lambda-k}}{t-z} - \frac{(t-z)^{\lambda-k}}{t-\bar{z}} \right) dt$$

$$- \frac{(-1)^{n-k-1}}{(n-k-1)!} \frac{1}{2\pi i} \int_{-\infty}^{\infty} [\gamma_{n-1}(t) + f(t)](z^{n-k-1} - \bar{z}^{n-k-1}) \log(t-i) dt$$

$$+ \frac{(-1)^{n-k}}{(n-k-1)!} \frac{1}{\pi} \int_{\mathbb{H}} f(\zeta) \left\{ \frac{(\overline{\zeta - z})^{n-k-1}}{(\zeta - z)} - \frac{\overline{Z}^{n-k-1}}{(\zeta - i)} - \frac{(\bar{\zeta} - z)^{n-k-1}}{(\zeta - z)} \right.$$

$$\left. + \frac{\bar{z}^{n-1}}{(\zeta - i)} \right\} d\xi d\eta = 0. \tag{3.10}$$

Since, $\lim_{z \to t_0} \frac{1}{\pi} \int_{-\infty}^{\infty} \gamma_\lambda(t) \frac{y}{|t-z|^2} dt = \gamma_\lambda(t_0)$ so $\partial_{\bar{z}}^{k} w(z) = \gamma_\lambda$ on $\mathbb{R}$.

Also, it can be verified at $z = i$ that $\partial_{\bar{z}}^{n-1} w(i) = c$.

Now, taking derivative of (3.10) w.r.t z, we get

$$\partial_z(\partial_{\bar{z}}^{n-1} w(z)) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} [\gamma_{n-1}(t) + f(t)] \frac{dt}{t-z} + \pi f(z) \tag{3.11}$$

subtracting (3.6) from (3.11), we have

$$\partial_z(\partial_{\bar{z}}^{n-1} w(z)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \gamma_{n-1}(t) \frac{y dt}{|t-z|^2} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(t) \frac{dt}{t-z} + \pi f(z) - T f_\zeta(\bar{z}).$$

Now, letting z tend to $t \in \mathbb{R}$

$$\partial_z(\partial_{\bar{z}}^{n-1} w)^+(t) = \gamma_{n-1}(t) + (\pi f)^+(t) - T f_\zeta(t) + \lim_{\substack{z \to t \\ z \in H}} \left[\frac{1}{2\pi i} \int_{-\infty}^{\infty} f(t) \frac{dt}{t-z} \right].$$

Using Plemelj-Sokhotzki formula [4], we have

$$\lim_{\substack{z \to t \\ z \in H}} \left[\frac{1}{2\pi i} \int_{-\infty}^{\infty} f(t) \frac{dt}{t - z} \right] = \frac{1}{2} f(t) + \int_{\infty}^{\infty} f(s) \frac{ds}{s - t}$$

$$= -(\pi f)^{+}(t) + Tf_{\zeta}(t). \quad \text{[using 2.5]}$$

Therefore, $\partial_z (\partial_{\bar{z}}^{n-1} w)^{+}(t) = \gamma_{n-1}(t)$.

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