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ATTESTING FINITE NUMBER OF INTEGER SOLUTIONS OR NO INTEGER SOLUTIONS TO FOUR MORDELL KINDS EQUATIONS

$Y^2 = X^3 + C, C = \pm 9, 36, -16$

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Abstract

In this text, four types of Mordell equationsc $Y^2 = X^3 + C, C = \pm 9, -16, 36$ are considered and showed that two of the equations $Y^2 = X^3 - 9, Y^2 = X^3 - 16$ have no integer solutions and the remaining two equations $Y^2 = X^3 + 9, Y^2 = X^3 + = 36$ have restricted number of integer solutions by mainly using the perceptions of properties of congruences.

1. Introduction

W. J. Ellison, F. Ellison, J. Pesek, C. E. Stahl, And D. S. Stall [7] discovered all integral solutions to the equation $y^2 = x^3 - 28$ using the effective technique of Baker and congruences. Ray P. Steiner [8] completely

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resolved the Stolarsky's conundrum $y^2 - k = x^3$. For more investigation of these types of equations, one can go through [1-6].

In this text, four groups of Mordell equations $Y^2 = X^3 + C, C = \pm 9, -16, 36$ are preferred and exposed that two of the equations $Y^2 = X^3 - 9, Y^2 = X^3 - 16$ among them have no integer solutions and the lingering two equations $Y^2 = X^3 + 9, Y^2 = X^3 + 36$ have partial integer solutions by mostly focused on the ideas of properties of congruences.

2. Procedure of Investigation

The intension of each phase is to treasure comprehensive solutions for Mordell type Diophantine equations of the form $Y^2 = X^3 + C$ where $C = \pm 9, -16, 36$

Applicable Theorem I. In [5], Let p denote a prime. Then $x^2 = -1 (\text{mod } p)$ has solutions if and only if $p \equiv 2 (\text{mod } 4)$ or $p \equiv 1 (\text{mod } 4)$.

Theorem 2.1. *If* $X, Y \in \mathbb{Z}$, *then there is no solution to* $Y^2 = X^3 - 9$.

Proof. Initially let us assume that $Y^2 = X^3 - 9$ has an integer solution (X, Y).

If $X = 0 (\text{mod } 2)$ and $X = 3 (\text{mod } 4)$, then $Y^2 \equiv -1 (\text{mod } 4)$ and $Y^2 \equiv 2 (\text{mod } 4)$ respectively. But, both of them are impossible. Hence, it is possible that $X \equiv 1 (\text{mod } 4)$ since it leads to $Y^2 \equiv 0 (\text{mod } 4)$.

Now, the implicit equation can be considered as

$$Y^2 + 1 = X^3 - 8 = (X - 2)(X^2 + 2X + 4).$$

As $X \equiv 1 (\text{mod } 4), X^2 + 2X + 4 \equiv 3 (\text{mod } 4)$.

This implies that $(Y^2 + 1)$ is divisible by a prime number which is congruent to 3 modulo 4.

That is $Y^2 \equiv -1 (\bmod p)$ where $p \equiv 3 (\bmod 4)$.

This is a contradiction to theorem I.

Hence, $Y^2 = X^3 - 9$ has no integer solution.

Theorem 2.2. *The feasible integral solutions to the particular Mordell equation* $Y^2 = X^3 + 9$ *are* $(X, Y) = \{(0, \pm 3), (3, \pm 6), (6, \pm 15), (40, \pm 253), (-2, \pm 1)\}$.

Proof. Rewrite the proposed equation as $X^3 = Y^2 - 9 = (Y + 3)(y - 3)$

If X is even, then Y is odd. If X is odd, then Y is even. If d is the common divisor of $(Y + 3)$ and $(Y - 3)$, then also divides their difference $(Y + 3) - (Y - 3) = 6$. Therefore, must be any one of the values 1, 2, 3, 6.

Case 1. Suppose Y is even

Then, $(Y + 3)$ and $(Y - 3)$ are both odd. So, $\gcd(Y + 3, Y - 3)$ is either 1 or 3.

If $\gcd(Y + 3, Y - 3) = 1$, then they are relatively prime. Since their product is a cube, they both are cube.

That is $(Y + 3) = a^3$ and $(Y - 3) = b^3$

$$\Rightarrow a^3 - b^3 = 6$$

However, no two odd cubes produce a difference 6.

Hence, $\gcd(Y + 3, Y - 3) = 3$

Since Y is even, $Y \equiv 0 (\bmod 4)$ or $Y \equiv 2 (\bmod 4)$

Subcase 1.1. Suppose $Y \equiv 0 (\bmod 4)$

Then, $Y + 3 \equiv 3 (\bmod 4)$ and $Y - 3 \equiv 1 (\bmod 4)$.

Dividing (1) by 27, it is emblazoned as

$$\left(\frac{X}{3}\right)^3 = \left(\frac{Y + 3}{3}\right)\left(\frac{Y - 3}{9}\right)$$

Due to the fact that division of each component by a multiple of $3 = \gcd(Y+3, Y-3)$, the right-hand side of the preceding equation comprehends relatively prime factors and therefore each factor is a cube.

That is, $\frac{Y+3}{3} = a^3$ and $\frac{Y-3}{9} = b^3$

$$\Rightarrow 3a^3 - 3 = 9b^3 + 3$$

$$\Rightarrow a^3 - 3b^3 = 2$$

This is factual only for $a = -1$, $b = -1$

If $a = -1$ or $b = -1$, then $Y = -6$ and $X - 3$.

Therefore, the integral solution in this situation is $(X, Y) = (3, -6)$

Subcase 1.2. Suppose $Y \equiv 2 \pmod 4$

However, $Y + 3 \equiv 1 \pmod 4$ and $Y - 3 \equiv 3 \pmod 4$.

By dividing (1) by 27, it is converted into

$$\left(\frac{X}{3}\right)^3 = \left(\frac{Y+3}{9}\right)\left(\frac{Y-3}{3}\right)$$

As the description specified in subcase 1.1, it is possible to designate the components on the right-hand side of the previous equation as

$\frac{Y+3}{9} = a^3$ and $\frac{Y-3}{3} = b^3$

$$\Rightarrow 9a^3 - 3 = 3b^3 + 3$$

$$\Rightarrow 3a^3 - b^3 = 2$$

The only possibility of the above equation is $a = 1$, $b - 1$.

Either of the above choices of a and b delivers $Y = 6$ and $X = 3$.

Therefore, in this circumstance the comprehensive solution is $(X, Y) = (3, 6)$

As an effect, this case grants two integral solutions $(X, Y) = (3, \pm 6)$.

Case 2. Presume Y is odd

Then, both the factors in (1) are even and $\gcd(Y+3, Y-3)$ is either 2 or 6. Also, $Y \equiv 1 (\bmod 4)$ or $Y \equiv 3 (\bmod 4)$

Subcase 2.1. If $\gcd(Y+3, Y-3) = 2$ and $Y \equiv 1 (\bmod 4)$, then $Y + 3 \equiv 0 (\bmod 4)$ and $Y - 3 \equiv 2 (\bmod 4)$.

Divide both sides of (1) by 8, it is received that

$$\left(\frac{X}{2}\right)^3 = \left(\frac{Y+3}{4}\right)\left(\frac{Y-3}{2}\right)$$

As in case 1, it is taken as

$\frac{Y+3}{4} = a^3$ and $\frac{Y-3}{2} = b^3$

$$\Rightarrow 4a^3 - 3 = 2b^2 + 3$$

$$\Rightarrow 2a^3 - b^3 = 3$$

The preparable chances of and are declared by

$a = 1, b = -1$ and $a = 4, b = 5$

If $a = 1$ or $b = -1$, then $Y = 1$ and $X = -2$.

If $a = 4$ or $b = 5$, then $Y = 253$ and $X = 4$.

Therefore, the integral solutions achieved in this instance are $(X, Y) = \{(-2, 1), (4, 253)\}$.

Subcase 2.2. If $Y \equiv 3 (\bmod 4)$, then $Y + 3 \equiv 2 (\bmod 4)$ and $Y - 3 \equiv 0 (\bmod 4)$ and proceeding as in case 1, it is concluded that $(X, Y) = \{(-2, -1), (4, -253),\}$.

Similarly, if $\gcd(Y+3, Y-3) = 6$, then integer solutions $(X, Y) = (0, \pm 3)$ of (1) are accomplished for both selections of $Y \equiv 1 (\bmod 4)$ and $Y \equiv 3 (\bmod 4)$ by means of the same methodology in case 1.

Case 3. In all the above two cases, the fact about the unique factorization domain in the ring of integers is considered. But one of the factors in the right-hand side of (1) divided by d^2 can be stretched to a fractional number. That is, $\frac{Y+3}{d^2}$ or $\frac{Y-3}{d^2}$ may be a fractional number. The product of an integer and a fractional number is a cube of an integer means that it fulfils the following conditions.

$$\frac{Y+3}{d^2} = \frac{a}{b} \text{ and } \frac{Y-3}{d} = a^2 b \text{ or } \frac{Y+3}{d^2} = \frac{a^2}{b} \text{ and } \frac{Y-3}{d} = ab$$

These choices afford the succeeding equations

$$a^2 b^2 - da - b = 0 \text{ or } ab^2 - da^2 - b = 0.$$

Solving them for b, it is acquired that

$$b = \frac{1 \pm \sqrt{1+4a^2 d}}{2a^2} \text{ or } b = \frac{1 \pm \sqrt{1+4a^2 d}}{2a}.$$

In these two equations, the discriminant is a positive integer if and only if $a = 1$, $d = 6$. Implementing this condition, a diverse solution is obtained as follows:

$$\frac{Y+3}{36} = \frac{1}{b} \text{ and } \frac{Y-3}{6} = b$$

$$\Rightarrow \frac{36}{b} - 3 = 6b + 3$$

$$\Rightarrow b^2 + b - 6 = 0$$

$$\Rightarrow b = 2 \text{ or } -3$$

$$\Rightarrow y = \pm 15 \text{ and hence } X = 6.$$

So, all the solutions assimilated for the preferred equation are

$$(X, Y) = \{(-2, \pm 1), (4, \pm 253), (0, \pm 3), (6, \pm 15)\}$$

Theorem 2.3. *If* $X, Y \in Z$, *then the equation* $Y^2 = X^3 - 16$ *has no solution.*

Proof. Redraft the suggested equation as

$$X^3 = Y^2 + 16 = (Y + 4i)(Y - 4i)$$

Since $Y^2 \equiv X^3 (\bmod 4)$, either X and Y are both even or they both odd.

Case 1. Suppose X and Y are both odd

If a is a common divisor of $(Y + 4i)$ and $(Y - 4i)$, then a divides their difference

$$(Y + 4i) - (Y - 4i) = 8i.$$

Hence, there exists some $b \in Z[i]$ such that $a = b \times 8i$.

Therefore, norm of 'a' denoted by $N(a)$ is provided by

$N(a) = N(b \times 8i) = N(b)N(8i)$, since norm is multiplicative in $Z[i]$.

This implies that $N(a)$ divides $N(8i) = 64$ where $N(x + iy) = x^2 + y^2$ is the norm of the Gaussian integer $x + iy$.

As well as $N(a) = 1$ divides $N(Y + 4i) = Y^2 + 16 = X^3$ which is odd.

Then, $N(a) = 1$ and thus a is a unit in $Z[i]$. It is well known that every unit is a cube in $Z[i]$.

This means that $(Y + 4i)$ and $(Y - 4i)$ are relatively prime.

From (2), both $(Y + 4i)$ and $(Y - 4i)$ are cubes.

Contemplate

$$Y + 4i = (M + Ni)^3$$

$$\Rightarrow Y = M(M^2 - 3N^2) \text{ and } 4 = N(3M^2 - N^2)$$

From the second of the above equation, N must be ± 1 or ± 2 or ± 4. None of these values of N gives an integer value for M.

Hence, the projected equation has no solution when X and Y are both odd.

Case 2. Suppose X and Y are both even.

Transcribe $X = 2X'$ and $Y = 2Y'$

Then, (2) can be swatted into the following equation

$$Y'^2 = 2X'^3 - 4$$

$$\Rightarrow Y' \text{ is even}$$

If X' is odd, then $Y'^2 \equiv 2 \pmod 4$. This is not possible, since even squares are always congruent to 0 modulo 4.

Hence, X' is even.

Now, designate $X' = 2X''$ and $Y' = 2Y''$.

Engaging these two alternations in (3), it can be extolled by

$Y''^2 = 4X''^3 - 1$ implies that Y'' is odd

If X'' is odd or even, then $Y''^2 \equiv 3 \pmod 4$ which is impossible because $Y''^2 \equiv 1 \pmod 4$.

Hence, this case offers no solution to the predictable equation.

Theorem 2.4. *The limited number of integral solutions to the definite type of Mordell equation* $Y^2 = X^3 + 36$ are $(X, Y) = \{(0, \pm 6), (4, \pm 10), (12, \pm 42), (-3, \pm 3)\}$.

Proof. The considered equation can be rephrased as

$$X^3 = Y^2 - 36 = (Y + 6)(Y - 6)$$

Here both X and Y are either even or odd. If $(Y + 6)$ and $(Y - 6)$ have a common divisor d', then their difference $(Y + 6) - (Y - 6) = 12$ is also divided by d'. Therefore, d' essentially be any one of the values 1, 2, 3, 4, 6, 12.

Case 1. Suppose X and Y are even

Then, $(Y + 6)$ and $(Y - 6)$ are together even and $\gcd(Y + 6, Y - 6) = 2$ or 4 or 6 or 12.

If $\gcd(Y + 6, Y - 6) = 2$, then by dividing (4) by 8, the following conjectures can be made

$$\left(\frac{X}{2}\right)^3 = \left(\frac{Y+6}{2}\right)\left(\frac{Y-6}{4}\right) = a^3b^3 \text{ or } \left(\frac{X}{2}\right)^3 = \left(\frac{Y+6}{4}\right)\left(\frac{Y-6}{2}\right) = a^3b^3$$

$$\Rightarrow a^3 - 2b^3 = 6 \text{ or } 2a^3 - b^3 = 6$$

It is scrutinized that both equations do not have solution in integers.

When $\gcd(Y+6, Y-6) = 4$, then dividing (4) by 64 and utilizing the prior technique as revealed earlier, the subsequent equations are grasped

$$a^3 - 4b^3 = 3 \text{ or } 4a^3 - b^3 = 3$$

$$\Rightarrow a = -1, b = -1 \text{ or } a = 1, b = 1$$

If $a = -1$ and $b = -1$, then $(X, Y) = (4, -10)$. If $a = 1$ and $b = 1$, then $(X, Y) = (4, 10)$

Hence, the needed solution is $(X, Y) = (4, \pm 10)$

If $\gcd(Y+6, Y-6) = 6$, then dividing (4) by 6^3 and using the identical method as acknowledged in earlier, theorems, it is scrutinized that the ensuing equations procures no integer values for both a and b.

$$a^3 - 6b^3 = 2 \text{ or } 6a^3 - b^3 = 2$$

If $\gcd(Y+6, Y-6) = 12$, then dividing (4) by 12^3 and by the similar procedures as given above, the succeeding equations are detected

$$a^3 - 12b^3 = 1 \text{ or } 12a^3 - b^3 = 1$$

$$\Rightarrow a = 1, b = 0 \text{ or } a = 0, b = -1$$

If $a = 1$ and $b = 0$, then $(X, Y) = (0, 6)$. If $a = 0$ and $b = -1$, then $(X, Y) = (0, -6)$.

Hence, the necessary solution is $(X, Y) = (0, \pm 6)$

This scenario thus provides the ideal solutions $(X, Y) = \{(4, \pm 10), (0, \pm 6)\}$.

Case 2. Suppose X and Y are odd

Then both the factors in right hand side of (4) are odd and $\gcd(Y+6, Y-6)$ is either 1 or 3.

If $\gcd(Y+6, Y-6) = 1$, then it is evidenced that

$$Y + 6 = a^3 \text{ and } Y - 6 = b^3$$

$$\Rightarrow a^3 - b^3 = 12.$$

But, the difference of no two odd cubes is 12.

Hence, $\gcd(Y+6, Y-6) = 3$. Then for both selections $Y \equiv 1 (\bmod 4)$ and $Y \equiv 3 (\bmod 4)$ and by utilizing the undistinguishable approach, it is noted that

$$a^3 - 3b^3 = 4 \text{ or } 3a^3 - b^3 = 4$$

$$\Rightarrow a = 1, b = -1$$

$$\Rightarrow Y = \pm 3, X = -3$$

Henceforth, the outcome in this case is $(X, Y) = (-3, \pm 3)$.

Case 3. Appeal the concept of case 3 in theorem 2.1, an exclusive solution is perceived when $a = 1, d = 12$ as follows

$$\frac{Y+6}{144} = \frac{1}{b} \text{ and } \frac{Y-6}{12} = b$$

$$\Rightarrow \frac{144}{b} - 6 = 12b + 6$$

$$\Rightarrow b^2 + b - 12 = 0$$

$$\Rightarrow b = -4 \text{ or } 3 \Rightarrow y = \pm 42 \text{ and hence } x = 12.$$

The solution obtained in this case is $(X, Y) = (12, \pm 42)$.

3. Conclusion

In this article, all existing integral solutions to the Mordell's equation $Y^2 = X^3 + K$, where K are square numbers $\pm 9, -16, 36$ are exposed.

Similarly, one can explore integer solutions or coprime integer solutions for the pair (X, Y) by choosing numerous values of K in the Mordell's equations by using the concepts of divisibility.

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